

Books versus Triangles

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Abstract

A book of size b in a graph is an edge that lies in b triangles. Consider a graph G with n vertices and $\lfloor n^2/4 \rfloor + 1$ edges. Rademacher proved that G contains at least $\lfloor n/2 \rfloor$ triangles, and Erdős conjectured and Edwards proved that G contains a book of size at least $n/6$.

We prove the following “linear combination” of these two results. Suppose that $\alpha \in (1/2, 1)$ and the maximum size of a book in G is less than $\alpha n/2$. Then G contains at least

$$\alpha(1 - \alpha) \frac{n^2}{4} - o(n^2)$$

triangles as $n \rightarrow \infty$. This is asymptotically sharp. On the other hand, for every $\alpha \in (1/3, 1/2)$, there exists $\beta > 0$ such that G contains at least βn^3 triangles. It remains an open problem to determine the largest possible β in terms of α . Our proof uses the Ruzsa-Szemerédi theorem.

1 Introduction

A book in a graph is a collection of triangles sharing a common edge. The size of a book is the number of triangles. Let $b(G)$ be the size of the largest book in graph G and $t(G)$ be the number of triangles in G . Throughout this note, unless otherwise specified, we let G be a graph with n vertices and $\lfloor n^2/4 \rfloor + 1$ edges. All asymptotic notation is to be taken as n grows.

Mantel’s theorem states that G contains a triangle, i.e. $t(G) \geq 1$. Rademacher (unpublished) proved in the 1950’s that in fact $t(G) \geq \lfloor n/2 \rfloor$. Erdős conjectured [2] in 1962 that $b(G) > n/6$ and this was proved soon after by Edwards (unpublished, see also Khadžiivanov and Nikiforov [4] for an independent proof). Both Rademacher’s and Edwards’ results are sharp. In the former, $t(G) = \lfloor n/2 \rfloor$ is achieved by adding an edge to one part in the complete balanced bipartite graph (note that this also yields $b(G) = \lfloor n/2 \rfloor$). In the latter, every known construction achieving $b(G) = \lfloor n/6 \rfloor + 1$ has $t(G) = \Omega(n^3)$.

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In this note, we study the relationship between $t(G)$ and $b(G)$. Intuitively, one would suspect that as $t(G)$ decreases, so does $b(G)$. However, this naive intuition is false. As $t(G)$ becomes smaller, this places greater restrictions on G and $b(G)$ becomes larger, approaching $n/2$. Indeed, when $t(G)$ is minimized, we saw in the construction above that $b(G) = \lfloor n/2 \rfloor$ which is much larger than $n/6$. On the other hand, when $b(G) = \lfloor n/6 \rfloor + 1$, which is as small as possible, then $t(G) = \Omega(n^3)$, which is much larger than $\lfloor n/2 \rfloor$.

Our first result shows that as $b(G)$ decreases from $\lfloor n/2 \rfloor$ to $(1 - \gamma)n/2$, the number of triangles increases from $\lfloor n/2 \rfloor$ to $\Omega_\gamma(n^2)$.

Theorem 1 *Fix $\alpha \in (1/2, 1)$ and $\varepsilon > 0$. Then there exists n_0 such that the following holds for $n > n_0$: Every n vertex graph G with at least $\lfloor n^2/4 \rfloor + 1$ edges and $b(G) < \alpha n/2$ satisfies*

$$t(G) > (\alpha(1 - \alpha) - \varepsilon) \frac{n^2}{4}.$$

Theorem 1 is asymptotically sharp, as there are examples of graphs with $t(G) = \alpha(1 - \alpha)n^2/4 - o(n^2)$ and $b(G) < \alpha n/2$. Indeed, take the balanced complete bipartite graph (for n even) with one vertex removed and make this vertex adjacent to $\lfloor \alpha n/2 \rfloor - 1$ vertices in one part $n/2 - \lfloor \alpha n/2 \rfloor + 2$ vertices in the other part.

