When is an almost monochromatic K_4 guaranteed?

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Abstract

Suppose that $n > (\log k)^{ck}$, where c is a fixed positive constant. We prove that no matter how the edges of K_n are colored with k colors, there is a copy of K_4 whose edges receive at most two colors. This improves the previous best bound of $k^{c'k}$, where c' is a fixed positive constant, which follows from results on classical Ramsey numbers.

1 Introduction

Let p, q be positive integers with $2 \le q \le \binom{p}{2}$. A (p,q)-coloring of K_n is an edge-coloring such that every copy of K_p receives at least q distinct colors on its edges. Let $f(n,p,q)$ denote the minimum number of colors in a (p,q)-coloring of K_n. This parameter, introduced in [1] and subsequently investigated by Erdős and Gyárfás [2] is a generalization of the classical Ramsey numbers. Indeed, if $R_k(p)$ denotes the minimum n so that every k-edge-coloring of K_n results in a monochromatic K_p, then determining all $R_k(p)$ is equivalent to determining all $f(n,p,2)$. Many special cases of $f(n,p,q)$ lead to nontrivial problems (see, e.g. [3, 5, 7, 8]). One particular interesting case is $f(n,4,3)$. In [1] it was observed that an easy application of the probabilistic method yields $f(n,4,3) = o(n)$. This was subsequently improved in [2] to $f(n,4,3) = O(\sqrt{n})$ via the Local Lemma. The second author [4] then improved the upper bound further to $e^{O(\sqrt{\log n})} = n^{o(1)}$, and this is the current best known upper bound. The lower bound follows from the well-known fact $R_k(4) < k^{O(k)}$, which implies that there is a constant c such that

$$f(n,4,3) \ge f(n,4,2) > \frac{c \log n}{\log \log n}.$$

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Here we give the first improvement of this lower bound.

Theorem 1 *Let $a \geq 1$ be fixed. There is a constant c depending on a such that for all $n \geq 2a$,*

$$f(n, 2a, a+1) > \frac{c \log n}{\log \log \log n}.$$

Let $R_k(p, q)$ be the minimum n so that every k-edge-coloring of K_n yields a copy of K_p with at most $q-1$ colors. Then $R_k(p, q) \leq n$ implies that every k-edge coloring of K_n yields a copy of K_p with at most $q-1$ colors. Therefore, in order to edge-color K_n with every copy of K_p receiving at least q colors, we need at least $k+1$ colors. This means that $f(n, p, q) > k$. Our main result is

$$R_k(2a, a+1) \leq c'(\log k)^{c'k} \tag{1}$$

where c' is a positive constant depending only on a.

Let us argue that Theorem 1 follows from (1). First observe that (1) implies that

$$f(\lfloor c'(\log k)^{c'k} \rfloor, 2a, a+1) > k.$$

Now suppose that $a \geq 1$ is fixed and n is sufficiently large. Let k be the largest integer such that $n \geq \lfloor c'(\log k)^{c'k} \rfloor$. Then

$$f(n, 2a, a+1) \geq f(\lfloor c'(\log k)^{c'k} \rfloor, 2a, a+1) > k.$$

Note that as $n \to \infty$, we also have $k \to \infty$. All asymptotic notation below is taken as both of these parameters approach infinity. It suffices to solve for k in terms of n. By definition of k, we clearly have $n = c'(\log k)^{c'k+O(1)}$. Taking logs this yields $\log n = \Theta(k \log \log k)$ or

$$k = \Theta\left(\frac{\log n}{\log \log k}\right). \tag{2}$$

Taking logs of the previous expresssion yields $\log \log n = \Theta(\log k + \log \log \log k) = \Theta(\log k)$ and taking logs once again gives $\log \log \log n = \Theta(\log \log k)$ or

$$\log \log k = \Theta(\log \log \log n).$$

Plugging this into (2) gives us a constant c such that $k > c \log n / \log \log \log n$ and this proves Theorem 1.

2 The setup of the proof

Let $a \geq 1$ be a positive integer throughout the rest of the paper.

Clearly, $f(n,2,2)=0$ for $n\geq 2$. The idea of our proof is to run induction on something related to a, but not on a itself, since in this case the scale would be too rough. To facilitate the induction, we introduce some definitions.

