Counting substructures II: triple systems

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Abstract

For various triple systems F, we give tight lower bounds on the number of copies of F in a triple system with a prescribed number of vertices and edges. These are the first such results for hypergraphs, and extend earlier theorems of various authors who proved that there is one copy of F.

A sample result is the following: Füredi-Simonovits [10] and independently Keevash-Sudakov [15] settled an old conjecture of Sós [28] by proving that the maximum number of triples in an n vertex triple system (for n sufficiently large) that contains no copy of the Fano plane is $p(n) = \binom{\lceil n/2 \rceil}{2}\lfloor n/2 \rfloor + \binom{\lfloor n/2 \rfloor}{2}\lceil n/2 \rceil$.

We prove that there is an absolute constant c such that if n is sufficiently large and $1 \le q \le cn^2$, then every n vertex triple system with $p(n) + q$ edges contains at least

$$6q\left(\binom{\lfloor n/2 \rfloor}{4} + (\lceil n/2 \rceil - 3)\binom{\lfloor n/2 \rfloor}{3}\right)$$

copies of the Fano plane. This is sharp for $q \le n/2 - 2$.

Our proofs use the recently proved hypergraph removal lemma and stability results for the corresponding Turán problem.

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1 Introduction

Many mathematical problems enjoy the supersaturation phenomenon which, broadly speaking, says that once we have sufficiently many objects of a particular type to guarantee the existence of a specific configuration, then we find not just one but many copies of this configuration. Perhaps the early examples of this phenomenon came from graph theory. Mantel proved that a graph with n vertices and $\lfloor n^2/4 \rfloor + 1$ edges contains a triangle. Rademacher extended this by showing that there are at least $\lfloor n/2 \rfloor$ copies of a triangle. Subsequently, Erdős [4, 5] proved that if $q < cn$ for some small constant c, then $\lfloor n^2/4 \rfloor + q$ edges guarantees at least $q\lfloor n/2 \rfloor$ triangles. Later Lovász and Simonovits [16] proved that the same statement holds with $c = 1/2$, thus confirming an old conjecture of Erdős. They also proved similar results for complete graphs.

In this paper (the second in a series) we initiate the study of this phenomenon to k-uniform hypergraphs (k-graphs for short). In the first paper of this series [18], we had extended the results of Erdős and Lovász-Simonovits in two ways. First, we proved such statements for the broader class of color critical graphs. Second, we showed that all the copies of the required subgraph were incident to a small number of edges or vertices. For example, in a graph with n vertices and $\lfloor n^2/4 \rfloor + q$ edges, [4, 5, 16] do not give information about how the $q\lfloor n/2 \rfloor$ triangles are distributed. In [18], we proved that as long as $q = o(n)$ there are $(1 - o(1))qn/2$ triangles incident with at most q vertices.

The main new tool we have at our disposal is the recently proved hypergraph removal lemma, which is a consequence of the hypergraph regularity lemma (see Gowers [11], Nagle-Rödl-Schacht [23], Rödl-Skokan [26], Tao [29]). The novelty in this project is the use of the removal lemma to count substructures in hypergraphs rather precisely.

Theorem 1. (Hypergraph Removal Lemma [11, 23, 26, 29]) *Fix $k \geq 2$ and a k-graph F with f vertices. Suppose that an n vertex k-graph $\mathcal{H}$ has at most $o(n^f)$ copies of F. Then there is a set of edges in $\mathcal{H}$ of size $o(n^k)$ whose removal from $\mathcal{H}$ results in a k-graph with no copies of F.*

Given a k-graph F, let $\mathrm{ex}(n, F)$, the Turán number of F, be the maximum number of edges in an n vertex k-graph with no copy of F. For $k > 2$, determining the Turán number is a very difficult problem, and there are only sporadic results. Many of these were obtained recently by using the so-called stability approach first introduced by Erdős and Simonovits [27] in the late 1960's. Here we take this project one step further by giving asymptotically sharp results on the number of copies of a k-graph F in a k-graph with n vertices and $\mathrm{ex}(n, F) + q$ edges. In two cases we are able to count the exact minimum number of copies even though this number is quite complicated (see the abstract).

In essentially all cases where $\mathrm{ex}(n, F)$ is known (when $k = 3$), it turns out that one is guaranteed many copies of F as long as there are $\mathrm{ex}(n, F)+1$ edges, so we extend all previous results that determine $\mathrm{ex}(n, F)$. It is somewhat surprising that although determining $\mathrm{ex}(n, F)$ for these hypergraphs F is quite difficult (in some cases they were decades old conjectures that were only recently settled), we are able to count quite precisely the number of copies of F as long as the number of extra edges q is not too large. Typically we can allow $q = o(n^2)$ for the 3-graphs we consider.

Here we should also mention the relationship between this project and recent work of Nikiforov [22] and Razborov [25] that gives asymptotically sharp estimates on the minimum number of triangles in a graph with n vertices and $\lfloor n^2/4 \rfloor + q$ edges, where $q = \Omega(n^2)$. There are at present no such results for k-graphs for $k > 2$, and little hope of achieving them. Moreover, even if such results were to be proved, they would apply only when $q = \Omega(n^k)$, so the results of the type [22, 25] will not overlap with ours.

Our proofs all have the following basic structure: Suppose we are given $\mathcal{H}$ with sufficiently many edges and we wish to find many copies of F in $\mathcal{H}$. First we observe that if the number of copies of F is very large, then we already have the bound sought. Consequently, we can use the hypergraph removal lemma to delete a small proportion of edges of $\mathcal{H}$ so that the resulting triple system has no copies of F. Next we use the stability results that guarantee the approximate structure of $\mathcal{H}$. At this point the techniques depend highly on the particular structure of F and of $\mathcal{H}$. The technical details are more involved than for the usual Turán problem, since it is not enough to find just one copy of F. At the end of the analysis, we are able to describe quite precisely how the copies of F are distributed within $\mathcal{H}$.

We illustrate our approach on four excluded hypergraph problems, each of which has been studied quite a lot.

Definition 2. *Let F have the property that for sufficiently large n, there is a unique (up to isomorphism) 3-graph $\mathcal{H}(n)$ with $ex(n, F)$ edges. Let $c(n, F)$ be the minimum number of copies of F in the 3-graph obtained from $\mathcal{H}(n)$ by adding an edge, where the minimum is taken over all possible ways to add an edge.*

Our theorems all say that if $\mathcal{H}$ is an n vertex 3-graph with $\mathrm{ex}(n, F) + q$ edges, then the number of copies of F in $\mathcal{H}$ is essentially at least $qc(n, F)$. In the next subsections we will state our results precisely.

There remain a few more exact 4-graph results in the literature where we could possibly use this approach for the counting problem. We will give the corresponding counting results for all of these problems in a forthcoming paper [19], the third in this series.

Notation: We associate a hypergraph with its edge set. The number of edges in a hypergraph $\mathcal{H}$ is $|\mathcal{H}|$. Given hypergraphs $F, \mathcal{H}$ (F has f vertices), a copy of F in $\mathcal{H}$ is a subset of f vertices and $|F|$ edges of $\mathcal{H}$ such that the subhypergraph formed by this set of vertices and edges is isomorphic to F. In other words, if we denote $Aut(F)$ to be the number of automorphisms of F, then the number of copies of F in $\mathcal{H}$ is the number of edge-preserving injections from $V(F)$ to $V(\mathcal{H})$ divided by $Aut(F)$. For a set S of vertices, define $d_{\mathcal{H}}(S)$ to be the number of edges of $\mathcal{H}$ containing S. If $S = \{v\}$, we simply write $d_{\mathcal{H}}(v)$. We will omit floor and ceiling symbols whenever they are not crucial, so that the presentation is clearer.

1.1 Fano plane

Let $\mathbf{F}$ be the projective plane of order two over the finite field of order two. An explicit description of $\mathbf{F}$ is $\{124, 235, 346, 457, 561, 672, 713\}$, obtained from the difference set $\{1, 2, 4\}$ over Z_7. It is well known that $\mathbf{F}$ is not 2-colorable, hence it cannot be a subgraph of any 2-colorable 3-graph. Say that a 3-graph $\mathcal{H}$ is bipartite (or 2-colorable) if it has a vertex partition $A \cup B$ such that every edge intersects both parts. Let $P^3(n)$ be the bipartite 3-graph with the maximum number of edges. Note that

$$p^3(n) := |P^3(n)| = \max_a \left\{ \binom{a}{2}(n-a) + \binom{n-a}{2} a \right\} = (3/4 + o(1))\binom{n}{3}$$

is uniquely achieved by choosing $a \in \{\lfloor n/2 \rfloor, \lceil n/2 \rceil\}$.

Sós [28] conjectured, and Keevash-Sudakov [15] and Füredi-Simonovits [10] independently proved that among all n vertex 3-graphs (n sufficiently large) containing no copy of $\mathbf{F}$, the unique one with the maximum number of edges is $P^3(n)$. Thus $c(n, \mathbf{F})$ is defined and in fact

$$c(n, \mathbf{F}) := 6\left(\binom{\lfloor n/2 \rfloor}{4} + (\lceil n/2 \rceil - 3)\binom{\lfloor n/2 \rfloor}{3}\right) = (20 + o(1))(n/4)^4.$$

This is achieved by adding an edge to the part of size $\lceil n/2 \rceil$. Indeed, if we add a triple 123 to this part, then one way to make a copy of $\mathbf{F}$ is to take four points a, b, c, d from the other part, partition the six pairs among $\{a,b,c,d\}$ into three perfect matchings m_1, m_2, m_3, and use the edges $\{i\} \cup p$ where $p \in m_i$, for each i to form a copy of $\mathbf{F}$. There are $\binom{\lfloor n/2 \rfloor}{4}$ ways to pick a,b,c,d and for each such choice there are six ways to choose m_1, m_2, m_3. The only other way to form a copy of $\mathbf{F}$ using 123 is to pick four points a,b,c,d with a in the same part as 1 and b,c,d in the other part. Then proceeding as before, we obtain $6(\lceil n/2 \rceil - 3)\binom{\lfloor n/2 \rfloor}{3}$ copies of $\mathbf{F}$. Altogether we obtain $c(n,\mathbf{F})$ copies.