Our second result shows that if $b(G) < (1/2 - \gamma)n/2$, then the number of triangles increases from $\Theta(n^2)$ to $\Omega_\gamma(n^3)$.

Theorem 2 *For every $\alpha \in (1/3, 1/2)$, there exists $\beta > 0$ such that the following holds for all sufficiently large n : Every n vertex graph G with at least $\lfloor n^2/4 \rfloor + 1$ edges and $b(G) < \alpha n/2$ has at least βn^3 triangles.*

Note that Theorems 1 and 2 cover all ranges of α except for $\alpha = 1 + o(1)$, $1/2 + o(1)$, and $1/3 + o(1)$. In particular, $\alpha < 1/3$ is impossible due to Edwards' theorem.

It seems likely that Theorem 2 can be strengthened by replacing β by an explicitly defined number (in terms of α) that is optimal, but this seems very hard. One plausible conjecture is that for every $\alpha \in (1/3, 1/2)$ one can take

$$\beta = \frac{\alpha(1 - \alpha)^2}{16} + o(1).$$

If true, this would be sharp due to the following natural generalization of the example achieving equality in Edwards' theorem: Partition the vertex set into two almost equal parts X, Y , and partition each of X and Y into three parts roughly of sizes $\alpha n/2, (1 - \alpha)n/4, (1 - \alpha)n/4$. Call them X_1, X_2, X_3 and Y_1, Y_2, Y_3 respectively. Add all edges between the three parts within X and within Y . Finally add all edges between X_i and Y_i for each i .

2 Tools

We need the following two results in our proof. The first is a very special case of the Erdős-Simonovits stability theorem [7]. The proof, which we include here for convenience, is inspired by a recent approach of Füredi. We write $e(G)$ for the number of edges in graph G .

Lemma 3 (Triangle Stability Lemma) *Let G be a triangle-free graph with n vertices and at least $\lfloor n^2/4 \rfloor - k$ edges. Then G has a vertex partition $X \cup Y$ such that $e(G[X]) + e(G[Y]) \leq k$.*

Proof: Let v be a vertex of maximum degree. Since G is triangle-free, there are no edges in $Y := N(v)$. Let $X = V(G) - N(v)$ and consider the partition $X \cup Y$ of $V(G)$. Let us change G as follows: for each vertex $w \in X$, delete all s edges incident to w contained in X and add s edges from w to Y that were not previously in G . Since $d(w) \leq d(v) = |Y|$ this is always possible. Let G' be the graph that results. Now suppose that G has t edges within X . Then $e(G') = e(G) + t$ as for every deleted edge within X , we add two new edges between X and Y . Since G' is bipartite, we have

$$\lfloor n^2/4 \rfloor - k + t = e(G) + t = e(G') \leq \lfloor n^2/4 \rfloor.$$

Consequently, $t \leq k$ as desired. □

Our second tool is the triangle removal lemma, first proved by Ruzsa and Szemerédi [6]. It is an easy consequence of the Regularity Lemma.

Lemma 4 (Triangle Removal Lemma [6]) *For every $\delta > 0$ there exists $\beta > 0$ and n_0 such that the following holds for all $n > n_0$: Every n vertex graph with at most βn^3 triangles can be made triangle-free by deleting a set of at most δn^2 edges.*

3 Proofs

In this section we prove Theorems 1 and 2. Crucial to our proof of Theorem 1 is an assumption on minimum degree, so the theorem that we actually prove is the following:

Theorem 5 *Let $\alpha' \in (1/2, 1)$. For every $\varepsilon' \in (0, (1 - \alpha)/3)$, there exists, $\delta > 0$ and n'_0 such that the following holds for all $n > n'_0$: Every n vertex graph G with at least $\lfloor n^2/4 \rfloor + 1$ edges, minimum degree at least $(1 - \delta)n/2$ and $b(G) < \alpha'n/2$ satisfies $t(G) > (\alpha'(1 - \alpha') - 4\varepsilon')n^2/4$.*

Before proceeding to the proof of Theorem 5 let us argue that it implies Theorem 1. We will need the Erdős-Stone theorem [3], which states that for every $\gamma > 0$ there exists $\delta' > 0$ and n_1 such that every graph with $n > n_1$ vertices and at least $n^2/4 + \gamma n^2$ edges contains at least $\delta'n^3$ triangles.