Definition 2 *A k-edge-coloring χ of K_n is a $(\gamma_1,\ldots,\gamma_k)$-*coloring *if, for each $i\in[k]$, color i does not appear in any subgraph $K_{2\gamma_i+2}$ whose edges are colored with at most γ_i+1 colors. In particular, if $\gamma_i=0$, then color i does not appear in any subgraph K_2 whose edges are colored with 1 color, that is, does not appear at all.*

Note that a k-edge-coloring of K_N is a $(2a,a+1)$-coloring iff it is an $(a-1,\ldots,a-1)$-coloring. Consequently, Equation (1) states that if K_N admits an $(a-1,\ldots,a-1)$-coloring with k colors, then $N\leq c'(\log k)^{c'k}$, where c' depends only on a.

Definition 3 *For an edge-coloring χ of K_n and a color i, the* weakness *$\gamma_i(\chi)$ of i is the minimum p such that color i does not appear in a K_{2p+2} with at most $p+1$ colors. In particular, $\gamma_i(\chi)=0$ iff color i is not present in χ at all. Then $\gamma(\chi)=\sum_{i=1}^{k}\gamma_i(\chi)$ is called the* weakness of χ.

Note that by definition, each edge-coloring χ of K_n is a $(\gamma_1(\chi),\ldots,\gamma_k(\chi))$-coloring. Also by definition, the weakness of any $(a-1,\ldots,a-1)$-coloring with k colors is at most $(a-1)k$. Then (1) will follow from the following fact.

Theorem 4 *There is a positive constant c_1 such that if χ is an edge-coloring of K_N, then*

$$N\leq c_1(\log\gamma(\chi))^{c_1\gamma(\chi)}.$$

In everything that follows, let γ_0 be sufficiently large so that for $\gamma\geq\gamma_0$, we have $\log\log\gamma>1$,

$$\left(\frac{\log\gamma}{1000\log\log\gamma}\right)^{15}>\frac{\log\gamma}{4500\log\log\gamma},\qquad\text{and}\qquad 10^4\left(\frac{\log\gamma}{1000\log\log\gamma}\right)^{5}\log\log\gamma>\log 2\gamma.$$

Let

$$\epsilon=\epsilon_\gamma=\frac{1000\log\log\gamma}{\log\gamma}<\frac{1}{100},\qquad t=t_\gamma=\lceil\epsilon^{-10}\rceil,\qquad s=s_\gamma=\left\lceil\frac{(t-1)^{1/4}}{\sqrt{20}}\right\rceil>\frac{40}{\epsilon}.\tag{3}$$

Let $c=R_{\gamma_0}(2\gamma_0)$ and define $g(\gamma)=c(\log\gamma)^{1000\gamma}=c\gamma^{\epsilon\gamma}$.

We will prove Theorem 4 by showing the following:

Suppose that χ is a $(\gamma_1,\ldots,\gamma_k)$-coloring of K_N and $\gamma=\sum_i\gamma_i$. Then $N<g(\gamma)$. $\quad(*)$

We will prove $(*)$ by induction on γ and k. If $0\leq\gamma\leq\gamma_0$, then certainly $N<c\leq g(\gamma)$, so we may assume that $\gamma>\gamma_0$. If some $\gamma_i=0$, then color i cannot appear at all, so we apply induction on k since the bound does not depend on k. Thus, we may assume that each γ_i is positive; in particular, $k \le \gamma$. We will also assume that $N \ge g(\gamma) = c(\log\gamma)^{1000\gamma} = c\gamma^{\epsilon\gamma}$ and proceed to get a contradiction.

For a vertex x in a colored K_n and a color i, let $d_i(x)$ denote the number of edges of color i incident to x.

Claim 5 *For $\gamma > \gamma_0$ and ϵ, t, s defined as above, we have $2t^{2s} < \gamma^{0.1s\epsilon-2}$.*

Proof. Since $2 < t^s$ and $s > 400/\epsilon$, the result follows from $t^{3s} < \gamma^{s\epsilon/20}$, which is equivalent to $60\log t < \epsilon\log\gamma$. Since $t < \epsilon^{-11}$, we have

$$\frac{60\log t}{\epsilon} < \frac{660\log\epsilon^{-1}}{\epsilon} < \frac{660\log\gamma}{1000\log\log\gamma}\log\left[\frac{\log\gamma}{1000\log\log\gamma}\right] < \frac{\log\gamma}{\log\log\gamma}\log\log\gamma = \log\gamma. \qquad \square$$

In the next section we prove the technical statement that every dense bipartite graph $F(V_1, V_2; E)$ contains a 'large' subset M of V_1 in which every t-element subset has 'many' common neighbors in V_2. In Section 4 we prove the main result.

