On the chromatic number and independence number of hypergraph products

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Abstract

The hypergraph product $G \Box H$ has vertex set $V(G) \times V(H)$, and edge set $\{e \times f : e \in E(G), f \in E(H)\}$, where $\times$ denotes the usual cartesian product of sets. We construct a hypergraph sequence $\{G_n\}$ for with $\chi(G_n) \to \infty$ and $\chi(G_n \Box G_n) = 2$ for all n. This disproves a conjecture of Berge and Simonovits [2]. On the other hand, we show that if G and H are hypergraphs with infinite chromatic number, then the chromatic number of $G \Box H$ is also infinite.

We also provide a counterexample to a "dual" version of their conjecture, by constructing a graph sequence $\{G_n\}$ with $\alpha(G_n)/|V(G_n)| \to 0$ and $\alpha(G_n \Box G_n)/|V(G_n)|^2 \to 1/2$. The constant $1/2$ cannot be replaced by a larger number.

1 Introduction

The direct product of hypergraphs G and H is the hypergraph $G \Box H$, whose vertex set is $V(G) \times V(H)$, and edge set is $\{e \times f : e \in E(G), f \in E(H)\}$, where $\times$ denotes the usual cartesian product of sets. The chromatic number $\chi(G)$ of a hypergraph G is the minimum number of colors into which $V(G)$ can be partitioned so that every edge contains two vertices with different colors. Berge and Simonovits [2] conjectured that if both $\chi(G)$ and $\chi(H)$ go to infinity, then so does $\chi(G \Box H)$ (see also Problem 15.14 in [5]). In this note we disprove their conjecture. Our only tool is the following result whose $k = 2$ case is due to Erdős [3, 4]. The proof for general k is essentially the same as for $k = 2$ and we repeat it here for completeness.

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Theorem 1. *Fix integers $k \geq 2$ and $n \geq 8$. Then the minimum number $m_k(n)$ of edges in an n-uniform hypergraph that is not k-colorable satisfies*

$$k^{n-1} \leq m_k(n) < n^2 k^{n+2}.$$

Proof. For the lower bound, observe that in a random k-coloring of an n-uniform hypergraph with m edges, the expected number of monochromatic edges is m/k^{n-1}. When $m < k^{n-1}$ this is less than 1, and hence the hypergraph admits a k-coloring. For the upper bound, we must produce an n-uniform hypergraph with at most $m = n^2 k^{n+2}$ edges that is not k-colorable. We do this by picking m edges randomly (possibly with repetition) on a vertex set of size $v = (k-1)n^2 + n$. Fix a k-coloring χ. The probability that a randomly chosen edge is monochromatic under χ is at least

$$\frac{k\binom{v/k}{n}}{\binom{v}{n}} > \frac{1}{k^{n-1}}\left(\frac{v-kn}{v-n}\right)^n = \frac{1}{k^{n-1}}\left(1-\frac{1}{n}\right)^n.$$

The probability that none of the m independently chosen random n-sets is monochromatic under χ is at most

$$\left(1-\frac{1}{k^{n-1}}\left(1-\frac{1}{n}\right)^n\right)^m.$$

Consequently, the probability that there exists a k-coloring under which none of the m edges is monochromatic is at most

$$k^v\left(1-\frac{1}{k^{n-1}}\left(1-\frac{1}{n}\right)^n\right)^m.$$

Since $n \geq 8$, $(1-1/n)^n > 1/3$. Therefore an upper bound for the expression above is

$$k^v\left(1-\frac{1}{3k^{n-1}}\right)^m < k^v e^{-m/(3k^{n-1})}.$$

When $m > 3k^{n-1} v \log k$, this is less than 1, and moreover, a short calculation shows that for our choice of v, $3k^{n-1} v \log k < 3n^2 k^n \log k < n^2 k^{n+2}$ hence the result follows. $\square$

Theorem 2. *For every integer $s \geq 2$ and $k = 2^s$, there exists a hypergraph G_k satisfying $\chi(G_k) > k$ and $\chi(G_k \square G_k) = 2$.*

Proof. Set $n = 4\log_2 k$. By the upper bound of Theorem 1, there exists an n-uniform hypergraph G_k with $\chi(G_k) > k$, and at most $n^2 k^{n+2}$ edges. Now $G_n \square G_n$ is an n^2-uniform hypergraph with at most $n^4 k^{2n+4}$ edges. By the choice of n, this is at most 2^{n^2-1}. By the lower bound in Theorem 1 applied with $k = 2$, we conclude that $\chi(G_n \square G_n) = 2$. $\square$

In spite of Theorem 2, the intuition behind Berge and Simonovits' conjecture is correct, since their conjecture holds for infinitely chromatic hypergraphs. In what follows we assume that all hypergraphs have countable vertex sets.

