BGG CORRESPONDENCE, COHOMOLOGY OF COMPACT KÄHLER MANIFOLDS, AND NUMERICAL INVARIANTS

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Introduction

Let X be a compact Kähler manifold of dimension d, and consider the cohomology

$$Q_X \ = \ H^*\big(X, \omega_X\big) \ = \ \oplus H^i\big(X, \omega_X\big)$$

of the canonical bundle of X. Via cup product, we may view this as a graded module over the exterior algebra $E = \Lambda^* H^1\big(X, \mathcal{O}_X\big)$ on $H^1\big(X, \mathcal{O}_X\big)$. In recent years, there has been considerable interest in the study of graded modules over the exterior algebra of a vector space, and the so-called Bernstein-Gel'fand-Gel'fand (BGG) correspondence between these and linear complexes over a symmetric algebra (cf. [BGG], [EFS], [Eis], [Co]). Our purpose here is to show that a body of work involving generic vanishing theorems implies that this picture takes a particularly clean form in the case of Q_X, and to deduce some surprising connections between the algebraic properties of this module and the geometry of X. We then use a vector bundle arising from the BGG correspondence to establish, under mild additional hypotheses, a number of inequalities on Hodge numbers and the holomorphic Euler characteristic of X.

Turning to details, we follow the degree conventions of [EFS] and [Eis]: we take E to be generated in degree -1, and declare that the summand $H^i\big(X, \omega_X\big)$ of Q_X has degree $-i$. By analogy with the Castelnuovo-Mumford regularity of a module over the symmetric algebra, we say that a graded E-module M generated in non-positive degrees is m-regular if for every $p \geq 0$, the p^{th} module of syzygies of M has all its generators in degrees $\geq -(p+m)$ (Definition 1.1).

Our first main result asserts that the regularity of Q_X is governed by the generic fibre dimension of the Albanese mapping $\text{alb}_X : X \longrightarrow \text{Alb}(X)$ of X.

Theorem A. *Set*

$$k \ = \ k(X) \ = \ \dim X - \dim \text{alb}_X(X).$$

Then Q_X is k-regular as an E-module. In particular, if X has maximal Albanese dimension (i.e. if $k = 0$), then Q_X is generated in degree 0 and has a linear free resolution.

At least when X is projective, results of Kollár [Ko] then imply that the regularity of Q_X is exactly k, i.e. that Q_X is not $(k-1)$-regular. We remark that other natural E-modules – for instance $H^*\big(X, \mathcal{O}_X\big)$ – do not enjoy such simple regularity properties.

It is natural to ask what additional geometric data Q_X determines. Of course the dimensions of its graded pieces are the Hodge numbers $h^{d,i}(X)$ of X, but the module in question turns out to contain also more subtle information. Recall that in classical terminology, a *paracanonical*

First author partially supported by NSF grant DMS-0652845.
Second author partially supported by NSF grant DMS-0601252 and a Sloan Fellowship.

divisor on X is an effective divisor algebraically equivalent to a canonical divisor. The set of all such is parametrized by the Hilbert scheme (or Douady space) $\mathrm{Div}^{\{\omega\}}(X)$, which admits an Abel-Jacobi mapping

$$u : \mathrm{Div}^{\{\omega\}}(X) \longrightarrow \mathrm{Pic}^{\{\omega\}}(X)$$

to the corresponding component of the Picard torus of X. The projective space $|\,\omega_X\,|$ parametrizing all canonical divisors sits as a subvariety of $\mathrm{Div}^{\{\omega\}}(X)$: it is the fibre of u over the point $[\,\omega_X\,] \in \mathrm{Pic}^{\{\omega\}}(X)$. Our second result asserts that Q_X dictates the infinitesimal geometry of $\mathrm{Div}^{\{\omega\}}(X)$ along $|\,\omega_X\,|$.

Theorem B. *One can read off from Q_X (and its structure as an E-module) the projectivized normal cone to $|\,\omega_X\,|$ inside $\mathrm{Div}^{\{\omega\}}(X)$.*

In fact, Q_X determines via the BGG-correspondence a coherent sheaf $\mathcal{F}$ on the projectivized tangent space $\mathbf{P}$ to $\mathrm{Pic}^{\{\omega\}}(X)$ at $[\,\omega_X\,]$, and we show that the normal cone in question is identified with $\mathbf{P}(\mathcal{F})$.

Under additional hypotheses the BGG sheaf $\mathcal{F}$ is locally free, and then the geometry of vector bundles on projective space yields some inequalities on numerical invariants of X. One says that X carries an *irregular fibration* if it admits a surjective morphism $X \longrightarrow Y$ with connected positive dimensional fibres onto a normal analytic variety Y with the property that (any smooth model of) Y has maximal Albanese dimension. These are higher-dimensional analogues of irrational pencils in the case of surfaces. Now consider the formal power series:

$$\gamma(X;t) \ =_{\mathrm{def}} \ \prod_{j=1}^{d} (1-jt)^{(-1)^j h^{d,j}} \ \in \ \mathbf{Z}[[t]],$$

where $h^{i,j} = h^{i,j}(X)$. Write $q = h^1\big(X, \mathcal{O}_X\big)$ for the irregularity of X (so that $q = h^{d,d-1}$), and for $1 \le i \le q-1$ denote by

$$\gamma_i = \gamma_i(X) \ \in \ \mathbf{Z}$$

the coefficient of t^i in $\gamma(X;t)$. Thus γ_i is a polynomial in the $h^{d,j}$. We prove:

Theorem C. *Assume that X does not carry any irregular fibrations (so that in particular X itself has maximal Albanese dimension). Then*

$$\gamma_i(X) \ \ge \ 0$$

for every $1 \le i \le q-1$, and more generally any Schur polynomial of weight $\le q-1$ in the γ_i is non-negative. Furthermore,

$$\gamma_i(X) \ = \ 0 \quad \mathit{for} \quad \chi\big(X, \omega_X\big) \ < \ i \ < \ q.$$

For example, when $i = 1$ this yields (under the assumption of the theorem) the inequality

$$(*) \qquad h^{d,1} - 2h^{d,2} + 3h^{d,3} - \ldots + (-1)^{d+1} \cdot d \cdot h^{d,d} \ \ge \ 0.$$

Supposing e.g. that $\dim X = 3$, (*) reduces to the classically known Castelnuovo-de Franchis-type inequality $h^{0,2} = h^{3,1} \ge 2q-3$, but the positivity of higher γ_i produces already some new statements. In fact, for threefolds satisfying the hypotheses of the theorem, the inequality $\gamma_2 \ge 0$ implies that asymptotically

$$h^{3,1} \ \succeq \ 2q + \sqrt{2q}$$

(see Corollary 3.5 for the precise statement), and a similar result holds for $h^{0,3}$. When X is a surface without irrational pencils, related methods applied to Ω^1_X yield a new inequality for $h^{1,1}$ as well. We also use the bundle $\mathcal{F}$ to derive a number of other inequalities, including a simple proof of the bound $\chi(\omega_X) \geq q(X) - \dim X$ due to Pareschi and the second author [PP3] for X satisfying the hypotheses of the Theorem. The method used here allows us to further analyze possible borderline cases when the Euler characteristic is small, and conjecture stronger inequalities. Examples show that all of the inequalities above can fail when X does carry irregular fibrations. Finally, under the same hypotheses, we show that Theorem B implies the somewhat surprising fact that whether or not the projective space $|\omega_X|$ is an irreducible component of $\mathrm{Div}^{\{\omega\}}(X)$ depends in most cases only on the Hodge numbers $h^{d,j}(X)$ (Proposition 3.8).

These results are simple consequences of work of several authors concerning generic vanishing theorems on irregular varieties. Specifically, let $S = \mathrm{Sym}\big(H^1\big(X, \mathcal{O}_X\big)^*\big)$ be the symmetric algebra on the vector space $H^1\big(X, \mathcal{O}_X\big)^*$. According to the general theory of [EFS], [Eis], the properties of Q_X as an E-module are governed by the linear complex $\mathbf{L}$

$$0 \longrightarrow S \otimes_{\mathbf{C}} H^0(X, \mathcal{O}_X) \longrightarrow S \otimes_{\mathbf{C}} H^1(X, \mathcal{O}_X) \longrightarrow \ldots \longrightarrow S \otimes_{\mathbf{C}} H^d(X, \mathcal{O}_X) \longrightarrow 0$$

of S-modules. Our basic observation is that $\mathbf{L}$ is an avatar of the *derivative complex* studied in [GL2]: in a word, $\mathbf{L}$ essentially computes locally the pushforward to $\mathrm{Pic}^0(X)$ of the Poincaré sheaf on $X \times \mathrm{Pic}^0(X)$. The exactness properties of $\mathbf{L}$ required for Theorem A then follow from the results of [Hac], [Pa], [PP3]. As for Theorems B and C, let $\mathbf{P}$ be the projective space of one-dimensional subspaces of $H^1\big(X, \mathcal{O}_X\big)$. Then $\mathbf{L}$ sheafifies to give a linear complex $\underline{\mathbf{L}}$ of bundles on $\mathbf{P}$:

$$\ldots \longrightarrow \mathcal{O}_{\mathbf{P}}(-1) \otimes H^{d-1}\big(X, \mathcal{O}_X\big) \longrightarrow \mathcal{O}_{\mathbf{P}} \otimes H^d\big(X, \mathcal{O}_X\big) \longrightarrow 0.$$

To prove Theorem B we establish that the cokernel of the displayed map is a sheaf $\mathcal{F}$ whose projectivization is the normal cone appearing in that statement. It follows from results of [GL2] that if X has no irregular fibrations, then $\underline{\mathbf{L}}$ is acyclic and $\mathcal{F}$ is locally free. Theorem C then expresses the fact that the Chern classes of $\mathcal{F}$ are non-negative. When the rank $\mathrm{rk}(\mathcal{F}) = \chi(X, \omega_X)$ of $\mathcal{F}$ is small compared compared to $q - 1 = \dim \mathbf{P}$, it is hard for such a bundle to exist, giving rise to lower bounds on $\chi(X, \omega_X)$. We remark that the sheafified complex $\underline{\mathbf{L}}$ has come up in passing in [EL] and [HP], but it has not up to now been exploited in a systematic fashion.

