## Solutions to HW 3

### Section 3.4:

1. Follow the proof on Page 30, except replace every instance of 21 by $k$, 22 by $k+1$, and 153 by $132+k$. One can make the same conclusion with 21 replaced by 22. Follow the argument above with $k = 22$. We obtain the same conclusion unless the 154 numbers

$$a_1, \ldots, a_{77}, a_1 + 22, \ldots, a_{77} + 22$$

are the 154 integers between 1 and 154. For this, $a_1$ through $a_{22}$ must be 1 through 22, then $a_{23}$ through $a_{44}$ must be 45 through 66, $a_{45}$ through $a_{66}$ must be 89 through 110, and $a_{67}$ through $a_{77}$ must be 133 through 143. But this is impossible, since $a_{77} \leq 132$.

2. We continue the argument from class. If 100 such integers are chosen to form a set $S$, then the odd part of these integers must consist of all 100 odd integers between 1 and 199. We give the sketch of a case analysis that prevents any number from $[15]$ in $S$.
Case 1: 3, 5, 7, 9, 11, 13, 15: Suppose $3 \in S$. As $3|27$ no integer of the form $27 \times 2^t$ is in $S$. For 5 and 15 we use 45, 7 we use 49, 11 we use 33, for 13 we use 39 and apply the same argument.
Case 2: 6, 10, 12, 14: Suppose $6 \in S$. Consider $27 \times 2^k$ and $81 \times 2^l$ both in $S$. Since $27|81$, $k > l$, so $k \geq 1$, and then $6|(27 \times 2^k)$. Similar arguments work for 10, 12, 14.
Case 3: 1, 2, 4, 8: Clearly $1 \notin S$. If $2 \in S$, then the remaining integers are all odd, which is impossible. If $4 \in S$, then no integer of the form $3 \times 2^k$ could be in $S$, since we have already excluded 3 and 6. If $8 \in S$, then the same argument works since we already excluded 12 as well.

4. Consider the $n$ intervals $[1,2], [3,4], \ldots, [2n-1, 2n]$. By the pigeonhole principle, two of the $n+1$ chosen integers must lie in the same interval, making them neighbors.

5. Apply the same argument as above with $[1,3], [4,6], \ldots, [3n-3, 3n]$.

8. Consider the long division process applied to $m/n$ to form its decimal expansion. At each stage, we write the highest factor of $n$ in $\alpha \times 10$, where $\alpha$ is the remainder from the previous division by $n$. Thus, in particular, $\alpha < n$. Consequently, by the pigeonhole principle, within $n$ divisions, the remainder must repeat, and therefore the decimal also repeats from this point. This argument also shows that the period is at most $n$.

22. Let $N = 2 + (k+1)(R_k(3) - 1)$, and suppose we have a $(k+1)$-coloring of $E(K_N)$. Pick a vertex $v$. By the pigeonhole principle, there is a color $i$ and a set $S$ of size $R_k(3)$ such that all $v, S$ edges have color $i$. If color $i$ appears anywhere within $S$, then we have a triangle in color $i$. Otherwise the edges within $S$ are $k$-colored, and since $|S| \geq R_k(3)$, we conclude that $S$ contains a monochromatic triangle in some other color.
We show by induction on $k$ that $R_k(3) \leq 1 + k!(\sum_{i=0}^{k} 1/i!) < 1 + k!e$. The base case follows by $R(3,3) \leq 6$. For the induction step,

$$R_k(3) \leq 2 + k(R_{k-1}(3) - 1) \leq 2 + k\left((k-1)!\sum_{i=0}^{k-1} 1/i!\right) = 1 + k!\sum_{i=0}^{k} 1/i!.$$

27. Partition the family of all subsets into pairs $\{A, [n] - A\}$. Then the collection cannot contain both $A$ and $[n] - A$ for any $A$, since these two sets are disjoint. Consequently, the collection has at most $2^n/2 = 2^{n-1}$ subsets.

**Section 4.6:**

5. We proceed by induction on $k$ (not $n$!), proving that if the sum is at least $k$, then we need at least $k$ switches. If $k = 0$, then the conclusion is trivial, so assume $k > 0$. Consider the switch $i_j, i_{j+1}$. Then $b_l$ remains unchanged for every $l$ except possibly $l = i_{j+1}$, which can change by at most one. Thus $k$ has reduced by at most one, so the sum of new $b_i$'s is at least $k - 1$, and by induction we need at least $k - 1$ more switches. Altogether, we therefore need at least $k$ switches.