Our first result shows that a 3-graph with $p^3(n) + q$ edges has at least as many copies of $\mathbf{F}$ as a 3-graph obtained from $P^3(n)$ by adding q edges in an optimal way. The precise number we can add is

$$q(n,\mathbf{F}) = \begin{cases} n & \text{if } n \text{ is even and } n/2 \equiv 0 \pmod 4 \\ n-2 & \text{if } n \text{ is even and } n/2 \equiv 1 \pmod 4 \\ n-4 & \text{if } n \text{ is even and } n/2 \equiv 2,3 \pmod 4 \\ \lceil n/2 \rceil & \text{if } n \text{ is odd and } \lceil n/2 \rceil \equiv 0 \pmod 4 \\ \lceil n/2 \rceil - 1 & \text{if } n \text{ is odd and } \lceil n/2 \rceil \equiv 1 \pmod 4 \\ \lceil n/2 \rceil - 2 & \text{if } n \text{ is odd and } \lceil n/2 \rceil \equiv 2,3 \pmod 4. \end{cases}$$

Theorem 3. *There exists an absolute constant $\varepsilon > 0$ such that if n is sufficiently large and $1 \le q \le \varepsilon n^2$, then the following holds:*

- *Every n vertex 3-graph with $p^3(n) + q$ edges contains at least $qc(n,\mathbf{F})$ copies of $\mathbf{F}$. This is sharp for all $q \le q(n,\mathbf{F})$.*
- *If $q > q(n,\mathbf{F})$, then every n vertex 3-graph with $p^3(n)+q$ edges contains at least $qc(n,\mathbf{F})+1$ copies of $\mathbf{F}$.*

Remark. For $q > q(n,\mathbf{F})$, our proof actually gives at least $qc(n,\mathbf{F}) + 2\binom{\lfloor n/2 \rfloor}{2}$ copies of $\mathbf{F}$.

To see that Theorem 3 is tight for $q \le q(n,\mathbf{F})$ observe that we may add q edges to $P^3(n)$ with every two edges sharing zero or two points. If n is even, we do this by adding to each part of $P^3(n)$ the maximum number of edge that pairwise share zero or two points. This is achieved by adding disjoint copies of K_4^3, the complete 3-graph on four points, or collections of edges that pairwise share the same two points. If n is odd, then we add edges only to the larger part. Each added edge lies in exactly $c(n,\mathbf{F})$ copies of $\mathbf{F}$ and no two added edges lie in any copy of $\mathbf{F}$. So the total number of copies of $\mathbf{F}$ is exactly $qc(n,\mathbf{F})$.

Theorem 3 is asymptotically sharp in a much larger range of q. In particular, we have the following.

Proposition 4. *For every $\varepsilon > 0$ there exists $\delta > 0$ and n_0 such that the following holds for all $n > n_0$ and $q < \delta n^2$. There is an n vertex 3-graph with $p^3(n) + q$ edges and at most $(1+\varepsilon)qc(n, \mathbf{F})$ copies of $\mathbf{F}$.*

1.2 Cancellative triple-systems

Say that a 3-graph is cancellative if whenever $A \cup B = A \cup C$ we have $B = C$. An equivalent definition is to simply say that the 3-graph does not contain a copy of two particular 3-graphs: $F_5 = \{123, 124, 345\}$ and $K_4^{3-} = \{123, 124, 234\}$. Write

$$t^3(n) = \left\lfloor \frac{n}{3} \right\rfloor \left\lfloor \frac{n+1}{3} \right\rfloor \left\lfloor \frac{n+2}{3} \right\rfloor$$

for the number of edges in $T^3(n)$, the complete 3-partite 3-graph with the maximum number of edges. It is easy to see that $T^3(n)$ is cancellative.

Katona conjectured, and Bollobás [2] proved, that the maximum number of edges in an n vertex cancellative 3-graph is $t^3(n)$, and equality holds only for $T^3(n)$. Later Frankl and Füredi [7] refined this by proving the same result (for $n > 3000$) even if we just forbid F_5. Recently, Keevash and the author [13] gave a new proof of the Frankl-Füredi result while reducing the smallest n value to 33.

It is easy to see that $c(n, F_5) = 3(n/3)^2 + \Theta(n)$ and this is achieved by adding a triple to $T^3(n)$ with two points in the largest part. In fact, even if we add a triple within one of the parts we get almost the same number of copies of F_5. Our second result shows that this is optimal, even when we are allowed to add as many as $o(n)$ edges.

Theorem 5. *For every $\varepsilon > 0$ there exists $\delta > 0$ and n_0 such that the following holds for $n > n_0$. Let $\mathcal{H}$ be a 3-graph with $t^3(n) + q$ edges where $q < \delta n$. Then the number of copies of F_5 in $\mathcal{H}$ is at least $q(1-\varepsilon)c(n, F_5)$. This is asymptotically sharp for $1 \le q < \delta n$. Moreover, if the number of copies is less than δn^3, then there is a collection of q distinct edges that each lie in $(1-\varepsilon)c(n, F_5)$ copies of F_5 with no two of these edges accounting for the same copy of F_5.*

1.3 Independent neighborhoods

The neighborhood of a $(k-1)$-set S of vertices in a k-graph is the set of vertices v whose union with S forms an edge. A set is independent if it contains no edge. We can rephrase Mantel's theorem as follows: the maximum number of edges in a 2-graph with independent neighborhoods is $\lfloor n^2/4 \rfloor$. This formulation can be generalized to $k > 2$ and there has been quite a lot of recent activity on this question. We focus here on $k = 3$, and observe that a 3-graph has independent neighborhoods if and only if it contains no copy of $B_5 = \{123, 124, 125, 345\}$. A 3-graph $\mathcal{H}$ has a $(2,1)$-partition if it has a vertex partition $A \cup B$ such that $|e \cap A| = 2$ for all $e \in \mathcal{H}$. Let $B^3(n)$ be the 3-graph with the maximum number of edges among all those that have n vertices and a $(2,1)$-partition. Note that

$$b^3(n) := |B^3(n)| = \max_a \binom{a}{2}(n-a) = (4/9 + o(1))\binom{n}{3}$$

is achieved by choosing $a = \lfloor 2n/3 \rfloor$ or $a = \lceil 2n/3 \rceil$.

The author and Rödl [21] conjectured, and Füredi, Pikhurko, and Simonovits [9] proved, that among all n vertex 3-graphs (n sufficiently large) containing no copy of B_5, the unique one with the maximum number of edges is $B^3(n)$.

It is easy to see that $c(n, B_5) = 2(n/3)^2 + \Theta(n)$ and this is achieved by adding a triple to $T^3(n)$ contained in the larger part. In fact, even if we add a triple within the smaller part we get almost the same number of copies of B_5. Our third result shows that this is optimal, even when we are allowed to add as many as $o(n^2)$ edges.

Theorem 6. *For every $\varepsilon > 0$ there exists $\delta > 0$ and n_0 such that the following holds for $n > n_0$. Let $\mathcal{H}$ be a 3-graph with $b^3(n) + q$ edges where $q < \delta n^2$. Then the number of copies of B_5 in $\mathcal{H}$ is at least $q(1-\varepsilon)c(n, B_5)$. This is asymptotically sharp for $1 \le q < \delta n^2$. Moreover, if the number of copies is less than δn^4, then there is a collection of q distinct edges that each lie in $(1-\varepsilon)c(n, B_5)$ copies of B_5 with no two of these edges accounting for the same copy of B_5.*

1.4 Expanded Cliques

Let L_r be the 3-graph obtained from the complete graph K_r by enlarging each edge with a new vertex. These new vertices are distinct for each edge, so L_r has $l_r = r + \binom{r}{2} = \binom{r+1}{2}$

vertices and $\binom{r}{2}$ edges. Write $T_r^3(n)$ for the complete r-partite 3-graph with the maximum number of edges. So $T_r^3(n)$ has vertex partition $V_1 \cup \cdots \cup V_r$, where $n_i := |V_i| = \lfloor (n+i-1)/r \rfloor$, and all triples with at most one point in each V_i. Define

$$t_r^3(n) := |T_r^3(n)| = \sum_{S \in \binom{[r]}{3}} \prod_{i \in S} n_i.$$

Every set of $r+1$ vertices in $T_r^3(n)$ contains two vertices in the same part, and these two vertices lie in no edge. Consequently, $L_{r+1} \not\subset T_r^3(n)$.

The author [17] conjectured, and Pikhurko [24] proved, that among all n vertex 3-graphs containing no copy of L_{r+1} ($r \geq 3$ fixed, n sufficiently large), the unique one with the maximum number of edges is $T_r^3(n)$. Thus $c(n, L_{r+1})$ is defined and in fact

$$c(n, L_{r+1}) = (1+o(1)) \left(\left(1 - \frac{2}{r}\right) n \right)^{\binom{r+1}{2}-1} \times \left(\frac{n}{r}\right)^{r-1} = \Theta(n^{l_{r+1}-3})$$

and this is achieved by adding a triple with exactly two points in a largest part. Our final results shows that this is asymptotically optimal, even when we are allowed to add as many as $o(n^2)$ edges.

Theorem 7. (Asymptotic Counting) *Fix $r \geq 3$. For every $\varepsilon > 0$ there exists $\delta > 0$ and n_0 such that the following holds for $n > n_0$. Let $\mathcal{H}$ be a 3-graph with $t_r^3(n) + q$ edges where $q < \delta n^2$. Then the number of copies of L_{r+1} in $\mathcal{H}$ is at least $q(1-\varepsilon)c(n, L_{r+1})$. The expression q is sharp for $1 \leq q < \delta n^2$. Moreover, if the number of copies is less than $\delta n^{\binom{r+1}{2}-1}$, then there is a collection of q distinct edges that each lie in $(1-\varepsilon)c(n, L_{r+1})$ copies of L_{r+1} with no two of these edges accounting for the same copy of L_{r+1}.*

Our next result improves the asymptotic counting result above to an exact result, with a more restricted range for q.