Going back to Theorem A, we note that a statement of a similar type was established in [EPY] for the singular cohomology of the complement of an affine complex hyperplane arrangement: in fact this cohomology always has a linear resolution over the exterior algebra in its first cohomology (though it is not 0-regular). But whereas the result of [EPY] is of combinatorial genesis and does not involve using the BGG correspondence, as explained above Theorem A is ultimately based on translating Hodge-theoretic information via BGG. We note also that Catanese suggested in [Ca2] that it might be interesting to study the BGG correspondence for holomorphic cohomology algebras.

Concerning the organization of the paper, in §1 we combine the generic vanishing package with the BGG machine to prove Theorem A and some related results. In the second section, we introduce the BGG sheaf $\mathcal{F}$, and relate it to the space of paracanonical divisors. Finally, in §3, we use this sheaf to prove some inequalities on Hodge numbers and other numerical invariants of X.

Acknowledgements. The paper owes a lot to discussions with G. Pareschi – some of the material we discuss would not have existed without his input. We are also grateful to F. Catanese, I. Coandă, Ch. Hacon, Ch. Schnell, F.-O. Schreyer and C. Voisin for various comments and suggestions.

1. The canonical cohomology module

In this section we study the regularity of the canonical cohomology module over the exterior algeba. After a brief review of the BGG correspondence, we prove Theorem A. In the final subsection we discuss some variants.

Review of the BGG correspondence. We recall briefly from [EFS] and [Eis] some basic facts concerning the BGG correspondence. Let V be a q-dimensional complex vector space,[1] and let $E = \bigoplus_{i=0}^{q} \bigwedge^i V$ be the exterior algebra over V. Denote by $W = V^*$ be the dual vector space, and by $S = \mathrm{Sym}(W)$ the symmetric algebra over W. Elements of W are taken to have degree 1, while those in V have degree -1.

Let $P = \oplus_i P_i$ be a graded module over E. The BGG correspondence associates to P the complex $\mathbf{L}(P)$ of graded modules over the symmetric algebra S given by

$$\ldots \longrightarrow S \otimes_{\mathbf{C}} P_{j+1} \longrightarrow S \otimes_{\mathbf{C}} P_j \longrightarrow S \otimes_{\mathbf{C}} P_{j-1} \longrightarrow \ldots$$

with differential induced by

$$s \otimes p \mapsto \sum_i x_i s \otimes e_i p,$$

where $x_i \in W$ and $e_i \in V$ are dual bases. The *dual* over E of the module P is defined to be the E-module $Q = \widehat{P} = \oplus_j P^*_{-j}$ (so positive degrees are switched to negative ones and vice versa).

It is natural to consider a notion of regularity for E-modules analogous to the theory of Castelnuovo-Mumford regularity for finitely generated S-modules. We limit ourselves here to modules concentrated in non-positive degrees.

Definition 1.1. (Regularity). A finitely generated graded E-module Q with no component of positive degree is called *m-regular* if it is generated in degrees 0 up to $-m$, and if its minimal free resolution has at most $m+1$ linear strands. Equivalently, Q is m-regular if and only if $\mathrm{Tor}_i^E(Q, \mathbf{C})_{-i-j} = 0$ for all $i \geq 0$ and all $j \geq m+1$.

As an immediate application of the results of Eisenbud-Fløystad-Schreyer, one has the following addendum to [EFS] Corollary 2.5 (cf. also [Eis] Theorems 7.7, 7.8), suggested to us by F.-O. Schreyer.

Proposition 1.2. *Let P be a finitely generated graded module over E with no component of negative degree, say $P = \oplus_{i=0}^d P_i$. Then $Q = \widehat{P}$ is m-regular if and only if $\mathbf{L}(P)$ is exact at the first $d-m$ steps from the left, i.e. if and only if the sequence*

$$0 \longrightarrow S \otimes_{\mathbf{C}} P_d \longrightarrow S \otimes_{\mathbf{C}} P_{d-1} \longrightarrow \ldots \longrightarrow S \otimes_{\mathbf{C}} P_m$$

of S-modules is exact. □

[1] The BGG correspondence works over any field, but in the interests of unity we stick throughout to $\mathbf{C}$.

Regularity of the canonical cohomology module. We propose now to apply the BGG machine to the canonical cohomology module.

As in the Introduction, let X be a compact Kähler manifold of dimension d, and $\mathrm{alb}_X : X \longrightarrow \mathrm{Alb}(X)$ its Albanese map. Set

$$V = H^1\big(X, \mathcal{O}_X\big) \quad , \quad E = \Lambda^* V \quad , \quad W = V^* \quad , \quad S = \mathrm{Sym}(W).$$

We are interested in the graded E-modules

$$P_X \;=\; \bigoplus_{i=0}^{d} H^i\big(X, \mathcal{O}_X\big) \quad , \quad Q_X \;=\; \bigoplus_{i=0}^{d} H^i(X, \omega_X),$$

the E-module structure arising from wedge product with elements of $H^1\big(X, \mathcal{O}_X\big)$. These become dual modules (thanks to Serre duality) provided that we assign $H^i\big(X, \mathcal{O}_X\big)$ degree $d-i$, and $H^i\big(X, \omega_X\big)$ degree $-i$.

The BGG functor applied to P_X gives a complex $\mathbf{L}(P_X)$ of graded S-modules in homological degrees 0 to d, which here takes the form:

$$0 \longrightarrow S \otimes_{\mathbf{C}} H^0\big(X, \mathcal{O}_X\big) \longrightarrow S \otimes_{\mathbf{C}} H^1\big(X, \mathcal{O}_X\big) \longrightarrow \ldots \longrightarrow S \otimes_{\mathbf{C}} H^d\big(X, \mathcal{O}_X\big) \longrightarrow 0. \tag{1.1}$$

Writing $\mathbf{P} = \mathbf{P}_{\mathrm{sub}}(V)$ for the projective space of one-dimensional subspaces of V, this complex sheafifies to yield a linear complex $\underline{\mathbf{L}}(P_X)$ of vector bundles on $\mathbf{P}$:

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_{\mathbf{P}}(-d) \otimes H^0\big(X, \mathcal{O}_X\big) \longrightarrow \mathcal{O}_{\mathbf{P}}(-d+1) \otimes H^1\big(X, \mathcal{O}_X\big) \longrightarrow \ldots \\ \ldots \longrightarrow \mathcal{O}_{\mathbf{P}}(-1) \otimes H^{d-1}\big(X, \mathcal{O}_X\big) \longrightarrow \mathcal{O}_{\mathbf{P}} \otimes H^d\big(X, \mathcal{O}_X\big) \to 0. \end{aligned} \tag{1.2}$$

Recall from Proposition 1.2 that the regularity of Q_X as an E-module is governed by the BGG complex $\mathbf{L}(P_X)$. Therefore Theorem A from the Introduction is a consequence of

Theorem 1.3. *Write $k = k(X) = \dim X - \dim \mathrm{alb}_X(X)$. Then $\mathbf{L}(P_X)$ and $\underline{\mathbf{L}}(P_X)$ are exact in the first $d-k$ terms from the left.*

Proof of Theorem 1.3. It is sufficient to prove the exactness for (1.1), as this implies the corresponding statement for its sheafified sibling. The plan for this is to relate $\mathbf{L}(P_X)$ to the *derivative complex* introduced and studied in [GL2].

Let $\mathbf{A} = \mathrm{Spec}(\mathrm{Sym}(W))$ be the affine space corresponding to V, viewed as an algebraic variety, so that a point in $\mathbf{A}$ is the same as a vector in V. Then there is a natural complex $\mathcal{K}^\bullet$ of trivial algebraic vector bundles on $\mathbf{A}$:

$$0 \longrightarrow \mathcal{O}_{\mathbf{A}} \otimes H^0\big(X, \mathcal{O}_X\big) \longrightarrow \mathcal{O}_{\mathbf{A}} \otimes H^1\big(X, \mathcal{O}_X\big) \longrightarrow \ldots \longrightarrow \mathcal{O}_{\mathbf{A}} \otimes H^d\big(X, \mathcal{O}_X\big) \longrightarrow 0,$$

with maps given at each point of $\mathbf{A}$ by wedging with the corresponding element of $V = H^1\big(X, \mathcal{O}_X\big)$. Recalling that $\Gamma\big(\mathbf{A}, \mathcal{O}_{\mathbf{A}}\big) = S$, one sees that $\mathbf{L}(P_X) = \Gamma\big(\mathbf{A}, \mathcal{K}^\bullet\big)$ is the complex obtained by taking global sections in $\mathcal{K}^\bullet$. As $\mathbf{A}$ is affine, to prove the stated exactness properties of $\mathbf{L}(P_X)$, it is equivalent to establish the analogous exactness for the complex $\mathcal{K}^\bullet$, i.e. we need to show the vanishings $\mathcal{H}^i(\mathcal{K}^\bullet) = 0$ of the cohomology sheaves of this complex in the range $i < d-k$. For this it is in turn equivalent to prove the vanishing

$$\mathcal{H}^i\big(\mathcal{K}^\bullet\big)_0 \;=\; 0 \tag{*}$$

of the stalks at the origin of these homology sheaves in the same range $i < d-k$. Indeed, (*) implies that $\mathcal{H}^i\big(\mathcal{K}^\bullet\big) = 0$ in a neighborhood of the origin. But the differential of $\mathcal{K}^\bullet$ scales linearly in radial directions through the origin, so we deduce the corresponding vanishing on all of $\mathbf{A}$.