8 a) The integer 1 can be brought to the first position by at most 5 adjacent switches, 2 can be brought to the second position by at most 4 adjacent switches etc. This gives a total of at most 15 inversions for any permutation. Equality holds only if all integers are as "out of order" as possible, giving 6,5,4,3,2,1, so there is only one permutation with 15 inversions.

20. 000, 001, 101, 100, 110, 111, 011, 010 is not the reflected Gray code since the first coordinate of the third vertex is 1.

35. Suppose that $A <_{lex} B$. Then the smallest element $x$ in $A \triangle B$ lies in $A$. Since $A \triangle B = \overline{A} \triangle \overline{B}$, the smallest element in $\overline{A} \triangle \overline{B}$ is also $x$, and it lies in $\overline{B}$. Therefore $\overline{B} <_{lex} \overline{A}$. This immediately implies the result.

37. Pick $x, y \in X$. Then $xR'x$ and $xR''x$, so $xRx$. If $xRy$ and $yRx$, then $xR'y, xR''y$, and $yR'x, yR''x$. Since $R'$ and $R''$ are partial orders, we get $x = y$. Finally, if $xRy$ and $yRz$, the $xR'y$ and $xR''y$, and $yR'z$, and $yR''z$. Since $R'$ is a partial order, $xR'z$, and similarly we get $xR''z$, which together yield $xRz$. Thus $R$ is an order relation.

47. a) We only show transitivity (the others are similar). Suppose that $\pi < \sigma$ and $\sigma < \tau$. Then each part of $\pi$ is a refinement of some part of $\sigma$, and each part of sigma is a refinement of some part of $\tau$, so each part of $\pi$, is a refinement of some part of $\tau$. Thus $\pi < \tau$.
b) Associate relations $R_\pi$ and $R_\sigma$ to partitions $\pi$ and $\sigma$. Define $R < R'$ if $aRb$ implies $aR'b$ for all $a, b \in [n]$. Then $\pi < \sigma$ if and only if $R_\pi < R_\sigma$. Indeed, let $\pi < \sigma$ and choose $a, b \in [n]$. Since every block of $\pi$ is a subset of a block of $\sigma$, we conclude that if $aR_\pi b$, then $aR_\sigma b$. Therefore $R_\pi < R_\sigma$. On the other hand, if $aR_\pi b$, then $a$ and $b$ are in the same block of $\pi$, so they are in the same block of $\sigma$, and hence $aR_\sigma b$.
c) Done in class

**Section 5.7:**

11. The LHS counts the number of $k$-subsets of $S$ containing one of $a, b, c$. The first term on the RHS counts the number of $k$-subsets of $S$ containing $a$. The second term counts the number omitting $a$, but containing $b$, and the third term counts the number omitting both $a$ and $b$, but containing $c$.

16.

$$\int_0^1 (1+x)^n dx = \int_0^1 \sum_{k=0}^{n} \binom{n}{k} x^k dx = \sum_{k=0}^{n} \binom{n}{k} \left[\frac{x^{k+1}}{k+1}\right]_0^1.$$

Simplifying yields $(2^{n+1}-1)/(n+1) = \sum_k \binom{n}{k}/(k+1)$ as desired.

18. Integrate $(1-x)^n$ from 0 to 1. Using the binomial theorem, we get the expression in the problem. On the other hand, integrating directly yields the answer $1/(n+1)$.

25. Divide a room of $m_1 + m_2$ people into one group of $m_1$ and another group of $m_2$. The RHS is the number of ways to choose $n$ people, while the LHS counts this depending on the number from each group.

48. If the maximum antichain has size at most $n$, then by Dilworth's Theorem, there is a chain partition with at most $n$ chains. Since all $mn+1$ elements are covered by these chains, by the pigeonhole principle, one of these chains must have size at least $m+1$.

49. Define a poset $P$ on the set $X$ of $mn+1$ real numbers given. Suppose $a < b$ in the natural ordering of reals. Then define $a <_R b$ if $a$ appears before $b$ in the given sequence. Now a chain in this poset is an increasing subsequence, and an antichain is a decreasing subsequence (Why??). Hence the result immediately follows from the previous problem.