Theorem 8. (Exact Counting) *Fix $r \geq 3$, $q > 0$ and let n be sufficiently large. Every n vertex triple system with $t_r^3(n) + q$ edges contains at least $qc(n, L_{r+1})$ copies of L_{r+1}.*

Theorem 8 is clearly tight, as we may add an appropriate set of q pairwise disjoint edges to $T_r^3(n)$ such that each edge lies in exactly $c(n, L_{r+1})$ copies of L_{r+1}.

Throughout the paper we will frequently use the notation $\delta \ll \varepsilon$, which means that δ, and any function of δ (that tends to zero with δ) used in a proof is smaller than any function of ε used in the proof. It is pretty difficult to write the precise dependence between δ and ε as one of the constraints comes from an application of the removal lemma.

2 Counting Fano's

In this section we will prove Theorem 3 and Proposition 4. We need some lemmas about binomial coefficients.

Lemma 9. *Let $x, y, t > 0$ be integers with $x + y = n$, $t < n^2$ and $s = \lceil \sqrt{2t/(n-2)} \rceil$. Suppose that n is sufficiently large and*

$$\binom{x}{2} y + \binom{y}{2} x \geq p^3(n) - t.$$

Then $\lfloor n/2 \rfloor - s \leq x \leq \lceil n/2 \rceil + s$ and if $t < (n-2)/2$, then $\lfloor n/2 \rfloor - s < x < \lceil n/2 \rceil + s$.

Proof. Suppose for contradiction that $x > \lceil n/2 \rceil + s$ (the upper bound on t ensures that $s < 2\sqrt{n}$ and hence $x < 3n/4$). Write

$$f(x) = \binom{x}{2}(n-x) + \binom{n-x}{2} x = \frac{1}{2}(n-2)x(n-x).$$

Note that $p^3(n) = f(\lfloor n/2 \rfloor) = f(\lceil n/2 \rceil)$. Our goal therefore is to obtain the contradiction $f(x) < f(\lceil n/2 \rceil) - t$. Observe that

$$f(a+1) = f(a) - \frac{1}{2}(n-2)(2a+1-n).$$

Applying this repeatedly beginning with $a = \lceil n/2 \rceil$ we obtain

$$\begin{aligned} f(x) < f(\lceil n/2 \rceil + s) &= f(\lceil n/2 \rceil) - \frac{1}{2}(n-2) \sum_{a=\lceil n/2 \rceil}^{\lceil n/2 \rceil + s - 1} (2a+1-n) \\ &= f(\lceil n/2 \rceil) - \frac{1}{2} s(n-2)(s + 2\lceil n/2 \rceil - n). \end{aligned}$$

The choice of s gives

$$\frac{1}{2} s(n-2)(s + 2\lceil n/2 \rceil - n) \geq \frac{1}{2} s^2 (n-2) \geq t \tag{1}$$

and therefore $f(x) < f(\lceil n/2 \rceil) - t$. We conclude that $x \leq \lceil n/2 \rceil + s$. Repeating this argument with x replaced by y gives $y \leq \lceil n/2 \rceil + s$ and hence $x \geq \lfloor n/2 \rfloor - s$.

If $t < (n-2)/2$, and $x \geq \lceil n/2 \rceil + s$, then we only have $f(x) \leq f(\lceil n/2 \rceil + s)$. However the last inequality in (1) is strict (since $s \geq 1$) and we again get the same contradiction. Therefore $x < \lceil n/2 \rceil + s$ and by a similar argument, $x > \lfloor n/2 \rfloor - s$. □

Recall that

$$c(n,\mathbf{F}) := 6\left(\binom{\lfloor n/2\rfloor}{4} + (\lceil n/2\rceil - 3)\binom{\lfloor n/2\rfloor}{3}\right).$$

Lemma 10. *Let x, y, s be positive integers with $x + y = n$ sufficiently large, $\lfloor n/2\rfloor - s \leq x \leq \lceil n/2\rceil + s$ and $s < n/10$. Then*

$$6\binom{y}{4} + 6(x-3)\binom{y}{3} \geq c(n,\mathbf{F}) - (s+3)n^3.$$

Proof. Define $f(y) = 6\binom{y}{4} + 6(n-y-3)\binom{y}{3}$ and $a = \lfloor n/2\rfloor$. Then $c(n,\mathbf{F}) = f(a)$. We first observe that $f(y)$ is increasing for $1 < y < n-4$. Indeed,

$$f(y+1) - f(y) = 6\left(\binom{y}{3} + (n-y-4)\binom{y+1}{3} - (n-y-3)\binom{y}{3}\right) = 6\binom{y}{2}(n-y-4)$$

and the condition on y shows that this is positive. The condition $\lfloor n/2\rfloor - s \leq x \leq \lceil n/2\rceil + s$ implies that $1 < a - s \leq y \leq a + 1 + s < n - 4$ and so $f(y) \geq f(a-s)$. Therefore

$$\begin{aligned}
c(n,\mathbf{F}) - f(y) &\leq f(a) - f(a-s)\\
&= 6\left(\binom{a}{4} + (n-a-3)\binom{a}{3} - \binom{a-s}{4} - (n-a+s-3)\binom{a-s}{3}\right)\\
&\leq 6\left(\frac{a^4}{4!} - \frac{(a-s-3)^4}{4!} + (n-a-3)\frac{a^3}{6} - (n-a+s-3)\frac{(a-s-2)^3}{6}\right)\\
&\leq 6\left(\frac{a^4}{4!} - \frac{a^4 - 4a^3s - 12a^3}{4!} + \frac{(n-a-3)}{6}(a^3 - (a^3 - 3a^2s - 6a^2)\right) \qquad (2)\\
&= 6\left(\frac{4a^3s + 12a^3}{4!} + \frac{(n-a-3)(3a^2s + 6a^2)}{6}\right)\\
&< a^3s + 3a^3 + 3a^2sn + 6a^2n\\
&< (s+3)n^3.
\end{aligned}$$

Note that (2) follows from the inequalities $(a-b)^4 > a^4 - 4a^3b$ and $(a-b)^3 > a^3 - 3a^2b$ which hold for $0 < b < 3a/2$; since $s < n/10$ we have $0 < s + 2 < 3a/2 - 1$. This completes the proof of the Lemma. □

We will need the following stability result proved independently by Keevash-Sudakov [15] and Füredi-Simonovits [10].

Theorem 11. (F Stability [10, 15]) *Let $\mathcal{H}$ be a 3-graph with n vertices and $p^3(n) - o(n^3)$ edges that contains no copy of $\mathbf{F}$. Then there is a partition of the vertex set of $\mathcal{H}$ into $X \cup Y$ so that the number of edges that are within X or within Y is $o(n^3)$. In other words, $\mathcal{H}$ can be obtained from $P^3(n)$ by adding and deleting a set of $o(n^3)$ edges.*

Remark. The $o(1)$ notation above should be interpreted in the obvious way, namely $\forall\beta, \exists\gamma, n_0$ such that if $n > n_0$ and $|\mathcal{H}| > p^3(n) - \gamma n^3$, then $\mathcal{H} = P^3(n) \pm \beta n^3$ edges. We will not explicitly mention the role of β, γ when we use the result, but it should be obvious from the context. A similar comment applies for all applications of Theorem 1.

Proof of Theorem 3. Let $0 < \delta \ll \varepsilon \ll 1$. Write $o_\delta(1)$ for any function that approaches zero as δ approaches zero and moreover, $o_\delta(1) \ll \varepsilon$. We emphasize that ε is an absolute constant. Let n be sufficiently large and let $\mathcal{H}$ be an n vertex 3-graph with $p^3(n) + q$ edges with $q < \varepsilon n^2$. Write $\#\mathbf{F}$ for the number of copies of $\mathbf{F}$ in $\mathcal{H}$.

If $\#\mathbf{F} \geq n^6$, then since $c(n, \mathbf{F}) < n^4$, we have $\#\mathbf{F} > \varepsilon n^2 c(n, \mathbf{F}) \geq qc(n, \mathbf{F})$ and we are done so assume that $\#\mathbf{F} < n^6 = (1/n)n^7$. Since n is sufficiently large, by the Removal lemma there is a set of at most δn^3 edges of $\mathcal{H}$ whose removal results in a 3-graph $\mathcal{H}'$ with no copies of $\mathbf{F}$. Since $|\mathcal{H}'| > p^3(n) - \delta n^3$, by Theorem 11, we conclude that there is a bipartition of $\mathcal{H}'$ (and also of $\mathcal{H}$) such that the number of edges contained entirely within a part is $o_\delta(n^3)$. Now pick a bipartition $X \cup Y$ of $\mathcal{H}$ that maximizes $e(X, Y)$, the number of edges that intersect both parts. We know that $e(X, Y) \geq p^3(n) - o_\delta(n^3)$, and an easy calculation also shows that each of X, Y has size $n/2 \pm o_\delta(n)$.

Let B be the set of edges of $\mathcal{H}$ that lie entirely within X or entirely within Y and let $G = \mathcal{H} - B$. Let M be the set of triples which intersect both parts that are not edges of $\mathcal{H}$. Then $G \cup M$ is bipartite so it has at most $p^3(n)$ triples. Consequently,

$$q + |M| \leq |B| \leq o_\delta(n^3).$$

Also, $|\mathcal{H}| = |G| + |B|$ so we may suppose that $|G| = p^3(n) - t$ and $|B| = q + t$ for some $t \geq 0$. For an edge $e \in B$, let $\mathbf{F}(e)$ be the number of copies of $\mathbf{F}$ in $\mathcal{H}$ containing the unique edge e from B.

If $t = 0$, then $G \cong P^3(n)$ and $\mathbf{F}(e) \geq c(n, \mathbf{F})$ for every $e \in B$ (by definition of $c(n, \mathbf{F})$) so we immediately obtain $\#\mathbf{F} \geq qc(n, \mathbf{F})$. If $q > q(n, \mathbf{F})$ and $\mathbf{F}(e) = c(n, \mathbf{F})$ for every $e \in B$, then there are two edges $e, e' \in B$ such that $|e \cap e'| = 1$. To see this when n is even, observe that if no two such edges exist, then every two edges of B within X intersect in zero or two points, and the same holds for the edges of B within Y. The maximum number of edges that one can add to $P^3(n)$ with this property is $q(n, \mathbf{F})$, as every component is either a subset of K_4^3 or a sunflower with core of size two. For n odd we can only have edges in the larger part and again the same argument applies.