Now let $\mathbf{V}$ be the vector space V, considered as a complex manifold, so that $\mathbf{V} = \mathbf{C}^q$, where $q = h^1(X, \mathcal{O}_X)$ is the irregularity of X. Then on $\mathbf{V}$ we can form as above a complex $(\mathcal{K}^\bullet)^{an}$

$$0 \longrightarrow \mathcal{O}_{\mathbf{V}} \otimes H^0(X, \mathcal{O}_X) \longrightarrow \mathcal{O}_{\mathbf{V}} \otimes H^1(X, \mathcal{O}_X) \longrightarrow \ldots \longrightarrow \mathcal{O}_{\mathbf{V}} \otimes H^d(X, \mathcal{O}_X) \longrightarrow 0,$$

of coherent analytic sheaves, which is just the complex of analytic sheaves determined by the algebraic complex $\mathcal{K}^\bullet$. This analytic complex was studied in [GL2], where it was called the *derivative complex* $D^\bullet_{\mathcal{O}_X}$ of $\mathcal{O}_X$. Since passing from a coherent algebraic to a coherent analytic sheaf is an exact functor (cf. [GAGA, 3.10]), one has $\mathcal{H}^i((\mathcal{K}^\bullet)^{an}) = \mathcal{H}^i((\mathcal{K}^\bullet))^{an}$. So it is equivalent for (*) to prove:

$$(**) \qquad \mathcal{H}^i\big((\mathcal{K}^\bullet)^{an}\big)_0 \ = \ 0 \ \text{ for } i < d-k.$$

But this will follow immediately from a body of results surrounding generic vanishing theorems.

Specifically, write $\mathrm{Pic}^0(X) = V/\Lambda$, let $\mathcal{P}$ be a normalized Poincaré line bundle on $X \times \mathrm{Pic}^0(X)$, and write

$$p_1 : X \times \mathrm{Pic}^0(X) \longrightarrow X \ \ , \ \ p_2 : X \times \mathrm{Pic}^0(X) \longrightarrow \mathrm{Pic}^0(X)$$

for the two projections. The main result of [GL2], Theorem 3.2, says that via the exponential map $\exp : V \to \mathrm{Pic}^0(X)$ we have the identification of the analytic stalks at the origin

$$(***) \qquad \mathcal{H}^i\big((\mathcal{K}^\bullet)^{an}\big)_0 \ \cong \ (R^i p_{2*}\mathcal{P})_0.$$

On the other hand, by [PP3] Theorem C, we have $R^i p_{2*}\mathcal{P} = 0$ for $i < d-k$, where p_2 is the projection onto the second factor.[2] In view of (***), this gives (**), and we are done. □

Remark 1.4. Note that the structure of $P_X = H^*(X, \mathcal{O}_X)$ as an E-module is often more complicated than that of Q_X. For instance, suppose that X is a d-dimensional hypersurface of very large degree in an abelian variety. Then the map

$$H^1(X, \mathcal{O}_X) \otimes H^{d-1}(X, \mathcal{O}_X) \longrightarrow H^d(X, \mathcal{O}_X)$$

is not surjective, meaning that P_X requires generators in different degrees.

We conclude this subsection by observing that in the projective case, the regularity bound in Theorem A is always optimal.

Proposition 1.5. *Assume that X is a smooth projective variety, and as above let $k = k(X)$ denote the dimension of the generic fibre of the Albanese map. Then Q_X is not $(k-1)$-regular.*

Proof. Writing $a : X \longrightarrow A = \mathrm{Alb}(X)$ for the Albanese map, a result of Kollár [Ko] asserts that one has a splitting

$$\mathbf{R}a_*\omega_X \ \cong \ \oplus_{j=0}^k R^j a_*\omega_X\,[-j]$$

in the derived category of A.[3] Therefore Q_X can be expressed as a direct sum

$$Q_X \ = \ \oplus_{j=0}^k Q^j[-j] \ \ , \ \ \text{with} \ \ Q^j \ = \ H^*\big(A, R^j a_*\omega_X\big).$$

[2] This was posed as a problem in [GL2], and first answered in the smooth projective case by Hacon [Hac] and Pareschi [Pa]. In [PP3] it was simply shown that the result is equivalent to the Generic Vanishing Theorem of [GL1], hence it holds also in the compact Kähler case.

[3] One would expect the same statement to hold for Kähler manifolds, but this seems not to be known.

Moreover this is a decomposition of E-modules: E acts on $H^*\big(A, R^j a_* \omega_X\big)$ via cup product through the identification $H^1\big(X, \mathcal{O}_X\big) = H^1\big(A, \mathcal{O}_A\big)$, and we again consider $H^i\big(A, R^j a_* \omega_X\big)$ to live in degree $-i$. We claim next that $Q^k \neq 0$. In fact, each of the $R^j a_* \omega_X$ is supported on the $(d-k)$-dimensional Albanese image of X, and hence has vanishing cohomology in degrees $> d-k$. Therefore $H^d\big(X, \omega_X\big) = H^{d-k}\big(A, R^k a_* \omega_X\big)$, which shows that $Q^k \neq 0$. On the other hand, $Q^k[-k]$ is concentrated in degrees $\leq -k$, and therefore Q_X must have generators in degrees $\leq -k$. □

Remark 1.6. Based on results in [CH] extending the picture in [GL2] to higher direct images of canonical bundles, one can further check that each one of the modules Q^j above is 0-regular, and the minimal E-resolution of Q_X splits into the direct sum of the linear resolutions of the modules $Q^j[-j]$.

Remark 1.7 (Exterior Betti numbers). The exterior graded Betti numbers of Q_X are computed as the dimensions of the vector spaces $\mathrm{Tor}_i^E(Q, \mathbf{C})_{-i-j}$. When X is of maximal Albanese dimension and $q(X) > \dim X$, Theorem A implies that these vanish for $i \geq 0$ and $j \geq 1$, and the i-th Betti number in the linear resolution of Q_X is

$$b_i = \dim_{\mathbf{C}} \mathrm{Tor}_i^E(Q, \mathbf{C})_{-i} = h^0(\mathbf{P}, \mathcal{F}(i))$$

where $\mathcal{F}$ is the cokernel of the end map

$$\mathcal{O}_{\mathbf{P}}(-1) \otimes H^{d-1}\big(X, \mathcal{O}_X\big) \longrightarrow \mathcal{O}_{\mathbf{P}} \otimes H^d\big(X, \mathcal{O}_X\big)$$

in $\underline{\mathbf{L}}(P_X)$. (The last equality follows from general machinery, cf. [Eis] Theorem 7.8.) On the other hand $\mathcal{F}$ is 0-regular in the sense of Castelnuovo-Mumford, by virtue of having a linear resolution, so the higher cohomology of its nonnegative twists vanishes. Hence $b_i = \chi(\mathbf{P}, \mathcal{F}(i))$, i.e. the exterior Betti numbers are computed by the Hilbert polynomial of $\mathcal{F}$. The sheaf $\mathcal{F}$ will play a central role in the next two sections.

Variants. We now discuss two variants, involving twisted modules and other holomorphic forms.

Fix an element $\alpha \in \mathrm{Pic}^0(X)$, and set

$$P_\alpha \ = \ \oplus H^i\big(X, \alpha\big) \quad , \quad Q_\alpha \ = \ \oplus H^j\big(X, \omega_X \otimes \alpha^{-1}\big).$$

With the analogous grading conventions as above, these are again dual modules over the exterior algebra E. Letting

$$t \ = \ t(\alpha) \ = \ \max\{\, i \mid H^i\big(X, \omega_X \otimes \alpha^{-1}\big) \neq 0 \,\},$$

the BGG complexes $\mathbf{L}(P_\alpha)$ and $\underline{\mathbf{L}}(P_\alpha)$ for P_α take the form

$$0 \to S \otimes H^{d-t}(X, \alpha) \to S \otimes H^{d-t+1}(X, \alpha) \to \ldots \to S \otimes H^{d-1}(X, \alpha) \to S \otimes H^d(X, \alpha) \to 0$$

and

$$0 \to \mathcal{O}_{\mathbf{P}}(-t) \otimes H^{d-t}(X, \alpha) \to \mathcal{O}_{\mathbf{P}}(-t+1) \otimes H^{d-t+1}(X, \alpha) \to \ldots \to$$
$$\to \mathcal{O}_{\mathbf{P}}(-1) \otimes H^{d-1}(X, \alpha) \to \mathcal{O}_{\mathbf{P}} \otimes H^d(X, \alpha) \to 0.$$

Writing as before $k = k(X)$ for the generic fibre dimension of the Albanese map, it follows as above from [Hac], [Pa], [PP3] that $\mathbf{L}(P_\alpha)$ is exact at the first $d-t-k$ steps from the left. Hence:

Variant 1.8. *The E-module Q_α is k-regular.* □

We will return to the sheafified complex $\underline{\mathbf{L}}(P_\alpha)$ later.