Proof of (Theorem 5 \rightarrow Theorem 1). Let us take inputs $\alpha \in (1/2, 1)$ and $\varepsilon > 0$ from Theorem 1. Now choose

$$\varepsilon' < \min \left\{ \frac{\varepsilon}{10}, \frac{1-\alpha}{2} \right\}.$$

Let $\alpha' = \alpha + \varepsilon'$. The choice of ε' ensures that $\alpha' \in (1/2, 1)$ and $\varepsilon' < (1 - \alpha')/3$. Let δ, n'_0 be the outputs of Theorem 5 with inputs α', ε' . We may assume that $\delta < \varepsilon'/2$. Let δ' and n_1 be the outputs of the Erdős-Stone theorem with input $\gamma = \delta^2/3$. Let n_2 be sufficiently large so that $\delta'n^3 > 4n^2$ for all $n > n_2$. Finally, let $n_0 > 2 \max\{n'_0, n_1, n_2, 100/\delta\}$.

Now suppose that $n > n_0$ and G is an n vertex graph with $e(G) \geq \lfloor n^2/4 \rfloor + 1$ and $b(G) < \alpha n/2$. Our goal is to show that $t(G) > (\alpha(1 - \alpha) - \varepsilon)n^2/4$.

Note that our constants satisfy the hierarchy $1/n_0 \ll \delta \ll \varepsilon' \ll \varepsilon, 1 - \alpha$.

If G has minimum degree $d < (1 - \delta)n/2$, then remove a vertex of degree less than d to form the graph G_1 with $n - 1$ vertices. Continue removing a vertex of degree less than $d_i = (1 - \delta)(n - i)/2$ in G_i to form the graph G_{i+1} if such a vertex exists. Then

$$e(G_k) \geq \left\lfloor \frac{n^2}{4} \right\rfloor + 1 - \frac{(1 - \delta)}{2} \sum_{i=0}^{k-1} (n - i) \geq \frac{n^2}{4} - \frac{(1 - \delta)}{2} \sum_{i=0}^{k-1} (n - i). \quad (1)$$

Suppose that this procedure continues until $k = \lceil \delta n \rceil < n/2$. Then by (1) and $n > n_0$ we have

$$e(G_k) \geq \frac{(n - k)^2}{4} + \left(\frac{\delta k n}{2} + \frac{\delta k}{4} \right) - \left(\frac{k}{4} + \frac{\delta k^2}{4} \right) > \frac{(n - k)^2}{4} + \frac{\delta k n}{2} - \frac{\delta k^2}{3} > \frac{(n - k)^2}{4} + \frac{\delta^2 (n - k)^2}{3}.$$

By the Erdős-Stone theorem, G_k (and therefore G) has at least $\delta'(n - k)^3$ triangles and by the choice of n_0 , this is greater than $4(n - k)^2 > n^2$ and we are done. Consequently, $k < \delta n$ and we may assume that this procedure stops at graph G_l with $n - l > (1 - \delta)n$ vertices and at least $(n - l)^2/4 + 1$ edges (since the expression in (1) with $k = l$ is always at least this large). Since $\delta < 1/2$, we have $n - l > n_0/2 = n'_0$, and the minimum degree of G_l is at least $(1 - \delta)(n - l)/2$, we may try and apply Theorem 5 to G_l . The inputs of Theorem 5 are α' and ε' . Because $\delta \ll \varepsilon'$ we have

$$b(G_l) \leq b(G) < \frac{\alpha n}{2} < \frac{(\alpha + \varepsilon')(1 - \delta)n}{2} < \frac{\alpha'(n - l)}{2}.$$