We deduce that the number of copies of $\mathbf{F}$ containing e or e' is at least $\mathbf{F}(e)+\mathbf{F}(e')+\mathbf{F}(e,e')$ where $\mathbf{F}(e,e')$ is the number of copies of $\mathbf{F}$ in $\mathcal{H}$ containing both e and e'. It is easy to see that $\mathbf{F}(e,e') \geq 1$ (in fact, we have $\mathbf{F}(e,e') \geq 2\binom{\lfloor n/2 \rfloor}{2}$).

We may therefore assume that $t \geq 1$ and we will now show that $\#\mathbf{F} > qc(n, \mathbf{F})$. Partition $B = B_1 \cup B_2$, where

$$B_1 = \{e \in B : \mathbf{F}(e) > (1-\varepsilon)c(n, \mathbf{F})\}.$$

A potential copy of $\mathbf{F}$ is a copy of $\mathbf{F}$ in $G \cup M \cup B$ that uses exactly one edge of B.

Claim 1. $|B_1| \geq (1-\varepsilon)|B|$

Proof of Claim. Suppose to the contrary that $|B_2| \geq \varepsilon|B|$. Pick $e = uvw \in B_2$. Write $B_2 = B_{XXX} \cup B_{YYY}$, where the subscripts have the obvious meaning. Assume by symmetry that $e \in B_{XXX}$. For each $Y' = \{y_1, \ldots, y_4\} \in \binom{Y}{4}$, we can form a copy of $\mathbf{F}$ as follows: Partition the six pairs of Y' into three perfect matchings $L_u = \{e_u, e'_u\}, L_v = \{e_v, e'_v\}, L_w = \{e_w, e'_w\}$ and for each $x \in e$, add the two triples $x \cup e_x$ and $x \cup e'_x$. There are six ways to choose the matchings L_u, L_v, L_w, so each choice of Y' gives six potential copies of $\mathbf{F}$ containing e. Altogether we obtain $6\binom{|Y|}{4}$ potential copies of $\mathbf{F}$. The only other way to form a copy of $\mathbf{F}$ using e is to pick four points a, b, c, d with $a \in X - e$ and $\{b, c, d\} \in \binom{Y}{3}$. Then proceeding as before, we obtain $6(|X|-3)\binom{|Y|}{3}$ copies of $\mathbf{F}$. This gives a total of $(1-o_\delta(1))c(n, \mathbf{F})$ potential copies of $\mathbf{F}$ containing e. At least $(\varepsilon/2)c(n, \mathbf{F})$ of these potential copies of $\mathbf{F}$ have a triple from M, for otherwise

$$\mathbf{F}(e) \geq (1 - o_\delta(1) - \varepsilon/2)c(n, \mathbf{F}) > (1-\varepsilon)c(n, \mathbf{F})$$

which contradicts the definition of B_2. The triple from M referenced above lies in at most $2(|X||Y| + \binom{|Y|}{2}) < n^2$ copies of $\mathbf{F}$, so the number of triples in M counted here is at least

$$\frac{(\varepsilon/2)c(n, \mathbf{F})}{n^2} > (\varepsilon/30)n^2.$$

At least a third of these triples from M are incident with the same vertex of e, so we conclude that there exists $x \in e$ such that $d_M(x) > (\varepsilon/100)n^2$. Let $V = X \cup Y$ and let

$$A = \{v \in V : d_M(v) > (\varepsilon/100)n^2\}.$$

We have argued above that every $e \in B_2$ has a vertex in A. Consequently,

$$3\sum_{v\in A} d_{B_2}(v) \geq 3|B_2| \geq 3\varepsilon|B| > 3\varepsilon|M| \geq \varepsilon \sum_{v\in A} d_M(v) > \varepsilon|A|(\varepsilon/100)n^2,$$

and there exists a vertex $u \in A$ such that $d_{B_2}(u) \geq (\varepsilon^2/300)n^2$. Assume wlog that $u \in X$ so that $d_{B_{XXX}}(u) \geq (\varepsilon^2/300)n^2$.

Let $\mathcal{H}_{XYY}$ be the set of edges in $\mathcal{H}$ with exactly one point in X. We may assume that $d_{H_{XYY}}(u) \geq d_{B_{XXX}}(u)$, for otherwise we may move u to Y and increase $e(X,Y)$, thereby contradicting the choice of X, Y. Consider

$$e = uvw, \quad f = uy_1y_2, \quad f' = uy_1'y_2',$$

with $e \in B_{XXX}$ and $f, f' \in \mathcal{H}_{XYY}$, $f \cap f' = \{u\}$. The number of choices of $(e, \{f, f'\})$ is at least

$$d_{B_{XXX}}(u) \times \left(\binom{d_{\mathcal{H}_{XYY}}(u)}{2} - n^3\right) > \varepsilon_1 n^6$$

where $\varepsilon_1 = \varepsilon^6/10^{10}$. If for at least half of the choices of $(e, \{f, f'\})$, these three edges span at least one copy of $\mathbf{F}$, then $\#\mathbf{F} > (\varepsilon_1/2)n^6 > qc(n, \mathbf{F})$, a contradiction. So for at least half of the choices of $(e, \{f, f'\})$ above, $e \cup f \cup f'$ do not span a copy of $\mathbf{F}$. This implies that at least one of the triples $xyy' \in M$ where $x \in e - \{u\}, y \in f - \{u\}, y' \in f' - \{u\}$. Since each such triple of M is counted at most $|X||Y|^2 < n^3$ times, we obtain the contradiction $(\varepsilon_1/2)n^6/n^3 < |M| = o_\delta(n^3)$. This concludes the proof of the Claim. $\square$

If $t \geq 4\varepsilon q$, then counting copies of $\mathbf{F}$ from edges of B_1 and using Claim 1 we get

$$\begin{aligned}\#\mathbf{F} \geq \sum_{e\in B_1}(1-\varepsilon)c(n,\mathbf{F}) &\geq |B_1|(1-\varepsilon)c(n,\mathbf{F}) \\ &\geq (1-\varepsilon)^2|B|c(n,\mathbf{F}) \\ &> (1-2\varepsilon)(q+t)c(n,\mathbf{F}) \\ &\geq (q+2\varepsilon q - 8\varepsilon^2 q)c(n,\mathbf{F}) \geq qc(n,\mathbf{F})\end{aligned}$$

and we are done. So we may assume that $t < 4\varepsilon q < 4\varepsilon^2 n^2$. Let $x = |X|, y = |Y|$ and $s = \left\lceil \sqrt{2t/(n-2)} \right\rceil$.

Claim 2. $\lfloor n/2 \rfloor - s \le x \le \lceil n/2 \rceil + s$ and if $t < (n-2)/2$, then $\lfloor n/2 \rfloor - s < x < \lceil n/2 \rceil + s$.

Proof of Claim. We know that

$$p^3(n) - t = |G| \le \binom{x}{2} y + \binom{y}{2} x.$$

Now the Claim follows immediately from Lemma 9.

Observe that $|M| \le t$ for otherwise $|G \cup M| > p^3(n)$ which is impossible. Pick $e \in B$ and assume wlog that $e \subset X$. Since $t > 0$, we have $1 \le s \le \sqrt{2t/(n-2)} + 1 < n/10$. The number of potential copies of $\mathbf{F}$ containing e, denoted $pot\mathbf{F}(e)$, is $6\binom{y}{4} + 6(x-3)\binom{y}{3}$. Now Claim 2, Lemma 10 and $s \ge 1$ imply that

$$pot\mathbf{F}(e) \ge c(n, \mathbf{F}) - (s+3)n^3 \ge c(n, \mathbf{F}) - 4sn^3.$$

Not all of these copies of $\mathbf{F}$ are in $\mathcal{H}$, in fact, a triple from M lies in at most $2n^2$ potential copies counted above (we pick either two more vertices in Y or one in each of Y and X, and there are two ways to complete a potential copy of $\mathbf{F}$ containing e). We conclude that

$$\mathbf{F}(e) \ge pot\mathbf{F}(e) - 2n^2|M| \ge c(n, \mathbf{F}) - 4sn^3 - 2n^2|M| \ge c(n, \mathbf{F}) - 4sn^3 - 2tn^2. \qquad (3)$$

Suppose first that $t < (n-2)/2$. Then Claim 2 gives $\lfloor n/2 \rfloor - s < x < \lceil n/2 \rceil + s$. Since $s = 1$ and x is an integer, $|x - n/2| < 1$. The definition of $c(n, \mathbf{F})$ now yields

$$pot\mathbf{F}(e) \ge \min \left\{ 6\binom{y}{4} + 6(x-3)\binom{y}{3} : x \in \{\lfloor n/2 \rfloor, \lceil n/2 \rceil\} \right\} \ge c(n, \mathbf{F}).$$

Consequently, we can refine the bound in (3) to

$$\mathbf{F}(e) \ge c(n, \mathbf{F}) - 2tn^2.$$

Altogether,

$$\#\mathbf{F} \ge \sum_{e \in B} \mathbf{F}(e) \ge (q+t)(c(n, \mathbf{F}) - 2tn^2) = qc(n, \mathbf{F}) + tc(n, \mathbf{F}) - 2qtn^2 - 2t^2n^2.$$

Let us recall that $q \le \varepsilon n^2$ and $0 < t < 4\varepsilon q$. Then $2qtn^2 < 2\varepsilon tn^4 < (t/2)c(n, \mathbf{F})$ and $2t^2n^2 = 2t(tn^2) < (8\varepsilon q)tn^2 < 8\varepsilon^2 tn^4 < (t/2)c(n, \mathbf{F})$. Consequently, $\#\mathbf{F} > qc(n, \mathbf{F})$ as required.