The second variant involves E-modules associated to other bundles of holomorphic forms. Specifically, fix $0 \le p \le d$, and set

$$P_X^{d-p} = \bigoplus_{i=0}^{d} H^i\big(X, \Omega_X^{d-p}\big) \quad \text{and} \quad Q_X^{d-p} = \bigoplus_{i=0}^{d} H^i\big(X, \Omega_X^{p}\big)$$

with $H^i\big(X, \Omega_X^{d-p}\big)$ and $H^i\big(X, \Omega_X^{p}\big)$ in degrees $d-i$ and $-i$ respectively. These are again dual modules, and the BGG complex $\mathbf{L}(P_X^{d-p})$ takes the form

$$0 \longrightarrow S \otimes H^0\big(X, \Omega_X^{d-p}\big) \longrightarrow S \otimes H^1\big(X, \Omega_X^{d-p}\big) \longrightarrow \ldots \longrightarrow S \otimes H^d\big(X, \Omega_X^{d-p}\big).$$

Just as above, results from [GL1], [GL2], [CH] and [PP2] on Nakano-type generic vanishing give regularity bounds on these modules, but for higher forms the vanishings are weaker than what holds for the canonical bundle. We only briefly sketch the proof, since it is completely analogous to that of Theorem A.

Variant 1.9. *Let X be a compact Kähler manifold of dimension d, and as above write $k = k(X)$ for the dimension of the generic fibre of the Albanese mapping of X.*

(i). *Assume X is projective. If f is the maximal dimension of a fiber of the Albanese map, and $\ell = \max\{k, f-1\}$, then Q_X^{d-p} is $(d-p+\ell)$-regular over E.*

(ii). *Let $m(X) := \min\ \{\mathrm{codim}\ Z(\omega) \mid 0 \neq \omega \in H^0\big(X, \Omega_X^1\big)$ be the least codimension of the zero-locus of a holomorphic 1-form on X. Then Q_X^{d-p} is $(2d-p-m(X))$-regular over E.*

Proof. (i). Theorems 5.11(a) and 3.7 in [PP2] imply that $R^i {p_2}_*({p_1}^*\Omega_X^{d-p} \otimes \mathcal{P}) = 0$ for all $i < p - \ell$, where as before p_1 and p_2 denote the projections onto X and $\mathrm{Pic}^0(X)$ from $X \times \mathrm{Pic}^0(X)$, and $\mathcal{P}$ denotes a normalized Poincaré bundle on this product. On the other hand, there is a corresponding derivative complex

$$\ldots \longrightarrow \mathcal{O}_V \otimes H^{i-1}(X, \Omega_X^{d-p}) \longrightarrow \mathcal{O}_V \otimes H^{i}(X, \Omega_X^{d-p}) \longrightarrow \mathcal{O}_V \otimes H^{i+1}(X, \Omega_X^{d-p}) \longrightarrow \ldots$$

on V from which by taking global sections one obtains $\mathbf{L}(P_X^{d-p})$. In analogy with the case of the structure sheaf, locally around the origin this complex computes precisely $R^i {p_2}_*({p_1}^*\Omega_X^{d-p} \otimes \mathcal{P})$, as suggested in [GL2] §6 and shown in [CH] Theorem 6.2. Precisely as in Theorem 1.3, we obtain the exactness of $\mathbf{L}(P_X^{d-p})$ at the first $p - \ell$ steps from the left, which is equivalent to our statement by Proposition 1.2.

(ii). After conjugation, at the fiber corresponding to a form $0 \neq \omega \in H^0(X, \Omega_X^1)$, the derivative complex in (a) looks like

$$\ldots \longrightarrow H^{d-p}(X, \Omega_X^{i-1}) \xrightarrow{\wedge\omega} H^{d-p}(X, \Omega_X^{i}) \xrightarrow{\wedge\omega} H^{d-p}(X, \Omega_X^{i+1}) \longrightarrow \ldots$$

This was shown in [GL1] Proposition 3.4 to be exact as long as $d - p + i < m(X)$. Again, this implies the exactness of the BGG complex at the first $m(X) + p - d$ steps, which is what we want. □

2. The BGG Sheaf and Paracanonical Divisors

We show that the normal cone of the canonical linear series of X inside paracanonical space can be recovered directly from the cohomology algebra of X, via the BGG construction.

As before, let X be a compact Kähler manifold of dimension d. Denote by

$$\mathbf{P} \ = \ \mathbf{P}_{\text{sub}}\big(H^1\big(X, \mathcal{O}_X\big)\big)$$

the projective space of one-dimensional subpaces of $H^1\big(X, \mathcal{O}_X\big)$, so that $\dim \mathbf{P} = q(X) - 1$, where as usual $q = h^1\big(X, \mathcal{O}_X\big)$. Thus $\mathbf{P}$ is the projectivized tangent space to the Picard torus of X. We will be interested in the coherent sheaf $\mathcal{F} = \mathcal{F}_X$ on $\mathbf{P}$ arising as the cokernel of the right-most map in the BGG complex $\underline{\mathbf{L}}(P_X)$, so that one has an exact sequence:

$$\mathcal{O}_{\mathbf{P}}(-1) \otimes H^{d-1}\big(X, \mathcal{O}_X\big) \longrightarrow \mathcal{O}_{\mathbf{P}} \otimes H^d\big(X, \mathcal{O}_X\big) \to \mathcal{F} \longrightarrow 0. \tag{2.1}$$

We refer to $\mathcal{F}$ as the *BGG-sheaf* of X. We propose to relate $\mathcal{F}$ to the geometry of X.

To this end, note first that the projectivization $\mathbf{P}(\mathcal{F}) = \text{Proj}_{\mathbf{P}}\big(\text{Sym}(\mathcal{F})\big)$ sits naturally in $\mathbf{P}^{q-1} \times \mathbf{P}\big(H^d\big(X, \mathcal{O}_X\big)\big)$, giving rise to a morphism

$$\mathbf{P}(\mathcal{F}) \longrightarrow \mathbf{P}\big(H^d\big(X, \mathcal{O}_X\big)\big) \ = \ |\,\omega_X\,|. \tag{*}$$

On the other hand, denote by $\text{Div}^{\{\omega\}}(X)$ the Hilbert scheme (or Douady space) parametrizing all effective divisors on X algebraically (or deformation) equivalent to a canonical divisor, with Abel-Jacobi mapping

$$u : \text{Div}^{\{\omega\}}(X) \longrightarrow \text{Pic}^{\{\omega\}}(X).$$

The canonical linear series $|\,\omega_X\,| = \mathbf{P}_{\text{sub}}\big(H^0\big(X, \omega_X\big)\big)$ sits inside $\text{Div}^{\{\omega\}}(X)$ as the fibre of u over the point $[\omega_X] \in \text{Pic}^{\{\omega\}}(X)$.

The next result describes the geometric meaning of $\mathbf{P}(\mathcal{F})$.

Theorem 2.1. *With the notation just introduced,* $\mathbf{P}(\mathcal{F})$ *is identified via the morphism* (*) *with the projectivized normal cone to* $|\,\omega_X\,|$ *inside* $\text{Div}^{\{\omega\}}(X)$.

In other words, we may say that the E-module Q_X determines the infinitesimal behavior of the Hilbert scheme $\text{Div}^{\{\omega\}}(X)$ along the canonical linear series.

Corollary 2.2. *The canonical linear series* $|\,\omega_X\,|$ *is an irreducible component of* $\text{Div}^{\{\omega\}}(X)$ *if and only if the mapping* (*) *fails to be surjective.* □

Under some additional hypotheses, the criterion in the Corollary can be checked numerically: see Proposition 3.8.

Turning to the proof of Theorem 2.1, the first point is to relate $\text{Div}^{\{\omega\}}(X)$ to a suitable direct image of the Poincaré bundle on $X \times \text{Pic}^0(X)$.

Proposition 2.3. *Let* $\mathcal{P}$ *denote the normalized Poincaré bundle on* $X \times \text{Pic}^0(X)$. *Then*

$$\text{Div}^{\{\omega\}}(X) \ = \ \mathbf{P}\big((-1)^* R^d p_{2*}\mathcal{P}\big)$$

as schemes over $\text{Pic}^0(X)$*, where* $(-1) : \text{Pic}^0(X) \longrightarrow \text{Pic}^0(X)$ *is the morphism given by multiplication by* -1*, and* p_1, p_2 *are the projections of* $X \times \text{Pic}^0(X)$ *onto its factors.*

We will provide a formal proof shortly, but for a quick plausibility argument note that if $\alpha \in \text{Pic}^0(X)$ is any point, then the fibre of $\mathbf{P}((-1)^* R^d p_{2*}\mathcal{P})$ over α is the projective space of one-dimensional quotients of $H^d\big(X, \alpha^{-1}\big)$, which thanks to Serre duality is identified with the projective space parametrizing divisors in the linear series $|\,\omega_X \otimes \alpha\,|$.