3 A Probabilistic Lemma

One of our main tools is the following lemma, essentially Lemma 1 in [6]. The proof uses ideas of Sudakov [9]. By $N(A)$ we denote the set of common neighbors of all vertices in A.

Lemma 6 *Let positive integers m, n, h, d and reals α, β be such that*

$$m^{d/h} < \beta. \tag{4}$$

Let $F = (V_1, V_2; E)$ be a bipartite graph with $|V_1| = m$, $|V_2| = n$ such that

$$\deg_F(v) \ge n/\alpha \quad \text{for each } v \in V_1.$$

Then there is a subset V_1'' of V_1 with $|V_1''| > m/\alpha^h - 1$ such that every d-tuple D of vertices in V_1'' has at least n/β common neighbors.

Proof. Let $x_1, ..., x_h$ be a sequence of h not necessarily distinct vertices of V_2, which we choose uniformly and independently at random and denote $S = \{x_1, ..., x_h\}$. Denote by V_1' the set $N(S)$ of common neighbors of vertices in S. Note that the size of V_1' is a random variable and that $S \subseteq N(v)$ for every $v \in V_1'$. Then, using (4), we can estimate the expected size of V_1' as follows

$$\mathbf{E}(|V_1'|) = \sum_{v\in V_1}\mathbf{Pr}(v \in V_1') = \sum_{v\in V_1}\left(\frac{|N(v)|}{n}\right)^h \ge m\,\alpha^{-h}. \tag{5}$$

On the other hand, by definition, the probability that a given set of vertices $W \subset V_1$ is contained in V_1' equals $\big(|N(W)|/n\big)^h$. Denote by Z the number of subsets W of V_1' of size d with $|N(W)| < n/\beta$. Then by (4) the expected value of Z is at most

$$\mathbf{E}(Z) = \sum_{W \subseteq V_1 \,:\, |W|=d,\, |N(W)|<n/\beta} \mathbf{Pr}(W \subset V_1') \le \binom{m}{d}\left(\frac{1}{\beta}\right)^h \le m^d \left(\frac{1}{\beta}\right)^h < 1. \tag{6}$$

Hence, the expectation of $|V_1'| - Z$ is greater than $m\,\alpha^{-h} - 1$ and thus, there is a choice S_0 of S such that the corresponding value of $|V_1'(S_0)| - Z(S_0)$ is greater than $m\,\alpha^{-h} - 1$. For every d-tuple D of vertices of $V_1'(S_0)$, delete a vertex $v_D \in D$ from $V_1'(S_0)$. The resulting set V_1'' satisfies the lemma. $\square$

4 Proof of the Theorem

Call a t-set of vertices *rainbow* if its edges are colored with at least $10t^{3/2}$ colors.

Claim 7 *Suppose that $n \ge \gamma > \gamma_0$, the edges of K_n are colored (with any number of colors) and $d_i(x) \le 2n\gamma^{-\epsilon/10}$ for each $x \in V(K_n)$ and each color i. Then the number of t-sets that are not rainbow is at most $\binom{n}{t}/\gamma$.*

Proof. First, let us estimate $\nu(i,t,n)$ — the number of t-sets in K_n in which there is a vertex incident with at least s edges of color i in this t-set. We can first choose the vertex, then choose s incident edges of color i and include the other ends of these edges, and then add $n - s - 1$ other vertices. This gives

$$\nu(i,t,n) \le \sum_{x \in V(K_n)} \binom{d_i(x)}{s}\binom{n-1-s}{t-1-s} \le n\binom{\frac{2n}{\gamma^{\epsilon/10}}}{s}\binom{n-1-s}{t-1-s} \le \binom{n}{t}\gamma^{-s\epsilon/10}\, t^{2s}.$$