Theorem 3. *Let G and H be hypergraphs whose edges have finite size. Suppose that G and H each have infinite chromatic number. Then the chromatic number of $G \Box H$ is infinite.*

Proof. Suppose that $V(G) = \{x_1, x_2, \ldots\}$ and $V(H) = \{y_1, y_2, \ldots\}$. Imagine that x_i represents the point $(i, 0)$ in the xy plane, and y_j represents the point $(0, j)$. Think of a typical vertex (x_i, y_j) of $V(G) \times V(H)$ as representing the point (i, j) in the xy plane.

Let $Y_j = \{(x_1, y_j), (x_2, y_j), \ldots\}$ be the set of vertices on the line with equation $y = j$. For $S \subset Y_j$, let $S^{-1} = \{x_i \in V(G) : (x_i, y_j) \in S\}$. We will repeatedly use the easy fact that if the vertices of a hypergraph with infinite chromatic number are colored with finitely many colors, then some color class induces a hypergraph with infinite chromatic number.

Now suppose for contradiction, that f is a proper coloring of $V(G \Box H)$ with k colors, where k is some positive integer. Let f_1 be the restriction of f to Y_1. Then f_1 induces a coloring on $V(G)$ with k colors, defined by $f_1(x_i) = f((x_i, y_1))$. Consequently, there exists a color c_1 and $S_1 \subset Y_1$ so that $f((v, w)) = c_1$ for all $(v, w) \in S_1$ and $\chi(G_1) = \infty$, where $G_1 = G[S_1^{-1}]$ is the subhypergraph of G induced by S_1^{-1}. From now we restrict each Y_j to $V(G_1) \times V(H)$. Let f_2 be the restriction of f to $Y_2 \cap (V(G_1) \times V(H))$. In a similar way we obtain a color c_2 and $S_2 \subset Y_2 \cap (V(G_1) \times V(H))$ so that $f((v, w)) = c_2$ for all $(v, w) \in S_2$ and $\chi(G_2) = \infty$, where $G_2 = G_1[S_2^{-1}]$. Repeating this we get a sequence $S_1, S_2, \ldots$ and $G_1 \supset G_2 \supset \ldots$ where $S_j \subset Y_j \cap (V(G_{i-1}) \times V(H))$, $f((v, w)) = c_j$ for all $(v, w) \in S_j$ and $G_j = G_{j-1}[S_j^{-1}]$. Note also that the sets S_j are nested in the sense that $(x_i, y_{j+1}) \in S_{j+1}$ implies that $(x_i, y_j) \in S_j$.

Consider the coloring f^* of $V(H)$ defined by $f^*(y_j) = c_j$. By the fact above, there is a color c and a set $T \subset V(H)$ so that $f^*(y_l) = c$ for all $y_l \in T$ and $\chi(H[T]) = \infty$. The latter implies that there is an edge $Y = \{y_{j_1}, \ldots, y_{j_m}\} \in E(H[T])$ with $j_1 < j_2 < \cdots < j_m$. Consider the hypergraph $I = G_{j_m} \Box H[Y] \subset G \Box H$. By construction, $f((v, w)) = c$ for every $(v, w) \in I$. Since G_{j_m} contains an edge of G and $Y \in E(H[Y])$, we deduce that I contains an edge which is monochromatic in color c. Because $I \subset G \Box H$, this is a monochromatic edge in $G \Box H$ under f, a contradiction. $\Box$

We remark that Theorem 3 also applies to the case when precisely one of the hypergraphs, say G, has edges that are infinite. For the proof to work, we only need that $Y = \{y_{j_1}, \ldots, y_{j_m}\}$ is finite, which is guaranteed if all edges of H are finite.

On the other hand Theorem 3 does not hold when both hypergraphs have edges only of infinite size. Indeed, if we let G be the hypergraph with $V(G) = \{1, 2, \ldots\}$ and $E(G)$ comprising all infinite subsets of $V(G)$, then clearly $\chi(G) = \infty$, since any coloring with finitely many colors results in a monochromatic infinite set, which is a monochromatic edge. Now we can color vertex (i, j) in $V(G \Box G)$ red if $i \leq j$ and blue otherwise. It is easy to see (since all edges of G have infinite size) that this is a proper 2-coloring of $G \Box G$. This implies, more generally, that whenever G and H have only infinite size edges, and $\chi(G)$ and $\chi(H)$ are both ∞, then $\chi(G \Box H) = 2$.

More precisely, we can write $H = H_{FIN} \cup H_{INF}$ and $G = G_{FIN} \cup G_{INF}$, where the subscripts denote the subhypergraph induced by the edges that are finite (FIN) or infinite (INF). Note that $\chi(G) = \infty$ if and only at least one of $\chi(G_{FIN})$ or $\chi(G_{INF})$ is ∞, since if $\chi(G)$ is finite, then $\chi(G) \leq \chi(G_{FIN}) \cdot \chi(G_{INF})$. Theorem 3 and the preceding discussion imply the following result.