Next we suppose that $t \ge (n-2)/2 > n/4$. This implies that $s \le \sqrt{2t/(n-2)} + 1 \le 4\sqrt{t/n}$ and $\sqrt{t} \le 2t/\sqrt{n}$. Therefore

$$4qsn^3 < 16qn^3\sqrt{t/n} = 16q\sqrt{t}n^{2.5} \le 32qtn^2 \le 32\varepsilon tn^4 < (t/5)c(n, \mathbf{F}).$$

So we again use (3) to deduce that $\#\mathbf{F}$ is at least

$$\sum_{e\in B}\mathbf{F}(e) \ge (q+t)(c(n,\mathbf{F})-4sn^3-2tn^2) \ge qc(n,\mathbf{F})+tc(n,\mathbf{F})-4qsn^3-2qtn^2-4tsn^3-2t^2n^2.$$

As $t < q < \varepsilon n^2$ we have the bounds

$$2qtn^2 < (t/5)c(n,\mathbf{F}), \quad 4tsn^3 < 4qsn^3 < (t/5)c(n,\mathbf{F}), \quad 2t^2n^2 < (t/5)c(n,\mathbf{F}).$$

This shows that $\#\mathbf{F} > qc(n,\mathbf{F})$ and completes the proof of the theorem. $\square$

We end this section by proving that this result is asymptotically sharp.

Proof of Proposition 4. Let $0 < \delta \ll \varepsilon$. Consider the following construction: Add a collection of q edges to $P^3(n)$ within the part of size $\lceil n/2\rceil$ such that the following two conditions hold.

(1) every two added edges have at most one point in common and

(2) the added edges do not form a Pasch configuration, which is the six vertex 3-graph obtained from $\mathbf{F}$ by deleting a vertex.

It is well-known that such triple systems exist of size δn^2 (in fact such Steiner triple systems also exist [12]). Each new edge lies in at most $c(n,\mathbf{F})$ copies of $\mathbf{F}$ that contain a unique new edge. Now suppose that two of these new edges, say e, e' lie in a copy C of $\mathbf{F}$. Then there are at most n^2 choices for the remaining two vertices of C. So the number of copies of $\mathbf{F}$ containing two new edges is at most $q^2n^2 \le \delta qn^4 < \varepsilon qc(n,\mathbf{F})$. There are no copies of $\mathbf{F}$ using three new edges since three edges of $\mathbf{F}$ either span seven vertices or form a Pasch configuration. In either case we would have a Pasch configuration among the added edges. Consequently, the number of copies of $\mathbf{F}$ is at most $q(1+\varepsilon)c(n,\mathbf{F})$. $\square$

3 Counting F_5's

Theorem 5 follows from the following result. Recall that $c(n,F_5) = (3+o(1))(n/3)^2$.

Theorem 12. *For every $\varepsilon > 0$ there exists $\delta > 0$ and n_0 such that the following holds for $n > n_0$. Every n-vertex 3-graph with $t^3(n)+1$ edges contains either*

- *an edge that lies in at least $(3-\varepsilon)(n/3)^2$ copies of F_5, or*
- *at least δn^3 copies of F_5.*

Proof of Theorem 5. Remove $q-1$ edges from $\mathcal{H}$ and apply Theorem 12. If we find δn^3 copies of F_5, then since $q < \delta n$, the number of copies is much larger than $q(1-\varepsilon)c(n, F_5)$ and we are done. Consequently, we find an edge e_1 in at least $(3-\varepsilon)(n/3)^2 > (1-\varepsilon)c(n, F_5)$ copies of F_5. Now remove $q-2$ edges from $\mathcal{H} - e_1$ and repeat this argument to obtain e_2. In this way we obtain edges $e_1, \ldots, e_q$ as required.

Sharpness follows by adding a 3-partite triple system to one of the parts of $T^3(n)$. It is easy to see that each added edge lies in $c(n, F_5) - O(1)$ copies of F_5 and no copy of F_5 contains two of the new edges. Consequently, the copies of F_5 are counted exactly once. $\square$

We will need the following stability theorem for F_5 proved by Keevash and the first author [13].

Theorem 13. **(F_5 Stability [13])** *Let $\mathcal{H}$ be a 3-graph with n vertices and $t^3(n) - o(n^3)$ edges that contains no copy of F_5. Then there is a partition of the vertex set of $\mathcal{H}$ into three parts so that the number of edges with at least two vertices in some part is $o(n^3)$. In other words, $\mathcal{H}$ can be obtained from $T^3(n)$ by adding and deleting a set of $o(n^3)$ edges.*

Proof of Theorem 12. Given ε let $0 < \delta \ll \varepsilon$. Write $o_\delta(1)$ for a function that approaches zero as δ approaches zero and moreover, $o_\delta(1) \ll \varepsilon$ for the set of functions used in this proof. Let n be sufficiently large and let $\mathcal{H}$ be an n vertex 3-graph with $t^3(n)+1$ edges. Write $\#F_5$ for the number of copies of F_5 in $\mathcal{H}$.

We first argue that we may assume that $\mathcal{H}$ has minimum degree at least $d = (2/9)(1-\delta_1)\binom{n}{2}$, where $\delta_1 = \delta^{1/4}$. Indeed, if this is not the case, then remove a vertex of degree less than d to form the 3-graph $\mathcal{H}_1$ with $n-1$ vertices. Continue removing a vertex of degree less than d if such a vertex exists. If we could continue this process for $\delta_2 n$ steps, where $\delta_2 = \delta^{1/2}$, then the resulting 3-graph $\mathcal{H}'$ has $(1-\delta_2)n$ vertices and number of edges at least

$$\begin{aligned}\frac{2}{9}(1-\delta/2)\binom{n}{3} - (\delta_2 n)\frac{2}{9}(1-\delta_1)\binom{n}{2} &\geq \frac{2}{9}(1-\delta-3\delta_2(1-\delta_1))\binom{n}{3}\\ &> \frac{2}{9}(1+\delta)(1-\delta_2)^3\binom{n}{3}\\ &> \frac{2}{9}(1+\delta)\binom{(1-\delta_2)n}{3}.\end{aligned}$$

By the result of Keevash-Mubayi [13] and Erdős-Simonovits supersaturation we conclude that $\mathcal{H}$ has at least $\delta' n^5$ copies of F^5 (for some fixed $\delta' > 0$) and we are done. So we may assume that this process of removing vertices of degree less than d terminates in fewer than $\delta_2 n$ steps, and when it terminates we are left with a 3-graph $\mathcal{H}'$ on $n' > (1-\delta_2)n$ vertices and minimum degree at least d.

Now suppose that we could prove that there is an edge of $\mathcal{H}'$ that lies in at least $(3-\varepsilon/2)(n'/3)^2$ copies of F_5. Since $\delta \ll \varepsilon$, this is greater than $(3-\varepsilon)(n/3)^2$ and we are done. If on the other hand $\mathcal{H}'$ contains at least $2\delta n'^3$ copies of F_5, then again this is at least δn^3 and we are done. So if we could prove the result for $\mathcal{H}'$ with $2\delta, \varepsilon/2$, then we could prove the result for $\mathcal{H}$ (with δ, ε). Consequently, we may assume that $\mathcal{H}$ has minimum degree at least $(2/9 - o_\delta(1))\binom{n}{2} = (1-o_\delta(1))(n/3)^2$.

If $\#F_5 \geq \delta n^5$, then we are done so assume that $\#F_5 < \delta n^4$. Then by the Removal lemma, there is a set of at most $o_\delta(n^3)$ edges of $\mathcal{H}$ whose removal results in a 3-graph $\mathcal{H}'$ with no copies of F_5. Since $|\mathcal{H}'| > t^3(n) - o_\delta(n^3)$, by Theorem 13, we conclude that there is a 3-partition of $\mathcal{H}'$ (and also of $\mathcal{H}$) such that the number of edges with at least two points in a part is $o_\delta(n^3)$. Now pick a partition $X \cup Y \cup Z$ of $\mathcal{H}$ that maximizes $e(X,Y,Z) = \mathcal{H} \cap (X \times Y \times Z)$. We know that $e(X,Y,Z) \geq t^3(n) - o_\delta(n^3)$, and an easy calculation also shows that each of X, Y, Z has size $n/3 + o_\delta(n)$.

Let $B = \mathcal{H} - (X \times Y \times Z)$ be the set of edges of $\mathcal{H}$ that have at least two points in one of the partition classes and set $G = \mathcal{H} - B$. Let $M = (X \times Y \times Z) - \mathcal{H}$ be the set of triples with one point in each of X, Y, Z that are not edges of $\mathcal{H}$. Then $G \cup M = (\mathcal{H} - B) \cup M$ is 3-partite so it has at most $t^3(n)$ triples. Since $|\mathcal{H}| = t^3(n) + 1$, we conclude that

$$0 \leq |M| < |B| = o_\delta(n^3).$$

Claim. For every vertex v of $\mathcal{H}$ we have $d_M(v) < \varepsilon'(n/3)^2$ for $\varepsilon' = \varepsilon/10^6$.

Proof of Claim. Suppose for contradiction that $d_M(v) \geq \varepsilon'(n/3)^2$ for some vertex v. Then

$$(1-o_\delta(1))(n/3)^2 \leq d_{\mathcal{H}}(v) = d_G(v) + d_B(v) \leq (1+o_\delta(1))(n/3)^2 - \varepsilon'(n/3)^2 + d_B(v).$$

We conclude that $d_B(v) \geq (\varepsilon' - o_\delta(1))(n/3)^2 > (\varepsilon'/2)(n/3)^2$. Assume wlog that $v \in X$.

Case 1: $d_{B_{XXX}}(v) > (\varepsilon'/10)(n/3)^2$. Suppose that $e = uvw$ satisfies $v \in e \in B_{XXX}$ and $(y,z) \in Y \times Z$. The number of such choices for $(e,(y,z))$ is at least $d_{B_{XXX}}(v)|Y||Z| > (\varepsilon'/20)(n/3)^4$. If for at least half of these choices $e \cup \{y,z\}$ forms a copy of F_5 via the edges e, uyz, wyz then we have $\#F_5 > (\varepsilon'/40)(n/3)^4 > \delta n^3$, a contradiction. So for at least half of the choices of $(e,(y,z))$ above, $xyz \notin \mathcal{H}$ for some $x \in \{u,w\}$ (i.e. $xyz \in M$). Since each such triple of M is counted at most $|X| < n$ times (as v is fixed), we obtain the contradiction $(\varepsilon'/40n)(n/3)^4 < |M| = o_\delta(n^3)$. This concludes the proof in this case.