Similarly, let $\psi(i,t,n)$ be the number of t-sets in K_n in which there is a matching of color i of size at least s. Let e_i be the number of edges of color i. Since

$$e_i \le \frac{n}{2} \max_{x \in V(K_n)} d_i(x) \le n^2 \gamma^{-\epsilon/10},$$

we have

$$\psi(i,t,n) \le \binom{e_i}{s}\binom{n-2s}{t-2s} \le \binom{\frac{n^2}{\gamma^{\epsilon/10}}}{s}\binom{n-2s}{t-2s} \le \binom{n}{t} t^{2s}\gamma^{-s\epsilon/10}.$$

Now Claim 5 implies that

$$\nu(i,t,n) + \psi(i,t,n) \le 2\binom{n}{t} t^{2s}\gamma^{-s\epsilon/10} < \frac{1}{\gamma^2}\binom{n}{t}.$$

Suppose that a t-set T contains more than s^2 edges of color i and let G_i be the graph of these edges. Either G_i has a vertex incident with at least s edges, or Vizing's Theorem implies that G_i has a proper edge-coloring with at most s colors. In the latter case, G_i has a matching of size at least $s^2/s = s$. We have already shown that the number of t-sets that contain a monochromatic matching of size s or a vertex with s edges of the same color is at most $\binom{n}{t}/\gamma^2$. Consequently, the number of t-sets that contain more than s^2 edges of some color is at most

$$k\binom{n}{t}/\gamma^2 \leq \binom{n}{t}/\gamma.$$

Each t-set not included above has at most s^2 edges in each color and therefore at least $\binom{t}{2}/s^2$ colors. By the choice of s, this is at least $10t^{3/2}$. Hence the number of rainbow t-sets is at least $(1-1/\gamma)\binom{n}{t}$. $\square$

Claim 8 *Let $u \in V(K_N)$ and $S = S(u) = \{j \in [k] \ : \ d_j(u) \leq N/\gamma^{1+\epsilon/2}\}$. Then for every $i \in [k] - S$ and $j \in [k]$, the number of vertices $x \in N_i(u)$ for which*

$$|N_j(x) \cap N_i(u)| \geq 2d_i(u)/\gamma^{\epsilon/10} \tag{7}$$

is at most $\gamma^{\epsilon\gamma-3}$.

Proof. Suppose the contrary. Then there are colors $i \in [k] - S(u)$ and $j \in [k]$ such that $N_i(u)$ contains a set M of $\lceil \gamma^{\epsilon\gamma-3} \rceil$ vertices x such that (7) holds. Consider the bipartite graph $F(V_1, V_2; E)$ with partite sets $V_1 = M$ and $V_2 = N_i(u) - M$ whose edges are all edges of color j in our K_N connecting V_1 with V_2. By (7) and since $|M| = \lceil \gamma^{\epsilon\gamma-3} \rceil < \lceil N/\gamma^3 \rceil < d_i(u)/\gamma^{\epsilon/10}$, we have for every $v \in V_1$,

$$\deg_F(v) > \frac{2d_i(u)}{\gamma^{\epsilon/10}} - |M| > \frac{d_i(u)}{\gamma^{\epsilon/10}} > \frac{|V_2|}{\gamma^{\epsilon/10}}.$$

Observe that graph F satisfies the conditions of Lemma 6 with

$$m = |M|, \quad n = |V_2|, \quad h = \gamma/\sqrt{t}, \quad d = t, \quad \alpha = \gamma^{\epsilon/10}, \quad \beta = 2m^{t/h}.$$

Hence, there is a subset M' of V_1 with

$$|M'| > m/\alpha^h - 1 \geq \gamma^{\epsilon\gamma-3}\alpha^{-h} - 1 > \gamma^{\epsilon\gamma-3}\gamma^{-(\gamma/\sqrt{t})\epsilon/10} - 1 > \gamma^{0.9\epsilon\gamma} \tag{8}$$

such that every d-tuple D of vertices in M' has at least n/β common neighbors.