Granting Proposition 2.3 for the time being, we complete the

Proof of Theorem 2.1. It is enough to establish the stated isomorphism after pulling back by the exponential map $\exp : V = H^1\big(X, \mathcal{O}_X\big) \longrightarrow \mathrm{Pic}^0(X)$, which is étale. Then the results of [GL2] quoted in the proof of Theorem 1.3 imply that $\exp^*\big(R^d p_{2*}\mathcal{P}\big)$ is isomorphic in a neighborhood of the the origin to the cokernel of map

$$u : H^{d-1}\big(X, \mathcal{O}_X\big) \otimes \mathcal{O}_{\mathbf{V}} \longrightarrow H^d\big(X, \mathcal{O}_X\big) \otimes \mathcal{O}_{\mathbf{V}}$$

of sheaves on the affine space $\mathbf{V} = \mathbf{C}^q$ arising from the right-most terms of the BGG complex. Note that u is given by a matrix of linear forms, and pulling back by (-1) just multiplies the entries of this matrix by -1. The theorem then reduces to a general statement, appearing in the following lemma, concerning the projectivization of the cokernel of a map of trivial vector bundles on affine space defined by a matrix of linear forms. $\square$

Lemma 2.4. *Let u be an $a \times b$ matrix of linear forms on a vector space $\mathbf{C}^q$, defining maps*

$$u : \mathcal{O}^a_{\mathbf{C}^q} \longrightarrow \mathcal{O}^b_{\mathbf{C}^q} \quad , \quad \overline{u} : \mathcal{O}_{\mathbf{P}^{q-1}}(-1)^a \longrightarrow \mathcal{O}^b_{\mathbf{P}^{q-1}},$$

and set $\mathcal{A} = \mathrm{coker}(u)$, $\overline{\mathcal{A}} = \mathrm{coker}(\overline{u})$. Consider the subscheme

$$\mathbf{P}(\mathcal{A}) \ \subseteq \ \mathbf{C}^q \times \mathbf{P}^{b-1},$$

whose fibre F over the origin $0 \in \mathbf{C}^q$ is a copy of $\mathbf{P}^{b-1}$. Then the projectivized normal cone to F in $\mathbf{P}(\mathcal{A})$ is identified with $\mathbf{P}(\overline{\mathcal{A}})$ via the natural projection $\mathbf{P}(\overline{\mathcal{A}}) \longrightarrow \mathbf{P}^{b-1}$.

Proof. An $a \times b$ matrix u of linear forms on $\mathbf{C}^q$ gives rise to an $a \times q$ matrix $\overline{w}$ of linear forms on $\mathbf{P}^{b-1}$ having the property that if $\overline{\mathcal{B}}$ is the cokernel of the resulting map

$$\overline{w} : \mathcal{O}_{\mathbf{P}^{b-1}}(-1)^a \longrightarrow \mathcal{O}^q_{\mathbf{P}^{b-1}},$$

then $\mathbf{P}(\mathcal{A}) \cong \mathbf{V}(\overline{\mathcal{B}})$ as subschemes of $\mathbf{P}^{b-1} \times \mathbf{C}^q$. (In fact, $\overline{w}$ is constructed so that $\mathbf{P}(\mathcal{A})$ and $\mathbf{V}(\overline{\mathcal{B}})$ are defined in $\mathbf{P}^{b-1} \times \mathbf{C}^q$ by the same equations.) Under this identification, the issue is to determine the projectivized normal cone to $\mathbf{V}(\overline{\mathcal{B}})$ along its zero section; equivalently, one needs to determine the exceptional divisor in the blow-up of $\mathbf{V}(\overline{\mathcal{B}})$ along this zero-section. But by virtue of [EGAII, 8.7] this exceptional divisor is $\mathbf{P}(\overline{\mathcal{B}})$, and this in turn coincides with $\mathbf{P}(\overline{\mathcal{A}})$ as a subscheme of $\mathbf{P}^{b-1} \times \mathbf{P}^{q-1}$. $\square$

It remains only to give the

Proof of Proposition 2.3. As explained in [Kl], Theorem 3.13, there is a unique coherent sheaf $\mathcal{Q}$ on $\mathrm{Pic}^0(X)$ characterized by the property that

$$(*) \qquad \mathcal{H}om\big(\mathcal{Q}, \mathcal{E}\big) \ = \ p_{2*}(p_1^*\omega_X \otimes \mathcal{P} \otimes p_2^*\mathcal{E})$$

for any sheaf $\mathcal{E}$ on $\mathrm{Pic}^0(X)$, and then $\mathrm{Div}^{\{\omega\}}(X) = \mathbf{P}(\mathcal{Q})$. So we need to establish that (*) holds with $\mathcal{Q} = (-1)^* R^d p_{2*}\mathcal{P}$. To this end, denote as usual by $\mathbf{R}\Phi_{\mathcal{P}}(\mathcal{G}) \ = \ \mathbf{R}p_{2*}(p_1^*\mathcal{G} \otimes \mathcal{P})$ the Fourier-Mukai transform of a sheaf $\mathcal{G}$ on X. By the projection formula in the derived category, one has

$$\mathbf{R}p_{2*}(p_1^*\omega_X \otimes \mathcal{P} \otimes p_2^*\mathcal{E}) \ \cong \ \mathbf{R}\Phi_{\mathcal{P}}(\omega_X) \overset{\mathbf{L}}{\otimes} \mathcal{E},$$

and we claim that it suffices to prove the derived formula

$$(**) \qquad \mathbf{R}\Phi_{\mathcal{P}}\omega_X \overset{\mathbf{L}}{\otimes} \mathcal{E} \ \cong \ \mathbf{R}\mathcal{H}om\big((-1)^*\mathbf{R}\Phi_{\mathcal{P}}\mathcal{O}_X[d], \mathcal{O}_{\mathrm{Pic}^0(X)}\big) \overset{\mathbf{L}}{\otimes} \mathcal{E}.$$

Indeed, suppose that (**) is known. Now

$$(***)\qquad \mathbf{R}\mathcal{H}om\big((-1)^*\mathbf{R}\Phi_{\mathcal{P}}\mathcal{O}_X[d],\mathcal{O}_{\mathrm{Pic}^0(X)}\big)\overset{\mathbf{L}}{\otimes}\mathcal{E}\ \cong\ \mathbf{R}\mathcal{H}om\big((-1)^*\mathbf{R}\Phi_{\mathcal{P}}\mathcal{O}_X[d],\mathcal{E}\big),$$

so the right-hand side of (*) is computed as the 0^{th} cohomology sheaf of the right-hand side in (***). But there is a spectral sequence

$$E_2^{p,q}\ =\ \mathcal{E}xt^p\big((-1)^*R^{d-q}p_{2*}\mathcal{P},\mathcal{E}\big)\Rightarrow\ R^{p+q}\mathcal{H}om\big((-1)^*\mathbf{R}\Phi_{\mathcal{P}}\mathcal{O}_X[d],\mathcal{E}\big)$$

with $p\geq 0$ and $q\leq 0$. For degree reasons only $\mathcal{H}om\big((-1)^*R^dp_{2*}\mathcal{P},\mathcal{E}\big)$ contributes to the 0^{th} term, so we get the required identity of sheaves.

So it remains only to prove (**), for which it suffices to establish that

$$\mathbf{R}\Phi_{\mathcal{P}}\omega_X\ \cong\ \mathbf{R}\mathcal{H}om\big((-1)^*\mathbf{R}\Phi_{\mathcal{P}}\mathcal{O}_X[d],\mathcal{O}_{\mathrm{Pic}^0(X)}\big)\ \cong\ \big(\mathbf{R}\Phi_{\mathcal{P}^\vee}\mathcal{O}_X\big)^\vee[-d].$$

But this is the well-known consequence of the commutation of Grothendieck duality and the Fourier-Mukai functor: see for instance [PP2] Lemma 2.2.[4] □

3. Inequalities for Numerical Invariants

In this section, we use the BGG sheaf $\mathcal{F}$ to study numerical invariants of a compact Kähler manifold. The exposition proceeds in three parts. We begin by establishing a result that includes Theorem C from the Introduction, and gives a simplified proof of the main result of [PP3]. In the remaining two subsections we discuss examples, applications and variants.

Inequalities from the BGG bundle. As before, let X be a compact Kähler manifold of dimension d, and write

$$p_g\ =\ h^0\big(X,\omega_X\big)\ \ ,\ \ \chi\ =\ \chi\big(X,\omega_X\big)\ \ ,\ \ q\ =\ h^1\big(X,\mathcal{O}_X\big)\ \ ,\ \ n=q-1.$$

Denote by $\mathbf{P}=\mathbf{P}_{\mathrm{sub}}\big(H^1\big(X,\mathcal{O}_X\big)\big)$, so that $\mathbf{P}$ is a projective space of dimension $n=q-1$. As in the previous section, we denote by $\mathcal{F}=\mathcal{F}_X$ the BGG sheaf on the projective space $\mathbf{P}$ introduced in (2.1).

We start by writing down a criterion to guarantee that $\mathcal{F}$ is locally free, and that it is resolved by the BGG complex. Recall that an *irregular fibration* of X is a surjective morphism $f:X\longrightarrow Y$ with connected fibres from X onto a normal variety Y with $0<\dim Y<\dim X$ having the property that a smooth model of Y has maximal Albanese dimension.