Case 2: $d_{B_{XXY}}(v) > (\varepsilon'/10)(n/3)^2$ or $d_{B_{XXZ}}(v) > (\varepsilon'/10)(n/3)^2$. Assume by symmetry that $d_{B_{XXY}}(v) > (\varepsilon'/10)(n/3)^2$. We may assume that $d_G(v) \geq d_{B_{XXY}}(v)$ for otherwise we can move v to Z and contradict the choice of the partition. Suppose that $e = uvw$ satisfies $v \in e \in B_{XXY}$ with $u \in X, w \in Y$. Let $(y, z) \in (Y - \{w\}) \times Z$ be such that $vyz \in \mathcal{H}$. The number of such choices for $(e, (y, z))$ is at least $d_{B_{XXY}}(v)(d_G(v) - |Z|) > (\varepsilon'/11)^2(n/3)^4$. If for at least half of these choices $e \cup \{y, z\}$ forms a copy of F_5 via the triples e, uyz, vyz then we have $\#F_5 > (\varepsilon'/20)^2(n/3)^4 > \delta n^3$, a contradiction. So for at least half of the choices of $(e, (y, z))$ above, $uyz \notin \mathcal{H}$ (i.e. $uyz \in M$). Since each such triple of M is counted at most $|Y| < n$ times (as v is fixed), we obtain the contradiction $(\varepsilon'/20)^2(n/3)^4/n < |M| = o_\delta(n^3)$. This concludes the proof in this case.

Case 3: $d_{B_{XYY}}(v) > (\varepsilon'/10)(n/3)^2$ or $d_{B_{XZZ}}(v) > (\varepsilon'/10)(n/3)^2$. Assume by symmetry that $d_{B_{XYY}}(v) > (\varepsilon'/10)(n/3)^2$. Suppose that $e = uvw$ satisfies $v \in e \in B_{XXY}$ with $u, w \in Y$. Pick $(x, z) \in (X - \{v\}) \times Z$. The number of such choices for $(e, (x, z))$ is at least $d_{B_{XYY}}(v)(|X| - 1)|Z| > (\varepsilon'/11)^2(n/3)^4$. If for at least half of these choices $e \cup \{x, z\}$ forms a copy of F_5 via the triples xzu, xzw, e then we have $\#F_5 > (\varepsilon'/20)^2(n/3)^4 > \delta n^3$, a contradiction. So for at least $(\varepsilon'/20)^2(n/3)^4$ of the choices of $(e, (x, z))$ above, $xyz \notin \mathcal{H}$ for some $y \in \{u, w\}$ (i.e. $xyz \in M$). For at least half of these choices, we may assume that $y = u$. Since each such triple of M is counted at most $|Y| < n$ times (as v is fixed), we obtain the contradiction $(\varepsilon'/20)^2(n/3)^4/2n < |M| = o_\delta(n^3)$. This concludes the proof of the Claim.

Let $B_1 = B_{XXX} \cup B_{YYY} \cup B_{ZZZ} \subset B$, where the subscripts have the obvious meaning (B_{XXX} is the set of edges in B with three points in X etc.), and let $B_2 = B - B_1$, so B_2 consists of those edges of $\mathcal{H}$ that have two points in one part and one point in some other part.

Suppose that $e = uvw \in B_{XXX}$. For each $(y, z) \in Y \times Z$ the points u, v, w, y, z form a potential copy of F_5 via e and two triples involving y, z. For at least $(\varepsilon/2)(n/3)^2$ of these potential copies, $xyz \in M$ for $x \in e$, otherwise e lies in $(3 - o_\delta(1) - \varepsilon/2)(n/3)^2 > (3 - \varepsilon)(n/3)^2$ copies of F_5 and we are done. Each such triple of M is counted at most twice, hence the number of triples intersecting e is at least $(\varepsilon/4)(n/3)^2$, and at least a third of these triples contain the same vertex $x \in e$. We conclude that $d_M(x) > (\varepsilon/12)(n/3)^3 \geq \varepsilon'(n/3)^2$ which contradicts the Claim. The argument above works for any $e \in B_1$, so we have shown that

$B_1 = \emptyset$.

Let $e = uvw \in B_2 = B$, where u, v are in the same part, say X, and w is in another part, say Y. For each $(y, z) \in (Y - \{w\}) \times Z$, there are three types of potential copies of F_5 with vertices u, v, w, y, z:

Type 1: uyz, vyz, e

Type 2: uwz, e, vyz or vwz, e, uyz

The number of Type i potential copies of F_5 is $(|Y| - 1)|Z| = (1 - o_\delta(1))(n/3)^2$. We may assume that the number of Type 1 (real, not potential) copies of F_5 is at most $(1-\varepsilon/3)(n/3)^2$, or that the number of Type 2 (real, not potential) copies of F_5 is at most $(2 - 2\varepsilon/3)(n/3)^2$. Otherwise e lies in at least $(3 - \varepsilon)(n/3)^2$ copies of F_5 and we are done.

Suppose that the number of Type 1 copies of F_5 is at most $(1 - \varepsilon/3)(n/3)^2$. The number of pairs $(y, z) \in (Y - \{w\}) \times Z$ for which either $uyz \in M$ or $vyz \in M$ is at least

$$(|Y| - 1)|Z| - (1 - \varepsilon/3)(n/3)^2 > (1 - o_\delta(1) - 1 + \varepsilon/3)(n/3)^2 > (\varepsilon/4)(n/3)^2.$$

Hence there exists $x \in \{u, v\}$ such that $xyz \in M$ for at least $(\varepsilon/8)(n/3)^2$ pairs $(y, z) \in Y \times Z$. In other words, $d_M(x) > (\varepsilon/8)(n/3)^2 \geq \varepsilon'(n/3)^2$. This contradicts the Claim.

We may therefore suppose that the number of Type 2 copies of F_5 is at most $(2-2\varepsilon/3)(n/3)^2$. Assume by symmetry that there are at most $(1 - \varepsilon/3)(n/3)^2$ Type 2 copies of the form uwz, e, vyz. Arguing as above, the number of pairs $(y, z) \in Y \times Z$ for which either $uwz \in M$ or $vyz \in M$ is at least $(\varepsilon/4)(n/3)^2$. If at least half of the time we have $vyz \in M$, then we obtain $d_M(v) > (\varepsilon/8)(n/3)^2 \geq \varepsilon'(n/3)^2$ and contradict the Claim. We therefore conclude that for at least $(\varepsilon/8)(n/3)^2$ pairs $(y, z) \in Y \times Z$, we have $uwz \in M$. Consequently, the number of $z \in Z$ for which $uwz \in M$ is at least $(\varepsilon/10)(n/3)$. We write this as $d_M(uw) \geq (\varepsilon/10)(n/3)$.

We have argued that for every edge $e = uvw \in B$ with u, v in the same part and w in a different part, either $d_M(uw) \geq \varepsilon n/30$ or $d_M(vw) \geq \varepsilon n/30$. Form a bipartite graph with parts B and M. Let $e \in B$ be adjacent to $f \in M$ if $|e \cap f| = 2$. We have shown above that each $e \in B$ has degree at least $\varepsilon n/30$. Since $|B| > |M|$, we conclude that there exists $f \in M$ which is adjacent to at least $\varepsilon n/30$ different $e \in B$. Each of these $e \in B$ has two points in common point with f, so there is a pair of vertices u, v in different parts of $\mathcal{H}$ that lie is at least $\varepsilon n/90$ different $e \in B$. Assume wlog that $u \in X, v \in Y$, and also that there are $x_i \in X$ for $1 \leq i \leq \varepsilon n/180$ such that $uvx_i \in B$ for each i. For each x_i, consider

$(y, z) \in (Y - \{v\}) \times Z$ and triples x_ivz, x_ivu, uyz. The number of such choices for (i, y, z) is at least $(\varepsilon n/200)(n/3)^2$. If for at least half of these choices these three triples are edges of $\mathcal{H}$, then we obtain $\#F_5 \geq (\varepsilon n/400)(n/3)^2 > \delta n^3$ and we are done. So for at least half of these choices of (i, y, z) we have either $x_ivz \in M$ or $uyz \in M$. Each such triple of M is counted at most n times so we obtain at least $(\varepsilon/400)(n/3)^2$ triples from M incident to some vertex of e. At least one third of these triples are incident to the same vertex of e, so we obtain $x \in e$ with $d_M(x) \geq (\varepsilon/1200)(n/3)^2 \geq \varepsilon'(n/3)^2$. The contradicts the Claim and completes the proof. □

4 Counting B_5's

Theorem 6 follows from the following result. Recall that $c(n, B_5) = (2 + o(1))(n/3)^2$.

Theorem 14. *For every $\varepsilon > 0$ there exists $\delta > 0$ and n_0 such that the following holds for $n > n_0$. Every n-vertex 3-graph with $b^3(n) + 1$ edges contains either*

- *an edge that lies in at least $(2 - \varepsilon)(n/3)^2$ copies of B_5, or*
- *at least δn^4 copies of B_5.*

Proof of Theorem 6. Remove $q - 1$ edges from $\mathcal{H}$ and apply Theorem 14. If we find δn^4 copies of B_5, then since $q < \delta n^2$, the number of copies is much larger than $(1 - \varepsilon)c(n, B_5)$ and we are done. Consequently, we find an edge e_1 in at least $(2 - \varepsilon)(n/3)^2 > (1 - \varepsilon)c(n, B_5)$ copies of B_5. Now remove $q - 2$ edges from $\mathcal{H} - e_1$ and repeat this argument to obtain e_2. In this way we obtain edges $e_1, \ldots, e_q$ as required.

Sharpness follows by adding a partial Steiner triple system to $B^3(n)$ where each added edge is entirely within X. In other words, we are adding a collection of triples within X such that every two have at most one point in common. It is easy to see that each added edge lies in $c(n, B_5) - O(1)$ copies of B_5 and moreover, since these edges have at most one common point, these copies are counted exactly once. □

We will need the following stability theorem for B_5 proved by Füredi-Pikhurko-Simonovits [9].