We will construct a sequence $M_0 \subset M_1 \subset \cdots$ of subsets of M' as follows. Let $M_0 = M'$. Suppose that $M_0, M_1, \ldots, M_l$ are constructed. If there is a vertex $x_{l+1} \in M_l$ and a color j_{l+1} such that $|N_{j_{l+1}}(x_{l+1}) \cap M_l| \geq |M_l|\gamma^{-\epsilon/10}$, then we let $M_{l+1} = N_{j_{l+1}}(x_{l+1}) \cap M_l$, otherwise we stop. Suppose that we stop at Step q. Each color i appears at most $2\gamma_i + 1$ times in $\{j_1, \ldots, j_q\}$ since otherwise we have a monochromatic $K_{2\gamma_i+2}$ which is forbidden. Consequently, $q \le \sum_i (2\gamma_i + 1) = 2\gamma + k \le 3\gamma$. From this and (8),

$$|M_q| > |M_0|(\gamma^{-\epsilon/10})^{3\gamma} = |M_0|\gamma^{-3\gamma\epsilon/10} > \gamma^{0.9\epsilon\gamma}\gamma^{-3\gamma\epsilon/10} = \gamma^{0.6\gamma\epsilon} > \gamma.$$

Hence, by Claim 7, M_q contains a rainbow t-tuple D (in fact it contains many). Let $N_F(D) = U$. By Lemma 6, $|U| \ge n/\beta$. Now suppose ℓ is a color that appears in D. Then the weakness of ℓ within U is strictly smaller than γ_ℓ, since if ℓ appears in a K_{2p} within U that receives at most p colors, then this copy together with an edge of color ℓ from D yields a $K_{2(p+1)}$ with at most $p+1$ colors (the only new color is possibly j). Therefore, the weakness of χ when restricted to U is at most $\gamma' = \gamma - 10t^{3/2}$. Hence by the induction hypothesis, $|U| < g(\gamma') = c(\log\gamma')^{1000\gamma'}$. Since $|U| \ge n/\beta$,

$$n \le \beta c(\log\gamma')^{1000\gamma'}.$$

On the other hand, since $|M| < d_i(u)/2$,

$$n = |V_2| = d_i(u) - |M| > \frac{d_i(u)}{2} > \frac{N}{2\gamma^{1+\epsilon/2}}.$$

This gives

$$N < 2\gamma^{1+\epsilon/2}(2m^{t/h})c(\log\gamma')^{1000\gamma'} = 4\gamma^{1+\epsilon/2}m^{t\sqrt{t}/\gamma}c(\log\gamma')^{1000\gamma'} < \gamma^{2+\epsilon t^{3/2}}c(\log\gamma')^{1000\gamma'},$$

where the last inequality holds because $m = |M| < \gamma^{\epsilon\gamma}$. As $N \ge g(\gamma) = c(\log\gamma)^{1000\gamma}$, we get

$$(\log\gamma)^{1000\gamma} < \gamma^{2+\epsilon t^{3/2}}(\log\gamma')^{1000\gamma'} < \gamma^{2+\epsilon t^{3/2}}(\log\gamma)^{1000\gamma'}.$$

Taking logs, this reduces to

$$1000\gamma\log\log\gamma < (2+\epsilon t^{3/2})\log\gamma + 1000\gamma'\log\log\gamma.$$

Consequently,

$$(1000\log\log\gamma)10t^{3/2} < (2+\epsilon t^{3/2})\log\gamma = 2\log\gamma + 1000t^{3/2}\log\log\gamma.$$

Simplifying, we obtain $9000t^{3/2}\log\log\gamma < 2\log\gamma$. Finally, this yields

$$\left(\frac{\log\gamma}{1000\log\log\gamma}\right)^{15} = \epsilon^{-15} \le t^{3/2} < \frac{\log\gamma}{4500\log\log\gamma},$$

which contradicts our choice of γ. □

Claim 9 *For every $u \in V(K_N)$, the number of rainbow t-sets on $V(K_N) - \{u\}$ all of whose vertices are connected with u by edges of the same color is at least* $0.3\sum_{i=1}^{k}\binom{d_i(u)}{t}$.