Since $\varepsilon', \delta \ll \varepsilon$,

$$(\alpha'(1 - \alpha') - 4\varepsilon')(1 - 2\delta + \delta^2) > (\alpha(1 - \alpha) - 5\varepsilon')(1 - 3\delta) > \alpha(1 - \alpha) - \varepsilon.$$

Therefore Theorem 5 implies that

$$t(G) \geq t(G_l) \geq (\alpha'(1 - \alpha') - 4\varepsilon') \frac{(n - l)^2}{4} > (\alpha(1 - \alpha) - \varepsilon) \frac{n^2}{4}.$$

This completes the proof. \square

Proof of Theorem 5. For notational simplicity, let us replace $\alpha', \varepsilon', n'_0$ in Theorem 5 by α, ε, n_0 . So we suppose that $\alpha \in (1/2, 1)$ and $\varepsilon \in (0, (1-\alpha)/3)$ are given. Let $\delta = \varepsilon^2/50$ and n_0 be sufficiently large for all inequalities needed in the proof and for an application of Lemma 4. Let $n > n_0$.

Suppose for contradiction that G is a graph with n vertices, at least $\lfloor n^2/4 \rfloor + 1$ edges, minimum degree at least $(1 - \delta)n$, $b(G) < \alpha n/2$ and $t(G) \leq (\alpha(1 - \alpha) - 4\varepsilon)n^2/4$. Since $t(G) < n^2$ and $n > n_0$, by Lemma 4 we may remove less than δn^2 edges from G to make it triangle-free. The resulting graph G' has more than $n^2/4 - \delta n^2$ edges, so by Lemma 3, G' has a vertex partition A, B where $e(G'[A]) + e(G'[B]) < \delta n^2$. Now consider a vertex partition of G into X, Y that maximizes the number of X, Y -edges. Since one possibility is A, B , we are guaranteed that the number of X, Y -edges is at least $n^2/4 - \delta n^2$. Moreover, both X and Y have size $(1 \pm 3\sqrt{\delta})n/2$, otherwise we obtain the contradiction

$$\frac{n^2}{4} < e(G) \leq |X||Y| + \delta n^2 < (1 - 3\sqrt{\delta})(1 + 3\sqrt{\delta})\frac{n^2}{4} + \delta n^2 = (1 - 9\delta)\frac{n^2}{4} + \delta n^2 < \frac{n^2}{4}.$$

Now let B be the set of edges of G entirely contained within X or entirely contained within Y , i.e., $B = E(G[X]) \cup E(G[Y])$. Let M be the set of pairs in $X \times Y$ that are not edges of G . Then $E(G) - B \cup M$ is bipartite, so it has at most $n^2/4$ edges. As $e(G) \geq \lfloor n^2/4 \rfloor + 1$, we conclude that

$$|M| < |B| < \delta n^2.$$

In particular, $B \neq \emptyset$. Next, let $M' \subset M$ be the set of those pairs $\{x, y\} \in M$, such that x and y are each incident with at least εn edges of B (of course x and y are on opposite sides of the partition).

Claim. $|M'| < c := \lceil \alpha^2(1 - \alpha)^2/\varepsilon^4 \rceil$.

Proof of Claim. Otherwise, by the König-Hall theorem, there is either a matching or a star of size at least $s = \lfloor \alpha(1 - \alpha)/\varepsilon^2 \rfloor$ in M' . In the case of a matching, each pair $f = uv$ of the matching is incident with $\lceil \varepsilon n \rceil$ edges of B in both X and Y . Consider any set of $\lceil \varepsilon n \rceil$ edges of B incident to $u \in X$. By the choice of X, Y , each vertex has at least as many neighbors on the opposite side of the partition as its own side, hence u has at least εn neighbors in Y . Each edge between these two sets of neighbors of u forms a triangle, and the number of such edges is at least

$$\varepsilon^2 n^2 - |M| > \varepsilon^2 n^2 - \delta n^2 > \frac{\varepsilon^2}{2} n^2.$$

