Solutions to HW 6

Chapter 8:

1. Fix a point x on the circle. Moving clockwise, write 1 if a chord is encountered for the first time, and -1 otherwise. We obtain a sequence of ± 1's of length $2n$ satisfying the Catalan properties. On the other hand, given such a sequence, consider the first -1 that appears, say at the ith entry. This means that there is a chord between the $(i-1)$st and ith point on the circle, since if the chord with endpoint i has its other endpoint different from $i-1$, then the chord with endpoint $i-1$ will cross this chord. Removing the ith and $(i-1)$st entry from the sequence, we can argue by induction that the resulting sequence yields the remaining chords. Hence this bijection shows that the answer is the nth Catalan number $\binom{2n}{n}/(n+1)$.

2. The bijection to the Catalan sequences is as follows: We write 1 at the ith position in the sequence if i appears on the top row of the array, and -1 otherwise. Now consider the first k elements of the sequence formed. If there are more -1s than 1s, then some integer j appears on the bottom row of the array and the space above it is currently empty. Since $j < k$, the integer that will fill this space must be greater than j, a contradiction to the construction of the array. Now consider a Catalan ± 1 sequence $x_1, \dots, x_{2n}$. Beginning with x_1, we write i on the top row of the array if $x_i = 1$, and on the bottom row if $x_i = -1$. We proceed in both rows from left to right. This assures us that the obtained array is increasing along rows, and the Catalan property of the sequence assures us that it is increasing along columns.

9. We proceed by induction on k. The base case $k = 0$ is trivial. For the induction step,

$$\Delta^{k+1} h_n = \Delta^k h_{n+1} - \Delta^k h_n = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} h_{n+1+j} - \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} h_{n+j}$$

$$= \sum_{j=1}^{k+1} (-1)^{k+1-j} \binom{k}{j-1} h_{n+j} + \sum_{j=0}^{k} (-1)^{k+1-j} \binom{k}{j} h_{n+j} = \sum_{j=0}^{k+1} (-1)^{k+1-j} \binom{k+1}{j} h_{n+j},$$

where the last equality holds by Pascals identity for binomial coefficients.

12. (a) is trivial, (b) was done in class. For (c), if we have $n-1$ nonempty boxes, then one box contains 2 elements, and all the others contain a single element. Hence the number of partitions is the number of ways to choose the 2-element set, which is $\binom{n}{2}$. (d) is similar, going one step further: either one box contains 3 elements, and all the others contain 1, giving a total of $\binom{n}{3}$ ways, or two boxes each contain 2 elements, and all others contain one element. Now there are $\binom{n}{4}$ ways of choosing the 4 elements for the two boxes, and there are 3 ways to divide the chosen elements into the boxes: $(12, 34), (13, 24), (14, 23)$. This gives a total of $3\binom{n}{4}$ partitions with this distribution.

19. (a) follows since $(n-1)!$ is the number of circular permutations of $[n]$. For (b), observe that we must have one circular permutation of 2 elements, and all others of one element. The number of circular permutations (with 1 or 2 elements) is 1, so it suffices to count the number of ways to choose 2 elements, which is $\binom{n}{2}$.

25. The contribution to n from the summand t_i is either 0, or t_i, or $2t_i$, etc. There is no contribution to n from any other summands. Thus the generating function is

$$\prod_{k=1}^{\infty}(1 + x^{t_k} + x^{2t_k} + \cdots) = \prod_{k=1}^{\infty}(1 - x^{t_k})^{-1}.$$

26 d. The Ferrers diagram is just the upper half triangle, and the partition is self-conjugate.

Chapter 9:

19. We know that $\lambda = r(k-1)/(v-1)$ must be an integer, but $80/17$ is not an integer, so no such BIBD exists.

26. Suppose that each element in Z_n appears λ times as a difference of elements in B. Pick an element $x \in Z_n$. Then x can be expressed as $a - b, a, b \in B$, in λ ways. But $a - b = x$ iff $(a+k) - (b+k) = x$, and $a+k, b+k \in B+k$, so each element in Z_n appears λ times as a difference of elements in $B+k$ as well.

28. The differences are $\{-1, -3, -9, -2, -8, -6, 1, 3, 9, 2, 8, 6, 0\}$ which are all the elements in Z_{13}. The blocks are $\{i, i+1, i+3, i+9\}$ for each $i = 1, \ldots, 13$. Thus $\lambda = 1, b = v = 13, k = 4$ and $r = \lambda(v-1)/(k-1) = 4$.

32. It suffices to show that existence of a STS with 3 points, and one with 7 points. A single block of size 3 is a STS with 3 points, and the Fano plane is a STS with 7 points. Now the construction of Theorem 10.3.2 produces a STS with 21 points.

38. $a_{i,j} = 5i + j$ modulo 6, so we have

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 5 & 0 & 1 & 2 & 3 & 4 \\ 4 & 5 & 0 & 1 & 2 & 3 \\ 3 & 4 & 5 & 0 & 1 & 2 \\ 2 & 3 & 4 & 5 & 0 & 1 \\ 1 & 2 & 3 & 4 & 5 & 0 \end{pmatrix}$$