Theorem 15. **(B_5 stability [9])** *Let $\mathcal{H}$ be a 3-graph with n vertices and $b^3(n) - o(n^3)$ edges that contains no copy of B_5. Then there is a partition of the vertex set of $\mathcal{H}$ into $X \cup Y$ so that the number of edges that are not of the form XXY is $o(n^3)$. In other words, $\mathcal{H}$ can be obtained from $B^3(n)$ by adding and deleting a set of $o(n^3)$ edges.*

Proof of Theorem 14. Given ε let $0 < \delta \ll \varepsilon$. Write $o_\delta(1)$ for any function that approaches zero as δ approaches zero and moreover, $o_\delta(1) \ll \varepsilon$. Let n be sufficiently large and let $\mathcal{H}$ be an n-vertex 3-graph with $b^3(n) + 1$ edges. Write $\#B_5$ for the number of copies of B_5 in $\mathcal{H}$.

We first argue that we may assume that $\mathcal{H}$ has minimum degree at least $d = (4/9)(1-\delta_1)\binom{n}{2}$, where $\delta_1 = \delta^{1/4}$. Indeed, if this is not the case, then remove a vertex of degree less than d to form the 3-graph $\mathcal{H}_1$ with $n-1$ vertices. Continue removing a vertex of degree less than d if such a vertex exists. If we could continue this process for $\delta_2 n$ steps, where $\delta_2 = \delta^{1/2}$, then the resulting 3-graph $\mathcal{H}'$ has $(1-\delta_2)n$ vertices and number of edges at least

$$\frac{4}{9}(1 - \delta - 3\delta_2(1-\delta_1)\binom{n}{3} > \frac{4}{9}(1+\delta)\binom{(1-\delta_2)n}{3}.$$

By the result of Füredi-Pikhurko-Simonovits [9] and Erdős-Simonovits supersaturation we conclude that $\mathcal{H}$ has at least $\delta' n^5$ copies of B^5 (for some fixed $\delta' > 0$) and we are done. So we may assume that this process of removing vertices of degree less than d terminates in at most $\delta_2 n$ steps, and when it terminates we are left with a 3-graph $\mathcal{H}'$ on $n' > (1-\delta_2)n$ vertices and minimum degree at least d.

Now suppose that we could prove that there is an edge of $\mathcal{H}'$ that lies in at least $(2 - \varepsilon/2)(n'/3)^2$ copies of B_5. Since $\delta \ll \varepsilon$, this is greater than $(2-\varepsilon)(n/3)^2$ and we are done. If on the other hand $\mathcal{H}'$ contains at least $2\delta n'^4$ copies of B_5, then again this is at least δn^4 and we are done. So if we could prove the result for $\mathcal{H}'$ with $2\delta, \varepsilon/2$, then we could prove the result for $\mathcal{H}$ (with δ, ε). Consequently, we may assume that $\mathcal{H}$ has minimum degree at least $(4/9 - o_\delta(1))\binom{n}{2}$.

If $\#B_5 \geq \delta n^4$, then we are done so assume that $\#B_5 < \delta n^4$. Then by the Removal lemma, there is a set of at most $o_\delta(n^3)$ edges of $\mathcal{H}$ whose removal results in a 3-graph $\mathcal{H}'$ with no copies of B_5. Since $|\mathcal{H}'| > b^3(n) - o_\delta(n^3)$, by Theorem 15, we conclude that there is a partition $X \cup Y$ of the vertex set of $\mathcal{H}'$ (and also of $\mathcal{H}$) such that the number of edges with $0, 1$, or 3 points in X is $o_\delta(n^3)$. Now pick a partition $X \cup Y$ of $\mathcal{H}$ that maximizes $e(X, X, Y)$ the number of edges with exactly two points in X. We know that $e(X,X,Y) \geq b^3(n) - o_\delta(n^3)$, and an easy calculation also shows that $|X| = 2n/3 + o_\delta(n)$ and $|Y| = n/3 + o_\delta(n)$.

Let B be the set of edges of $\mathcal{H}$ that do not have exactly two points in X. Let M be the set of triples with exactly two points in X that are not edges of $\mathcal{H}$ and let $G = \mathcal{H} - B$ be the set of edges of $\mathcal{H}$ with exactly two points in X. Then $\mathcal{H} - B \cup M$ has a $(2,1)$-partition $X \cup Y$, so it has at most $b^3(n)$ edges. We conclude that

$$|M| < |B| = o_\delta(n^3).$$

In particular, $B \neq \emptyset$. Partition $B = B_{XXX} \cup B_{XYY} \cup B_{YYY}$, where $B_{X^iY^{3-i}}$ is the set of edges in B with i points in X and $3-i$ points in Y.

Claim 1. For every vertex v of $\mathcal{H}$ we have $d_{B_{XXX}}(v) < \varepsilon_1 n^2$, where $\varepsilon_1 = \varepsilon^2/10^6$.

Proof of Claim 1. Suppose for contradiction that $d_{B_{XXX}}(v) > \varepsilon_1 n^2$ for some vertex v. Let $B(v)$ be the set of edges in B_{XXX} that contain v, so $|B(v)| = d_{B_{XXX}}(v)$. First observe that $d_G(v) \geq d_{B_{XXX}}(v)$, for otherwise we can move v to Y and contradict the choice of the partition X, Y. Now for each $e = vab \in B(v)$ and $f = vxy \in \mathcal{H}$ with $\{a,b,x\} \in \binom{X-\{v\}}{3}$, $y \in Y$, consider the two triples axy, bxy. We see that e, f, axy, bxy forms a (potential) copy of B_5. For each e, the number of f is at least $d_G(v) - n \geq |B(v)| - n > |B(v)|/2$, since f must omit a, b and there are at most $|Y|$ pairs containing either of them. Hence the number of choices for (e,f) is at least $|B(v)|^2/2$. If for at least half of these choices of (e,f), we obtain a copy of B_5 in $\mathcal{H}$, then $\#B_5 > |B(v)|^2/4 > \delta n^4$, a contradiction. So for at least half of the choices of (e,f) above, one of the triples axy, bxy is in M. A given triple in M is counted at most $|X| < n$ times, so we obtain the contradiction $|B(v)|^2/(4n) < |M| = o_\delta(n^3)$. This finishes the proof of the Claim.

Case 1. $|B_{XXX}| \geq |B|/3$.

For each $e = uvw \in B_{XXX}$, and $(x,y) \in (X-e) \times Y$, there is a potential copy of B_5 consisting of vertices u, v, w, x, y and edges uxy, vxy, wxy, e. This gives a total of $(|X|-3)|Y| > (2 - o_\delta(1))(n/3)^2$ potential copies of B_5. At least $(2\varepsilon/3)(n/3)^2$ of these potential copies of B_5 have a triple from M, for otherwise e would lie in at least $(2 - o_\delta(1) - 2\varepsilon/3)(n/3)^2 > (2-\varepsilon)(n/3)^2$ copies of B_5 and we are done. The triple from M referenced above cannot be e (since $e \in \mathcal{H}$), and therefore lies in exactly one copy of B_5 that was counted above. At least a third of these triples from M are incident with the same vertex of e, hence there exists $z \in e$ such that $d_M(z) > (2\varepsilon/9)(n/3)^2$.

Let $V = X \cup Y$ and let

$$A = \{v \in V : d_M(v) > (2\varepsilon/9)(n/3)^2\}.$$

We have argued above that every $e \in B_{XXX}$ has a vertex in A. Consequently,

$$9 \sum_{v \in A} d_{B_{XXX}}(v) \geq 9|B_{XXX}| \geq 3|B| > 3|M| \geq \sum_{v \in A} d_M(v) > |A|(2\varepsilon/9)(n/3)^2,$$

and there exists a vertex $v \in X \cap A$ such that $d_{B_{XXX}}(v) > (\varepsilon/50)(n/3)^2 > \varepsilon_1 n^2$. This contradicts Claim 1 and concludes the proof in this case.

Case 2. $|B_{YYY}| \geq |B|/3$.

For each $e = uvw \in B_{YYY}$ and $x, x' \in X$, there is a potential copy of B_5 consisting of vertices u, v, w, x, x' and edges $xx'u, xx'v, xx'w, e$. This gives a total of $\binom{|X|}{2} > (2 - o_\delta(1))(n/3)^2$ potential copies of B_5. At least $(2\varepsilon/3)(n/3)^2$ of these potential copies of B_5 have a triple from M, for otherwise e would lie in at least $(2 - o_\delta(1) - 2\varepsilon/3)(n/3)^2 > (2 - \varepsilon)(n/3)^2$ copies of B_5 and we are done. The triple from M referenced above cannot be e (since $e \in \mathcal{H}$), and therefore lies in exactly one copy of B_5. At least a third of these triples from M are incident with the same vertex of e, hence there exists $z \in e$ such that $d_M(z) > (2\varepsilon/9)(n/3)^2$. As in Case 1, let $V = X \cup Y$ and $A = \{v \in V : d_M(v) > (2\varepsilon/9)(n/3)^2\}$. We have argued above that every $e \in B_{YYY}$ has a vertex in A. Consequently,

$$9 \sum_{v \in A} d_{B_{YYY}}(v) \geq 9|B_{YYY}| \geq 3|B| > 3|M| \geq \sum_{v \in A} d_M(v) > |A|(2\varepsilon/9)(n/3)^2,$$

and there exists a vertex $v \in Y \cap A$ such that $d_{B_{YYY}}(v) > (\varepsilon/50)(n/3)^2$. Let $B(v)$ be the set of edges in B_{YYY} that contain v, so $|B(v)| = d_{B_{YYY}}(v)$.