Proof. Fix some $u \in V(K_N)$. Let $S = \{i \in [k] \ : \ d_i(u) \leq N/\gamma^{1+\epsilon/2}\}$. Then

$$\sum_{i \in S} \binom{d_i(u)}{t} \leq k \binom{\left\lfloor \frac{N}{\gamma^{1+\epsilon/2}} \right\rfloor}{t} \leq k \binom{\lfloor \frac{N}{k} \rfloor}{t} \gamma^{-\epsilon t/2} \leq 2\gamma^{-\epsilon t/2} \sum_{i=1}^{k} \binom{d_i(u)}{t}.$$

We put the factor 2 since $d_1(u) + \ldots + d_k(u) = N - 1$ and not N. Since $t = \lceil \epsilon^{-10} \rceil$, we have $t\epsilon > 20$ and hence

$$\sum_{i \in S} \binom{d_i(u)}{t} \leq \gamma^{-10} \sum_{i=1}^{k} \binom{d_i(u)}{t}. \tag{9}$$

Now, let $i \notin S$. Let M be the set of vertices $x \in N_i(u)$ such that for some color j (7) holds. Let $\overline{M} = N_i(u) - M$. By Claim 8,

$$|M| < \gamma^{\epsilon\gamma - 2} < \frac{N}{\gamma^2} < \frac{|N_i(u)|}{t}.$$

Hence for the subgraph F of our K_N on $\overline{M}$, the conditions of Claim 7 are satisfied since $|\overline{M}| > (1 - 1/t)d_i(u) > 0.9d_i(u) > \gamma$. Thus by Claim 7, at least $(1 - 1/\gamma)\binom{|\overline{M}|}{t}$ t-sets in $\overline{M}$ are rainbow. Now

$$\frac{\gamma - 1}{\gamma} \binom{|\overline{M}|}{t} \geq \frac{\gamma - 1}{\gamma} \binom{d_i(u)(1 - 1/t)}{t}.$$

For large γ, the last expression is at least

$$0.9 \left(\frac{t-1}{t} \right)^t \binom{d_i(u)}{t} \geq \frac{1}{3} \binom{d_i(u)}{t}.$$

Combining this with (9), we finish the proof. □

By Claim 9, the total number of $(t+1)$-sets $\{u_0, u_1, \ldots, u_t\}$ of vertices of $V(K_N)$ such that the t-set $\{u_1, \ldots, u_t\}$ is rainbow and all edges from u_0 to $u_1, \ldots, u_t$ are of the same color is at least

$$0.3 \sum_{u \in V(K_N)} \sum_{i=1}^{k} \binom{d_i(u)}{t} \geq 0.3N \cdot k \binom{(N-1)/k}{t} \geq N \cdot (2k)^{1-t} \binom{N}{t}.$$

It follows that some rainbow t-set $\{u_1, \ldots, u_t\}$ is contained in at least $N \cdot (2k)^{1-t}$ such $(t+1)$-sets. Let U be the set of all vertices u_0 in these $(t+1)$-sets containing our chosen $\{u_1, \ldots, u_t\}$. Then, for some $1 \leq i \leq k$ the size of the subset U_i of U that is connected with each of $u_1, \ldots, u_t$ by an edge of color i is at least $2N \cdot (2k)^{-t}$. Since $\{u_1, \ldots, u_t\}$ is rainbow, it contains edges of at least $10t^{3/2}$ colors. For every color ℓ that appears within $\{u_1, \ldots, u_t\}$, the weakness of ℓ when restricted to U_i is at most $\gamma_\ell - 1$. Hence by the induction hypothesis, $|U_i| \le g(\gamma') = c(\log\gamma')^{1000\gamma'}$, where $\gamma' = \gamma - 10t^{3/2}$. Since $|U_i| \ge 2N/(2k)^t$ and $N \ge g(\gamma)$, we obtain

$$c(\log\gamma)^{1000\gamma} \le N < (2k)^t c(\log\gamma')^{1000\gamma'} < (2k)^t c(\log\gamma)^{1000\gamma'}.$$

Dividing by c and taking logs,

$$1000\gamma \log\log\gamma < t\log 2\gamma + 1000\gamma' \log\log\gamma.$$

Consequently,

$$(1000\log\log\gamma)10t^{3/2} < t\log 2\gamma.$$

Plugging in the values of t and ϵ, we obtain

$$10^4\left(\frac{\log\gamma}{1000\log\log\gamma}\right)^5 \log\log\gamma = 10^4\epsilon^{-5}\log\log\gamma < 10^4\sqrt{t}\log\log\gamma < \log 2\gamma.$$

This contradicts our choice of γ and completes the proof. □

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