Every two such pairs $uv, u'v'$ in the matching of M' count at most $4n$ common triangles, so by Inclusion/Exclusion, we obtain the contradiction

$$t(G) > 2s\frac{\varepsilon^2}{2}n^2 - \binom{2s}{2}4n > s\varepsilon^2 n^2 - 8s^2 n \geq \alpha(1 - \alpha)n^2 - \varepsilon^2 n^2 - \frac{8}{\varepsilon^4}n > (\alpha(1 - \alpha) - \varepsilon)\frac{n^2}{4}.$$

In the case of a star, the same argument works, since we count triangles starting from a vertex in the part corresponding to the leaf set of this star of M' . Indeed, suppose we have pairs

$xy_1, \dots, xy_s \in M'$. Then for each i , consider $\lceil \varepsilon n \rceil$ edges of B incident with y_i . Since y_i can be moved to the other part, there are at least εn edges of $E(G) - B$ incident with y_i . Now proceed to find many triangles as in the previous case. \square

Form the bipartite graph H with parts B and M , where $e \in B$ is adjacent to $f \in M$ if edge e is incident with the pair f . Since $b(G) \leq \alpha n/2$, every $e \in B$ is adjacent (in H) to at least

$$\min\{|X|, |Y|\} - \frac{\alpha n}{2} > (1 - 3\sqrt{\delta} - \alpha)\frac{n}{2}$$

vertices $f \in M$. Consequently

$$e(H) \geq |B|(1 - 3\sqrt{\delta} - \alpha)\frac{n}{2}.$$

The number of edges of H incident to M' is at most $|M'||B| < c|B|$ so the number of edges in H incident to some pair of $M - M'$ is at least

$$|B|(1 - \alpha - 3\sqrt{\delta})\frac{n}{2} - c|B| > |B|(1 - \alpha + \varepsilon)\frac{n}{2}.$$

Since $|M| < |B|$, we conclude that there is an $f = uv \in M - M'$ that is incident (in G) with at least $(1 - \alpha - \varepsilon)n/2 > \varepsilon n$ distinct $e \in B$. Since $f \notin M'$, we may assume (wlog) that at least

$$(1 - \alpha - \varepsilon)\frac{n}{2} - \varepsilon n > (1 - \alpha - 3\varepsilon)\frac{n}{2}$$

of these edges e lie in X , say they form a star in G with center u and leaf set $L_X = N(u) \cap X$. So we have $|L_X| > (1 - \alpha - 3\varepsilon)n/2$. Let $L_Y = N(u) \cap Y$.

If $|L_Y| < |L_X|$, then we could move u to Y and increase the number of edges between X and Y , thereby contradicting the choice of the partition $X \cup Y$. We therefore have $|L_Y| \geq |L_X|$. As G has minimum degree at least $(1 - \delta)n/2$, we have $d(u) = |L_X| + |L_Y| \geq (1 - \delta)n/2$. Consequently,

$$|L_Y| \geq \max \left\{ |L_X|, \frac{(1 - \delta)n}{2} - |L_X| \right\}.$$

Let $|L_X| = an/2$ and $|L_Y| = bn/2$. The number of edges in G between L_X and L_Y is at least

$$|L_X||L_Y| - |M| \geq ab\frac{n^2}{4} - \delta n^2 = (ab - 4\delta)\frac{n^2}{4},$$

where

$$b \geq a \geq 1 - \alpha - 3\varepsilon \quad \text{and} \quad a + b \geq 1 - \delta.$$

Now $ab - 4\delta$ is minimized by minimizing $a + b$ and then maximizing $b - a$. Since $\varepsilon < (1 - \alpha)/3$, the minimum occurs at

$$a = 1 - \alpha - 3\varepsilon > 0 \quad \text{and} \quad b = 1 - \delta - a = \alpha - \delta + 3\varepsilon$$

where it equals

$$(1 - \alpha - 3\varepsilon)(\alpha - \delta + 3\varepsilon) - 4\delta > \alpha(1 - \alpha) - 4\varepsilon.$$

Since each of these edges gives rise to a unique triangle, we conclude that $t(G) > (\alpha(1 - \alpha) - 4\varepsilon)n^2/4$, a contradiction. \square