Lemma 3.1. (i). *If X has maximal Albanese dimension, then $\underline{\mathbf{L}}(P_X)$ is a resolution of $\mathcal{F}$.*

(ii). *Suppose that $0\in\mathrm{Pic}^0(X)$ is an isolated point of the cohomological support loci*

$$V^i(\omega_X)\ =_{\mathrm{def}}\big\{\alpha\in\mathrm{Pic}^0(X)\ \big|\ H^i\big(X,\omega_X\otimes\alpha\big)\neq 0\,\big\}$$

for every $i>0$. Then $\mathcal{F}$ is a vector bundle on $\mathbf{P}$, with $\mathrm{rk}(\mathcal{F})=\chi$.

(iii). *The hypothesis of (ii) holds in particular if X does not carry any irregular fibrations.*

[4] This is proved in [PP2] in the context of smooth projective varieties, but as indicated in [PP3] the same proof works on complex manifolds, due to the fact that the analogue of Grothendieck duality holds by [RRV].

Proof. The first statement is the case $k = 0$ of Theorem 1.3, and (iii) follows from [GL2], Theorem 0.1. If the $V^i(\omega_X)$ are finite for $i > 0$, then the theory of [GL1] implies as in [EL], Theorem 1.2, or [HP], Proposition 2.11, that the vector bundle maps appearing in $\underline{\mathbf{L}}(P_X)$ are everywhere of constant rank. Thus $\mathcal{F}$ is locally free. □

Remark 3.2. We note for later reference that the main result of [GL2] asserts more generally that if X doesn't admit any irregular fibrations, then in fact $V^i(\omega_X)$ is finite for every $i > 0$.

Once one knows that the BGG sheaf $\mathcal{F}$ is locally free, more or less elementary arguments with vector bundles on projective space yield inequalities for numerical invariants. As in the Introduction, for $1 \leq i \leq q-1$ define $\gamma_i = \gamma_i(X)$ to be the coefficient of t^i in the formal power series:

$$\gamma(X;t) \ =_{\text{def}} \ \prod_{j=1}^{d} (1-jt)^{(-1)^j h^{d,j}} \ \in \ \mathbf{Z}[[t]].$$

Theorem 3.3. *Assume that X does not admit any irregular fibrations, or more generally that $0 \in \operatorname{Pic}^0(X)$ is an isolated point of $V^i(\omega_X)$ for every $i > 0$.*

(i). *Any Schur polynomial of weight $\leq q-1$ in the γ_i is non-negative. In particular*

$$\gamma_i(X) \ \geq \ 0$$

for every $1 \leq i \leq q-1$.

(ii). *If i is any index with $\chi < i < q$, then $\gamma_i(X) = 0$.*

(iii). *One has $\chi \geq q - d$.*

Theorem C from the Introduction is the content of the first two statements. Assertion (iii) is (a slightly special case of) the main result of [PP3].

Proof of Theorem 3.3. Thanks to the Lemma, the hypotheses guarantee that $\mathcal{F}$ is locally free, and has a linear resolution:

$$(3.1) \quad 0 \longrightarrow \mathcal{O}_{\mathbf{P}}(-d) \otimes H^0\big(X, \mathcal{O}_X\big) \longrightarrow \mathcal{O}_{\mathbf{P}}(-d+1) \otimes H^1\big(X, \mathcal{O}_X\big) \longrightarrow \dots$$
$$\dots \longrightarrow \mathcal{O}_{\mathbf{P}}(-1) \otimes H^{d-1}\big(X, \mathcal{O}_X\big) \longrightarrow \mathcal{O}_{\mathbf{P}} \otimes H^d\big(X, \mathcal{O}_X\big) \longrightarrow \mathcal{F} \longrightarrow 0.$$

Identifying as usual cohomology classes on $\mathbf{P}^n$ with integers, $\gamma(X;t)$ is then just the Chern polynomial of $\mathcal{F}$. On the other hand, as $\mathcal{F}$ is globally generated, the Chern classes $c_i(\mathcal{F})$ – as well as the Schur polynomials in these – and represented by effective cycles. Thus

$$\gamma_i(X) \ = \ \deg c_i(\mathcal{F}) \ \geq \ 0.$$

The second statement follows from the fact that $c_i(\mathcal{F}) = 0$ for $i > \operatorname{rank}\,(\mathcal{F})$.

Turning to (iii), we may assume that $q > d$ since in any event $\chi \geq 0$ by generic vanishing. If $q - d = 1$, then the issue is to show that $\chi = \operatorname{rank}\,(\mathcal{F}) \geq 1$, or equivalently that $\mathcal{F} \neq 0$. But this is clear, since there are no non-trivial exact complexes of length n on $\mathbf{P}^n$ whose terms are sums of line bundles of the same degree. So we may suppose finally that $q - 1 = n > d$. The quickest argument is note that chasing through (3.1) implies that $\mathcal{F}$ and its twists have vanishing cohomology in degrees $0 < j < n-d-1$. But if $\chi \leq n-d$ this means by a result of Evans–Griffith that $\mathcal{F}$ is a direct sum of line bundles, which as before is impossible: see [La], Example 7.3.10, for a quick proof of this splitting criterion due to Ein based on Castelnuovo-Mumford regularity and vanishing theorems for vector bundles.

For a more direct argument in the case at hand that avoids Evans–Griffith, let $s \in H^0\big(\mathbf{P},\mathcal{F}\big)$ be a general section, and let $Z = \text{Zeroes}(s)$. We may suppose that Z is non-empty – or else we could construct at a vector bundle $\mathcal{F}'$ of smaller rank having a linear resolution as in (3.1) – and smooth of dimension $n-\chi$. Splicing together the sequence (3.1) and the Koszul complex determined by s, we arrive at a long exact sequence having the shape:

$$(*)\quad 0 \longrightarrow \mathcal{O}_{\mathbf{P}}(-d) \longrightarrow \oplus\, \mathcal{O}_{\mathbf{P}}(-d+1) \longrightarrow \ldots \longrightarrow \oplus\, \mathcal{O}_{\mathbf{P}}(-1) \longrightarrow$$
$$\longrightarrow \oplus\, \mathcal{O}_{\mathbf{P}} \longrightarrow \Lambda^2\mathcal{F} \longrightarrow \ldots \longrightarrow \Lambda^{\chi-1}\mathcal{F} \longrightarrow \mathcal{O}_{\mathbf{P}}(c_1) \longrightarrow \mathcal{O}_Z(c_1) \longrightarrow 0,$$

where $c_1 = c_1(\mathcal{F})$. Observe that $\omega_Z = \mathcal{O}_Z(c_1-n-1)$ by adjunction. Since $\mathcal{F}$ is globally generated, a variant of the Le Potier vanishing theorem[5] yields that

$$H^i\big(\mathbf{P}, \Lambda^a\mathcal{F}\otimes\omega_{\mathbf{P}}(\ell)\big) = 0 \ \text{ for } \ i > \chi - a\ ,\ \ell > 0.$$

Now twist through in (*) by $\mathcal{O}_{\mathbf{P}}(d-n-1)$. Chasing through the resulting long exact sequence, one finds that $H^{n-d-(\chi-1)}\big(Z,\omega_Z(d)\big) \neq 0$. But if $\chi \leq n-d$, this contradicts Kodaira vanishing on Z. □

Remark 3.4 (Evans–Griffith Theorem). A somewhat more general form of (iii) appears in [PP3]: applying Variant 3.15 and using more carefully the results of [GL2], one can assume only that there are no irregular fibrations $f : X \longrightarrow Y$ such that Y is generically finite onto a *proper* subvariety of a complex torus (i.e. X has no higher irrational pencils in Catanese's terminology [Ca1].) The argument in [PP3] involved applying the Evans–Griffith syzygy theorem to the Fourier–Mukai transform of the Poincaré line bundle on $X \times \text{Pic}^0(X)$. The possibility mentioned in the previous proof of applying the Evans–Griffith–Ein splitting criterion to the BGG bundle $\mathcal{F}$ is related but substantialy quicker. As we have just seen the additional information that $\mathcal{F}$ admits a linear resolution allows one to bypass Evans–Griffith altogether, although as in Ein's proof we still use vanishing theorems for vector bundles.

Hodge-number inequalities. Here we give some examples and variants of the inequalities appearing in the first assertions of Theorem 3.3.

We start by unwinding some of the statements in the case of threefolds.

Corollary 3.5. *Let X be an irregular compact Kähler threefold with no irregular fibration. Then*

$$h^{0,2}(X) \ \geq\ 2q(X) + \frac{\sqrt{8q(X)-23}-7}{2} \quad \text{and} \quad h^{0,3}(X) \ \geq\ h^{0,2} - 2.$$

Proof. For simplicity write $b = h^{0,2}(X)$, $a = h^{0,3}(X)$, $q = q(X)$. As X must be of maximal Albanese dimension, we have $q \geq 3$. Then

$$\gamma(X;t) \ =\ \frac{(1-2t)^q}{(1-3t)(1-t)^b}.$$

[5] The statement we use is that if $\mathcal{E}$ is a nef vector bundle of rank e on a smooth projective variety V of dimension n, then

$$H^i\big(V, \Lambda^a\mathcal{E}\otimes\omega_V\otimes L\big) = 0$$

for $i > e-a$ and any ample line bundle L. In fact, it is equivalent to show that $H^j\big(V,\Lambda^a\mathcal{E}^*\otimes L^*\big) = 0$ for $j < n+a-e$. For this, after passing to a suitable branched covering as in [La], proof of Theorem 4.2.1, we may assume that $L = M^{\otimes a}$, in which case the statement follows from Le Potier vanishing in its usual form: see [La], Theorem 7.3.6.

