Solutions to HW 2

Chapter 2:

1. With no restriction, we have five choices for each position, giving 5^4 numbers. If a) holds, then there are five choices for the first digit, then four for the next and so on, giving $5 \cdot 4 \cdot 3 \cdot 2 = 120$. If b) holds, then the first three digits each have five possibilities, but the last can only be 2 or 4, giving a total of $5^3 \cdot 2 = 250$. If both a) and b) hold, then there are two choices for the last digit (either 2 or 4), four choices for the third digit, three for the second, and two for the first, giving a total of 48.

2. Given a suit, there are 13! orderings of its cards. There are 4! orderings of the suits. Thus we have a total of $4!(13!)^4$ orderings.

5. a) The highest power of 5 that divides 50! is 12, two for 25 and 50, and one for each of the other eight multiples of five between 5 and 50. The highest power of 2 that divides 50! is greater than 12, so the highest power of 10 that divides 50! is 12.
b) there are 200 integers between 1 and 1000 that 5 divides, ofthese there are 40 that 25 divides, of these there are 8 that 125 divides, of these there is one that 625 divides. Putting this all together, the highest power of 5 that divides 1000! is $200 + 40 + 8 + 1 = 249$. It is easy to see that 2^{249} also divides 1000! so the answer is 249.

8. There are 6! ways to seat the men. Now seat the women in each of the size spots between adjacent men. There are 5! ways to seat the women. So the answer is 6!5!.

11. We count the complement event. Suppose we must choose at least two consecutive numbers. If all three are consecutive, then there are 18 ways to choose the numbers; these are $\{1, 2, 3\}, \ldots, \{18, 19, 20\}$. Now suppose exactly two are consecutive. If these two are 1,2 or 19,20, then there are seventeen possibilities for the last number. If the two consecutive numbers are any other pair $\{i, i+1\}$, then both $i-1$ and $i+2$ are forbidden, so there are 16 possibilities. Thus the number of choices is $\binom{20}{3} - 18 - 2 \cdot 17 - 17 \cdot 16$.

16. To each r-set A of $[n]$, associate the $(n-r)$-set $A' = [n] - A$. This correspondence is clearly 1-1 hence the number of r-sets is equal to the number of $(n-r)$-sets. Each side of the equation counts one of these quantities.

28. a) He must travel 17 blocks in total, so his walk is determined by the choice of which 8 blocks he takes north. The number of these is $\binom{17}{8}$.
b) He must avoid the block from $(4, 3)$ to $(5, 3)$, so count the number of routes that pass through this block. There are $\binom{7}{3}$ ways of reaching this block, and $\binom{9}{5}$ of proceeding from the end of this block to work. So the total number of routes is $\binom{17}{8} - \binom{7}{3}\binom{9}{5}$.

38. Substitute $y_1 = x_1 - 2, y_2 = x_2, y_3 = x_3 + 5, y_4 = x_4 - 8$. Then this is the same as the number of solutions to $\sum_i y_i = 30 - 5 = 25$, where each $y_i \geq 0$. This number is $\binom{25-1+4}{25} = \binom{28}{3}$.

39. a) Clearly $\binom{20}{6}$
c) Count instead the possibilities of the remaining sticks. This is the number of solutions to $x_1 + \ldots + x_7 = 14$, where $x_i \geq 2$ for $i \neq 1, 7$, which is the number of solutions to $y_1 + \ldots + y_7 = 4, y_i \geq 0$, which is $\binom{7+4-1}{4} = \binom{10}{4}$.

42. First decide which two students get the lime and lemon. There are $P(4,2) = 12$ ways of doing this. Now give the other two students one orange drink each. Now distribute the eight remaining orange drinks to the four students. There are $\binom{8+4-1}{8} = \binom{11}{3}$ ways of doing this. Hence the answer is $12\binom{11}{3}$.

45. a) Think of the bookshelves lined in a row. Now first arrange the 20 books, for which there are 20! arrangements. Then fit the ends of the bookshelves to separate the books. The number of ways to do this is the same as the number of solutions to $x_1 + \ldots + x_5 = 20$, $x_i \geq 0$, which is $\binom{24}{4}$. So the answer is $20!\binom{24}{4}$.
b) This is clearly the number of solutions to $x_1 + \ldots + x_5 = 20$, $x_i \geq 0$, which is $\binom{24}{4}$.