Proof of Theorem 2. We use the notation from Theorem 5's proof. Let $\alpha \in (1/3, 1/2)$ be given and choose

$$\varepsilon = \min \left\{ \frac{1}{10}, \frac{1 - 2\alpha}{4} \right\}.$$

Note that $\alpha < 1/2$ implies that $\varepsilon > 0$. Let $\delta = \varepsilon^2/50$ and let β be sufficiently small so that we can apply Lemma 4 with input δ and output β . Our hierarchy of constants is

$$1/n_0 \ll \beta \ll \delta \ll \varepsilon \ll \alpha.$$

We do not need the minimum degree assumption on G . Suppose for contradiction that $b(G) < \alpha n/2$ and $t(G) < \beta n^3$. By Lemma 4 we can make G triangle-free by removing a set of at most δn^2 edges. Now follow the proof of Theorem 5 precisely to obtain the partition X, Y with the same properties and also $|M| < |B| < \delta n^2$. We may also assume the Claim from Theorem 5's proof holds. Then we find a pair $f = uv \in M - M'$ incident with at least $(1 - \alpha - \varepsilon)n/2$ distinct $e \in B$. Again form the sets L_X and L_Y whose vertices are neighbors of $u \in X$. By optimality of the partition, we have

$$|L_Y| \geq |L_X| \geq (1 - \alpha - 3\varepsilon)\frac{n}{2} > \frac{n}{4}.$$

Consider the subgraph K of edges of G between L_X and L_Y . Then

$$e(K) \geq |L_X||L_Y| - |M|$$

so there exists a vertex $w \in L_X$ with

$$d_K(w) \geq \frac{e(K)}{|L_X|} \geq |L_Y| - \frac{|M|}{|L_X|} \geq (1 - \alpha - 3\varepsilon)\frac{n}{2} - \frac{\delta n^2}{n/4} > (1 - \alpha - 4\varepsilon)\frac{n}{2}.$$

Since $\alpha < 1/2$ and $\varepsilon < (1 - 2\alpha)/4$, this is at least $\alpha n/2$. Therefore the edge uw lies in at least $d_K(w) \geq \alpha n/2$ triangles, contradicting the hypothesis $b(G) < \alpha n/2$. \square

4 Concluding Remarks

- We observed that as α decreases from 1 to $1/3$, the number of triangles increases from $\lfloor n/2 \rfloor$ to $\Omega(n^3)$. Theorem 1 shows that for $\alpha < 1$ we always have $t(G) = \Omega(n^2)$, Similarly, as α changes

from $1/2 + o(1)$ to $1/2 - o(1)$, the number of triangles changes from quadratic to cubic in n . There appear to be two phase transitions here, $\alpha = 1$ and $\alpha = 1/2$. It would be very interesting to understand the scaling window in these two ranges, namely, the rate at which $t(G)$ changes from linear to quadratic and from quadratic to cubic. Perhaps this is connected with the behavior of various parameters in the regularity Lemma, since this seems crucial to our argument.

- One could also ask the same questions for graphs with $\lfloor n^2/4 \rfloor + q$ edges for $q > 1$. Results of Erdős [2] and Lovász-Simonovits [5] determine the minimum number of triangles and Bollobás and Nikiforov [1] determine the minimum value of $b(G)$. Theorems 1 and 2 apply to this case, since the hypothesis is simply $e(G) \geq \lfloor n^2/4 \rfloor + 1$. Moreover, when $q = o(n)$, the results are asymptotically sharp, as evidenced by easy modifications of the constructions shown earlier. The situation when $q = \Omega(n)$ appears to be more complicated and our methods do not seem to apply.
- One could consider cliques of larger size and the appropriately defined books (collection of cliques that share an edge). Our proofs appear to be robust enough to address this situation in a similar fashion, in particular, the tools we need (removal lemmas, stability results, results for generalized books) are available. Nevertheless, the technical details would probably be quite complicated, and since the situation for triangles is not yet well understood, we have chosen not to address this.

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