One finds that $\gamma_2 = b^2/2+7b/2+9-8q-2bq+2q^2$, which must then be ≥ 0. Setting $x = b-2q$, this yields $x^2+7x+18-2q \geq 0$. A simple calculation shows that then

$$b \geq 2q + \frac{\sqrt{8q-23}-7}{2}.$$

For the second inequality, we use the inequality (iii) of Castelnuovo-de Franchis type, which gives

$$a-b+q-1 \ = \ \chi(\omega_X) \ \geq \ q-3,$$

or in other words $a \geq b-2$. □

Remark 3.6. We note that equality holds in Corollary 3.5 when X is birational to a complex torus of dimension 3, or to a principal polarization in a PPAV of dimension 4. Here $\mathrm{rk}(\mathcal{F}) = 0$ and 1 respectively, so of course $\gamma_2 = 0$.

Remark 3.7 (Inequalities in [Ca1] and [CP]). Catanese [Ca1] shows that if a compact Kähler manifold X admits no irregular fibrations, then the natural maps

$$\phi_k : \bigwedge^k H^i(X, \mathcal{O}_X) \longrightarrow H^k(X, \mathcal{O}_X)$$

are injective on primitive forms $\omega_1 \wedge \ldots \wedge \omega_k$, for $k < \dim X$. Since these correspond to the Plücker embedding of the Grassmannian $\mathbf{G}(k, V)$, one obtains the bounds $h^{0,k}(X) \geq k(q(X)-k)+1$, including the classical $h^{0,2}(X) \geq 2q(X)-3$. Hence in the case of threefolds, Corollary 3.5 provides a stronger inequality. On the other hand, Causin and Pirola [CP] provide a more refined study in the case of ϕ_2. Among other things, for $q(X) \leq 2\dim X - 1$ they show that ϕ_2 is injective, hence $h^{0,2}(X) \geq \binom{q(X)}{2}$. Thus for threefolds with $q(X) = 5$ and no irregular fibrations, they obtain the even stronger $h^{0,2}(X) \geq 10$.[6]

As another example, we consider the question of whether the canonical series $|\,\omega_X\,|$ is a component of the space $\mathrm{Div}^{\{\omega\}}(X)$ of paracanonical divisors: following Beauville [Be], one says that $|\,\omega_X\,|$ is *exorbitant* if this happens. A somewhat unexpected consequence of Corollary 2.2 is that in the setting of Theorem 3.3, the exorbitance of the canonical series actually depends only on the Hodge numbers of X. In fact:

Proposition 3.8. *Assume that the hypotheses of Theorem 3.3 are satisfied, and that*

$$(*) \qquad p_g - \chi \ \leq \ q-1.$$

Then $|\,\omega_X\,|$ is exorbitant if and only if the codimension $(p_g-\chi)$ Segre number of $\mathcal{F}^*$ vanishes, i.e. if and only if:

$$s_{1^{\times(p_g-\chi)}}\big(\gamma_1, \ldots, \gamma_{q-1}\big) \ = \ 0,$$

where the quantity in question indicates the Schur function associated to the partition $(1, \ldots, 1)$ $(p_g-\chi)$ *times.*

Observe that if $\chi > 0$ then $\mathrm{Div}^{\{\omega\}}(X)$ has a unique component of dimension $q+\chi-1$ dominating $\mathrm{Pic}^0(X)$, so if (*) fails in this case, then necessarily $|\,\omega_X\,|$ is exorbitant.

[6] We remark that the results of Catanase and Causin and Pirola actually only assume the absence of higher irrational pencils, cf. Remark 3.4.

Proof of Proposition 3.8. According to Corollary 2.2, $|\,\omega_X\,|$ is exorbitant if and only if the natural mapping

$$\mathbf{P}(\mathcal{F}) \longrightarrow \mathbf{P}\big(H^d\big(X, \mathcal{O}_X\big)\big) \ = \ \mathbf{P}^{p_g-1}$$

fails to be surjective. But the Segre number in question computes the degree in $\mathbf{P}^{q-1}$ of the preimage of a general point in the target, and the statement follows. □

Example 3.9. Suppose that X is a surface without irrational pencils. Then $p_g - \chi = q - 1$, and the BGG complex takes the form

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}}(-2) \longrightarrow \mathcal{O}_{\mathbf{P}}(-1)^q \longrightarrow \mathcal{O}_{\mathbf{P}}^{p_g} \longrightarrow \mathcal{F} \longrightarrow 0.$$

In this case the Segre number appearing in Proposition 3.8 is the coefficient of t^{q-1} in $(1 + t)^q/(1+2t)$, and this is $= 0$ if q is even, and $= 1$ if q is odd. Thus $|\,\omega_X\,|$ is exorbitant if and only if q is even, a fact observed by Beauville in [Be], §4.

Finally, we note that arguments similar to those leading to Theorem 3.3 (i) and (ii) yield a new inequality for $h^{1,1}$ on a surface. Specifically, let X be a compact Kähler surface with no non-constant morphism to a curve of genus at least 2. The classical Castelnuovo-de Franchis inequality asserts that $h^{0,2}(X) \geq 2q(X) - 3$. A related result based on the Castelnuovo-de Franchis Lemma and linear algebra also bounds $h^{1,1}$ in terms of the irregularity: it is shown in [BHPV], IV.5.4, that $h^{1,1}(X) \geq 2q(X) - 1$. The methods of the present paper lead to a strengthening of this:

Proposition 3.10. *If X is a compact Kähler surface without irrational pencils, then*

$$h^{1,1}(X) \ \geq \ \begin{cases} 3q(X) - 2 & \text{if } q(X) \text{ is even} \\ 3q(X) - 1 & \text{if } q(X) \text{ is odd} \end{cases}$$

Proof. It is well known that given any non-zero one-form $\omega \in H^0(X, \Omega^1_X)$ on such a surface X, the map $H^1(X, \mathcal{O}_X) \xrightarrow{\wedge\omega} H^1(X, \Omega^1_X)$ obtained by wedging with ω is injective. On the other hand, this map is naturally dual to the map $H^1(X, \Omega^1_X) \xrightarrow{\wedge\omega} H^1(X, \Omega^2_X)$, via Serre duality. Hence in the natural complex

$$0 \longrightarrow H^1(X, \mathcal{O}_X) \xrightarrow{\wedge\omega} H^1(X, \Omega^1_X) \xrightarrow{\wedge\omega} H^1(X, \Omega^2_X) \longrightarrow 0,$$

the first map is injective and the second is surjective. Globalizing as in Variant 1.9, i.e. considering the (sheafified) BGG complex associated to the module $\oplus H^i(X, \Omega^1_X)$, we obtain a monad of vector bundles on $\mathbf{P} := \mathbf{P}_{\text{sub}}(V)$:

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}}(-1)^q \longrightarrow \mathcal{O}_{\mathbf{P}}^{h^{1,1}(X)} \xrightarrow{\ \phi\ } \mathcal{O}_{\mathbf{P}}(1)^q \longrightarrow 0.$$

The cohomology E of this monad sits in an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}}(-1)^q \longrightarrow K \longrightarrow E \longrightarrow 0$$

where $K = \ker(\phi)$. A direct calculation shows that $\text{rk}(E) = h^{1,1}(X) - 2q$ and

$$c_t(E) = \frac{1}{(1-t^2)^q} = 1 + qt^2 + \binom{q+1}{2} t^4 + \ldots,$$

with non-zero terms in all even degrees $\leq \dim \mathbf{P} = q - 1$. This implies that $\text{rk}(E) \geq q - 2$ if q is even, and $\text{rk}(E) \geq q - 1$ if q is odd. □

Bounds involving the Euler characteristic. In this final subsection, we make some remarks surrounding the inequality

$$\chi(X, \omega_X) \;\geq\; q(X) - \dim X \tag{3.2}$$

established in [PP3] and Theorem 3.3 (iii) for compact Kähler manifolds that do not admit irregular fibrations.

Note to begin with that equality holds in (3.2) when X is birational to a complex torus (in which case $\chi = 0$) or to a theta divisor in a principally polarized abelian variety (in which case $d = q - 1$ and $\chi = 1$). It was essentially established by Hacon–Pardini [HP], §4, that in fact these are the only such examples with $\chi \leq 1$.

Proposition 3.11. *Let X be an irregular smooth projective complex variety with no irregular fibrations.*

(i). *If $\chi(\omega_X) = 0$, then X is birational to an abelian variety.*
(ii). *If $\chi(\omega_X) = 1 = q(X) - \dim X$, then X is birational to a principal polarization in a* PPAV.

Since the statement does not appear explicitly in [HP] we will briefly indicate the proof, but we stress that all the ideas are already present in that paper.

Sketch of Proof. We again focus on the exact sequence (3.1) of bundles on $\mathbf{P} = \mathbf{P}^{q-1}$. Note that in any event X has maximal Albanese dimension, and hence $q \geq d$. If $\chi = 0$, then $\mathcal{F} = 0$. In this case one reads off from (3.1) that $q = d$ and $P_1(X) = h^{0,d}(X) > 0$. On the other hand, the assumption of the theorem implies by [GL2] that for $V^i(\omega_X)$ has only isolated points when $i > 0$, and since $\chi(\omega_X) = 0$, this implies that $V^0(\omega_X)$ also consists only of isolated points. But a result of Ein-Lazarsfeld (cf. [ChH], Theorem 1.7) says that a variety of maximal Albanese dimension with $V^0(\omega_X)$ zero dimensional is birational to its Albanese.