Theorem 4. *Suppose that $\chi(G) = \infty$ and $\chi(H) = \infty$. Then $\chi(G \square H) = \infty$ if and only if $\chi(G_{FIN}) = \infty$ or $\chi(H_{FIN}) = \infty$.*

As the chromatic number of almost all graphs on n vertices is n divided by the independence number, one could ask whether the analogue of the Berge-Simonovits conjecture holds for independence number, i.e., if $\alpha(G_n)/|V(G_n)| \to 0$, then does it follow that $\alpha(G_n \square G_n)/|V(G_n)|^2 \to 0$ as well? This question was posed by Kostochka [6]. Here we construct a sequence $\{G_n\}$ with $\alpha(G_n)/|V(G_n)| \to 0$ and $\alpha(G_n \square G_n)/|V(G_n)|^2 \not\to 0$ by relating it to the subject of Ramsey-Turán theory (see [8] for a survey).

One of the seminal results in this field is due to Bollobás-Erdős [1]. They constructed an n vertex graph (n sufficiently large) with at least $(1/8 - o(1))n^2$ edges that contains no copy of K_4 and has independence number at most $o(n)$. In addition to this, we need the following lemma.

Lemma 5. *Fix $0 < \gamma < 1/2$ and $n \geq 2$. Let G be a graph with vertex partition $X \cup Y$, each of size n. Suppose that $\alpha(G[X])$ and $\alpha(G[Y])$ are both at most γn, and G contains no K_4 with two vertices in each of X, Y. Then the number of edges with one endpoint in X and the other in Y is at most $(1/2 + \gamma)n^2$.*

Proof. Suppose, for contradiction, that the number of X, Y edges is greater than $(1/2 + \gamma)n^2$. Let X' be the set of vertices in X with at least $(1/2 + \gamma/2)n$ neighbors in Y. Counting edges between X and Y we obtain

$$(1/2 + \gamma)n^2 < |X'|n + (n - |X'|)(1/2 + \gamma/2)n.$$

This yields $|X'| > \gamma n/(1 - \gamma) > \gamma n$. Hence there exist two adjacent vertices $u, v \in X'$. Considering their neighborhoods in Y, we obtain at least γn vertices in Y adjacent to both u and v. Two of these, say y, z must be adjacent. Now $\{u, v, y, z\}$ forms a copy of the forbidden K_4. $\square$

Theorem 6. *For every sufficiently large n, there exists a graph G_n on n vertices with $\alpha(G_n)/n \to 0$ and $\alpha(G_n \square G_n)/n^2 \to 1/2$. The constant $1/2$ cannot be replaced by a larger number.*

Proof. The Bollobás-Erdős graph BE on $2n$ vertices has a vertex partition into two sets X_1, X_2 each of size n, such that the number of edges within each part is at most $o(n^2)$. Let BE_i be the subgraph induced by X_i. Then the construction of Bollobás-Erdős yields $BE_1 \cong BE_2$. Let us call each of these graphs G_n. Note that $\alpha(G_n) = o(n)$ since $\alpha(BE) = o(n)$. Now we construct a large independent set in $G_n \square G_n$. Consider the set of edges in BE between X_1 and X_2. We know there are $(1/2 - o(1))n^2$ of them. Each of these edges corresponds to a vertex in $G_n \Box G_n$. An edge in $G_n \Box G_n$ containing four of these chosen vertices corresponds to a copy of K_4 in BE. Since no such copy of K_4 exists in BE, we conclude that the set of vertices chosen in $G_n \Box G_n$ is an independent set. The first claim of the theorem follows.

For the second part, suppose that we have an n vertex graph G_n with $\alpha(G_n)/n = o(1)$. Pick an independent set S in $G_n \Box G_n$. This results in a construction of a $2n$ vertex graph G, whose vertex set is partitioned into two copies of G_n, and whose edge set contains in addition the elements of S. The fact that S is independent is equivalent to the fact that G contains no copy of K_4 with two vertices in each part. By Lemma 5, the number of edges of G with endpoints in different copies of G_n is at most $(1/2 + o(1))n^2$. In other words, $|S| \leq (1/2 + o(1))n^2$ as required. □

Our construction in Theorem 2 has the property that the sizes of the edges of G_k are not bounded. Perhaps the following is true.

Conjecture 7. *For every $r \geq 2$ there exists $c \geq 2$ so that for every positive integer k, there exist r-graphs G_k and H_k for which $\chi(G_k) > k$, $\chi(H_k) > k$, and $\chi(G_k \Box H_k) \leq c$. Moreover, the above also holds for $c = r = 2$.*

A disproof of this conjecture would give a positive answer to a question of Poljak and Rödl [7] which is a weaker version of the well-known Hedetniemi conjecture on the chromatic number of graph products.

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