Now suppose that $\chi = 1$. Then $\mathcal{F}$ is a line bundle, and it follows that (3.1) is a twist of the standard Koszul complex, this being the unique linear complex of length $n+1$ on $\mathbf{P}^n$ whose outer terms have rank one. In particular $h^{0,d}(X) = q$. On the other hand we have an injection $H^0(A, \Omega_A^d) \to H^0(X, \Omega_X^d)$. Indeed, since $d = q - 1$, if the pullback map were not injective we would have a d-wedge of independent holomorphic 1-forms on X equal to 0, which by [Ca1] Theorem 1.14 would imply the existence of an irregular fibration. Now since the two dimensions are equal, the map is in fact an isomorphism. To prove (ii), one can then use a characterization of principal polarizations due to Hacon-Pardini (cf. [HP] Proposition 4.2), extending a criterion of Hacon, which says that the only other thing we need to check is $V^i(\omega_X) = \{0\}$ for all $i > 0$. But this follows from Remarks 3.2 and 3.16. □

On the other hand, one expects it to be very rare to find manifolds X with no irregular fibrations for which $\chi(X, \omega_X) = q(X) - \dim X \geq 2$.

Conjecture 3.12. *If X is an irregular compact Kähler manifold with no irregular fibrations and $\chi(\omega_X) \geq 2$, then*

$$\chi(\omega_X) \;>\; q(X) - \dim X$$

when $q(X)$ is very large compared to $\chi(\omega_X)$.

The thinking here is that if equality were to hold in (3.2), then the BGG sheaf $\mathcal{F}$ would be a non-split vector bundle of small rank on the projective space $\mathbf{P}$. But these should almost never exist. The fact that $\mathcal{F}$ admits a linear resolution, and the resulting relations in Theorem 3.3 (ii), should provide even more constraints.

As an example in this direction, one has the following, whose proof was shown to us by I. Coandă.

Proposition 3.13. *Let X be a compact Kähler manifold with no irregular fibrations, such that $\chi(\omega_X) = 2$ and $q(X) \geq 5$. Then $q(X) - \dim X < 2$.*

Proof. Assume for a contradiction that $q(X) - \dim X = 2$, and consider yet again the BGG resolution (3.1) of $\mathcal{F}$. This resolution shows first of all that $\mathcal{F}$ is 0-regular in the sense of Castelnuovo-Mumford. We next claim that $H^1\big(\mathbf{P}, \mathcal{F}(-2)\big) \neq 0$, while $H^1\big(\mathbf{P}, \mathcal{F}(i)\big) = 0$ for all $i \neq -2$. Grant this for the moment. Then the S-module $H^1_*(\mathcal{F})$ has a non-zero summand in degrees -2 and higher. But [MKR] Theorem 1.7 asserts that a 0-regular rank 2 bundle with this property cannot exist when $n \geq 4$. As for the claim, note that (3.1) starts on the left with a twist of the Euler sequence, and thus the cokernel of the injection $\mathcal{O}_{\mathbf{P}^n}(-n+1) \to \mathcal{O}_{\mathbf{P}^n}^{n+1}(-n+2)$ is $T_{\mathbf{P}^n}(-n+1)$. One then finds that

$$H^1\big(\mathbf{P}^n, \mathcal{F}(i)\big) \ = \ H^{n-1}\big(\mathbf{P}^n, T_{\mathbf{P}^n}(-n+1+i)\big),$$

and the assertion follows from the Bott formula (cf. [OSS] p.8–9) and Serre duality. □

Example 3.14 (Surfaces and the Tango bundle). The case of surfaces is particularly amusing from the present point of view. When $\dim X = 2$, (3.2) is equivalent to the classical Castelnuovo-de Franchis inequality

$$p_g(X) \ \geq \ 2q(X) - 3,$$

which holds for surfaces with no irrational pencils of genus at least 2. As soon as $q(X) \geq 4$ it has been suggested (cf. e.g. [MLP]) – and proved by Pirola for $q(X) = 5$ – that there should be no such surfaces satisfying $p_g(X) = 2q(X) - 3$. If such a surface were to exist, its BGG bundle $\mathcal{F}$ would have a resolution:

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^n}(-2) \longrightarrow \mathcal{O}_{\mathbf{P}^n}(-1)^{n+1} \longrightarrow \mathcal{O}_{\mathbf{P}^n}^{2n-1} \longrightarrow \mathcal{F} \longrightarrow 0$$

where $n = q(X) - 1 \geq 3$. On the other hand, for every $n \geq 3$ there does exist a vector bundle having this shape: it is the *Tango bundle* (cf. [OSS] Ch.I, §4.3). It would be quite interesting to decide one way or the other whether one can in fact realize the Tango bundle as the BGG bundle of a surface.

Finally, we discuss a strengthening of the inequality (3.2), also appearing in [PP3], which involves the twisted BGG complexes introduced in Variant 1.8.

As always, let X be a compact Kähler manifold, and fix any point $\alpha \in \mathrm{Pic}^0(X)$. Following [PP3], one defines the *generic vanishing index* of ω_X at $0 \in \mathrm{Pic}^0(X)$ to be the integer

$$\mathrm{gv}_0(X) \ = \ \min_{i>0} \{\mathrm{codim}_{\,0}\, V^i(\omega_X) - i\}.$$

The basic generic vanishing theorems assert that $\mathrm{gv}_0(X) \geq 0$ when X has maximal Albanese dimension, and if 0 is an isolated point of $V^i(\omega_X)$ for every $i > 0$ then

$$\mathrm{gv}_0(X) \ = \ q(X) \ - \ \dim(X).$$

The following statement, which appeared as Corollary 4.1 in [PP3], therefore generalizes Theorem 3.3 (iii).

Variant 3.15. *Assume that X has maximal Albanese dimension. Then*

$$\chi(X, \omega_X) \ \geq \ \mathrm{gv}_0(X).$$

Brief Sketch of Proof. The origin belongs to all the $V^i(\omega_X)$ (cf. [EL], Lemma 1.8), and hence there exists a largest index $t > 0$, and an irreducible component $Z \subseteq V^t(X)$, such that $\mathrm{gv}_0(X) = \mathrm{codim}\, Z - t$. Choose a general point $\alpha \in Z$ and consider the twisted BGG complex $\underline{\mathbf{L}}(P_\alpha)$, giving a resolution of the indicated sheaf $\mathcal{F}_\alpha$:

$$(*) \quad 0 \longrightarrow \mathcal{O}_{\mathbf{P}}(-t) \otimes H^{d-t}(X, \alpha) \longrightarrow \mathcal{O}_{\mathbf{P}}(-t+1) \otimes H^{d-t+1}(X, \alpha) \longrightarrow \dots$$
$$\longrightarrow \mathcal{O}_{\mathbf{P}}(-1) \otimes H^{d-1}(X, \alpha) \longrightarrow \mathcal{O}_{\mathbf{P}} \otimes H^d(X, \alpha) \longrightarrow \mathcal{F}_\alpha \longrightarrow 0.$$

The sheaf $\mathcal{F}_\alpha$ is typically not locally free. But if one chooses a subspace

$$W \ \subseteq \ T_\alpha \mathrm{Pic}^0(X) \ = \ H^1\big(X, \mathcal{O}_X\big)$$

transverse to the tangent space of Z at α, and restricts (*) to the projectivization $\mathbf{P'} = \mathbf{P}_{\mathrm{sub}} W$ of W, then it follows as in [EL], Theorem 1.2, that one gets an exact complex

$$(**) \quad 0 \longrightarrow \mathcal{O}_{\mathbf{P'}}(-t) \otimes H^{d-t}(X, \alpha) \longrightarrow \mathcal{O}_{\mathbf{P'}}(-t+1) \otimes H^{d-t+1}(X, \alpha) \longrightarrow \dots$$
$$\longrightarrow \mathcal{O}_{\mathbf{P'}}(-1) \otimes H^{d-1}(X, \alpha) \longrightarrow \mathcal{O}_{\mathbf{P'}} \otimes H^d(X, \alpha) \longrightarrow \mathcal{G} \longrightarrow 0$$

where $\mathcal{G}$ is a vector bundle, of rank $\chi(X, \omega_X)$. Note that

$$\dim \mathbf{P'} \ = \ \mathrm{codim}\, Z - 1 \ = \ \mathrm{gv}_0(\omega_X) + t - 1.$$

Now the argument proceeds much as in the proof of Theorem 3.3, using (**) in place of (3.1). □

Remark 3.16 (Non-trivial isolated points). A similar argument gives yet another variant, which was also noted in [PP3].

> Suppose that $\alpha \in \mathrm{Pic}^0(X)$ is a point having the property that for every $i > 0$ either $\alpha \notin V^i(\omega_X)$ or else α is an isolated point of $V^i(\omega_X)$. Assume furthermore $H^p\big(X, \alpha\big) \neq 0$ for some $p < d$, and let $p(\alpha)$ be the least index for which this holds. Then
> $$\chi(X, \omega_X) \ \geq \ q(X) - \dim X + p(\alpha).$$

Since evidently $p(\alpha) > 0$ if $\alpha \neq 0$, this means that non-trivial isolated points improve the basic lower bound for the Euler characteristic.

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