Next we observe that $d_{B_{XYY}}(v) \leq d_G(v)$ otherwise we can move v to X and contradict the choice of the partition X, Y. We also recall that $\mathcal{H}$ has minimum degree at least $(4/9 - o_\delta(1))\binom{n}{2}$, so

$$d_{B_{YYY}}(v) + d_G(v) + d_{B_{XYY}}(v) \geq (4/9 - o_\delta(1))\binom{n}{2}.$$

Since $d_{B_{YYY}}(v) \leq \binom{|Y|}{2} < (1/9 + o_\delta(1))\binom{n}{2}$, we conclude that

$$d_G(v) > \frac{1}{2}\left(\frac{4}{9} - \frac{1}{9} - o_\delta(1)\right)\binom{n}{2} = \left(\frac{1}{6} - o_\delta(1)\right)\binom{n}{2}.$$

Now for each $e = vyy' \in B(v)$ and $f = vxx' \in G$ $(x, x' \in X)$, consider the two triples $xx'y, xx'y'$. We see that $e, f, xx'y, xx'y'$ forms a potential copy of B_5. The number of choices of (e, f) above is at least $|B(v)|d_G(v) = d_{B_{YYY}}(v)d_G(v)$. If for at least half of these choices of (e, f), we obtain a copy of B_5 in $\mathcal{H}$, then $\#B_5 > d_{B_{YYY}}(v)d_G(v)/2 > \delta n^4$, a contradiction. So for at least half of the choices of (e, f) above, one of the triples $xx'y, xx'y'$ is in M. A given triple in M is counted at most $|Y| < n$ times, so we obtain the contradiction $d_{B_{YYY}}(v)d_G(v)/(2n) < |M| = o_\delta(n^3)$. This concludes the proof in this case.

Case 3. $|B_{XYY}| \geq |B|/3$.

Let

$$B_1 = \{e \in B_{XYY} : \text{there exists } v \in e \cap Y \text{ with } d_M(v) > \varepsilon(n/3)^2\}.$$

Subcase 3.1. $|B_1| \geq |B_{XYY}|/2$. Let

$$A = \{v \in Y : d_M(v) > (\varepsilon/2)(n/3)^2\}.$$

By definition, every $e \in B_1$ has a vertex in A. Therefore

$$18 \sum_{v \in A} d_{B_1}(v) \geq 18|B_1| \geq 9|B_{XYY}| \geq 3|B| > 3|M| \geq \sum_{v \in A} d_M(v) > |A|(\varepsilon/2)(n/3)^2,$$

and there exists a vertex $v \in Y$ such that

$$d_{B_{XYY}}(v) \geq d_{B_1}(v) > (\varepsilon/36)(n/3)^2.$$

Recall that G is the set of edges of $\mathcal{H}$ with exactly two points in X. Next observe that $d_G(v) \geq d_{B_{XYY}}(v)$ for otherwise we can move v to X which increases $e(X, X, Y)$ and contradicts the choice of X, Y. It follows that $d_G(v) > (\varepsilon/36)(n/3)^2$.

Now for each $e = uvw \in B_{XYY}$ and $f = xx'v \in G$ with $\{u, x, x'\} \in \binom{X}{3}$, and $w \in Y$, consider the two triples uwx, uwx'. We see that $e, uwx, uwx', xx'v$ forms a potential copy of B_5. The number of choices of (e, f) above is at least $d_{B_{XYY}}(v) \times (d_G(v) - |X|) > d_{B_{XYY}}(v)d_G(v)/2$. If for at least half of these choices of (e, f), we obtain a copy of B_5 in $\mathcal{H}$, then

$$\#B_5 > \frac{d_{B_{XYY}}(v)d_G(v)}{4} > \frac{\varepsilon^2}{10^5}\left(\frac{n}{3}\right)^4 > \delta n^4,$$

a contradiction. So for at least half of the choices of (e, f) above, one of the triples uwx, uwx' is in M. A given triple in M is counted at most $|X| < n$ times, so we obtain the contradiction

$$\frac{\varepsilon^2}{10^5}\left(\frac{n^3}{3^4}\right) < \frac{d_{B_{XYY}}(v)d_G(v)}{4n} < |M| = o_\delta(n^3).$$

This concludes the proof in this subcase.

Subcase 3.2. $|B_1| < |B_{XYY}|/2$. So in this subcase we have $|B_2| \geq |B_{XYY}|/2$, where

$$B_2 = \{e \in B_{XYY} : \text{for every } v \in e \cap Y \text{ we have } d_M(v) \leq (\varepsilon/2)(n/3)^2\}.$$

Fix $e = uvw \in B_2$ with $u \in X$ and $v, w \in Y$.

Claim 2. There exist sets $X_v, X_w \subset X$ such that

- $xuv \in M$ for every $x \in X_v$ and $xuw \in M$ for every $x \in X_w$ and
- $|X_v| > (\varepsilon/20)n$ and $|X_w| > (\varepsilon/20)n$

Proof of Claim 2. Let $X_v = \{x \in X : xuv \in M\}$. We will show that $|X_v| \geq (\varepsilon/20)n$. The same argument will apply to X_w.

Suppose for contradiction that $|X_v| < (\varepsilon/20)n$. Pick $x, x' \in X - X_v$ and consider u, v, w, x, x'. The triples $uvx, uvx', e, xx'w$ form a potential copy of B_5. Since $x, x' \in X - X_v$, we have $uvx \in \mathcal{H}$ and $uvx' \in \mathcal{H}$. So if these four edges do not form a copy of B_5 in $\mathcal{H}$ then $xx'w \in M$. Since $e \in B_2$, the number of pairs $\{x, x'\} \in \binom{X}{2}$ such that $xx'w \in M$ is at most $(\varepsilon/2)(n/3)^2$. Consequently, the number of pairs $x, x' \in X - X_v$ with $xx'w \in \mathcal{H}$ is at least

$$\begin{aligned} \binom{|X - X_v| - 1}{2} - \frac{\varepsilon}{2}\left(\frac{n}{3}\right)^2 &> \binom{(1 - o_\delta(1) - \frac{3\varepsilon}{40})\frac{2n}{3}}{2} - \frac{\varepsilon}{2}\left(\frac{n}{3}\right)^2 \\ &> \left(2\left(1 - \frac{\varepsilon}{10}\right)^2 - \frac{\varepsilon}{2}\right)\left(\frac{n}{3}\right)^2 \\ &= \left(2 - \frac{9\varepsilon}{10} + \frac{\varepsilon^2}{50}\right)\left(\frac{n}{3}\right)^2 \\ &> (2 - \varepsilon)\left(\frac{n}{3}\right)^2. \end{aligned}$$

This gives us the required number of copies of B_5 containing the edge e and concludes the proof of the Claim.

For each edge $e = uvw \in B_2$ with $u \in X, v, w \in Y$, Claim 2 shows that are at least $(\varepsilon/20)n$ triples of the form $xuv \in M$. Form the bipartite graph with parts B_2 and M, where $uvw \in B_2$ is adjacent to all such $xuv \in M$. Then since every vertex of B_2 has degree at least $(\varepsilon/20)n$, and $|B_2| \geq |B_{XYY}|/2 \geq |B|/6 > |M|/6$, we conclude that there exists $xuv \in M$ (with $v \in Y$) which is adjacent to at least $(\varepsilon/120)n$ edges in B_2. Each of these edges of B_2 contains v, and either x or u, so we may assume by symmetry that at least half of them contain u. So we have $uvw_i \in B_2$, where $u \in X$ and $v, w_i \in Y$ for $i = 1, \ldots, (\varepsilon/240)n$. For each w_i, consider the set X_{w_i} defined in Claim 2. We know that $x'uw_i \in M$ for each w_i and $x' \in X_{w_i}$. Since these triples are distinct for distinct w_i or distinct x', we conclude that $d_M(u) \geq (\varepsilon/240)n(\varepsilon/20)n = (\varepsilon^2/4800)n^2$. Recalling the minimum degree condition on $\mathcal{H}$, we have

$$(4/9 - o_\delta(1))\binom{n}{2} \leq d_{\mathcal{H}}(u) = d_G(u) + d_B(u) \leq (4/9 - o_\delta(1))\binom{n}{2} - d_M(u) + d_B(u).$$

We conclude that $d_B(u) \geq (\varepsilon^2/5000)n^2$. By Claim 1 we know that $d_{B_{XXX}}(u) < \varepsilon_1 n^2$ where $\varepsilon_1 = \varepsilon^2/10^5$. As $d_B(u) = d_{B_{XXX}}(u) + d_{B_{XYY}}(u)$, we obtain

$$d := d_{B_{XYY}}(u) = d_B(u) - d_{B_{XXX}}(u) > (\varepsilon^2/5000)n^2 - \varepsilon_1 n^2 \geq 2\varepsilon_1 n^2.$$

Say that $uyy' \in B_{XYY}$ is bad if

$$|\{x \in X : xuy \in M \text{ or } xuy' \in M\}| > (1 - \varepsilon_1)(2n/3).$$

Let

$$S = \left\{ \{y, y'\} \in \binom{|Y|}{2} : uyy' \text{ is bad} \right\}.$$

Now suppose that $|S| \geq (0.9)d$. For each $e = uyy' \in B_{XYY}$ with $\{y, y'\} \in S$ there is a set $X_e \subset X$ with $|X_e| \geq (1 - \varepsilon_1)(2n/3)$ such that $xuy \in M$ or $xuy' \in M$ for all $x \in X_e$. Each of these triples in M is counted at most $|Y|$ times so we obtain

$$d_M(u) \geq \frac{|S|(1 - \varepsilon_1)(2n/3)}{(1 + o_\delta(1))n/3} = 2(1 - 2\varepsilon_1)|S| \geq (1.8)(1 - 2\varepsilon_1)d > (1.7)d. \tag{4}$$

Again recalling the minimum degree condition on $\mathcal{H}$, we have

$$\begin{aligned}(4/9 - o_\delta(1))\binom{n}{2} \leq d_{\mathcal{H}}(u) &= d_{B_{XXX}}(u) + d_G(u) + d_{B_{XYY}}(u) \\ &= d_{B_{XXX}}(u) + \left((4/9 + o_\delta(1))\binom{n}{2} - d_M(u)\right) + d.\end{aligned}$$

Using (4) and $d > \varepsilon_1 n^2$ we obtain $d_{B_{XXX}}(u) > (0.7)d - o_\delta(n^2) > \varepsilon_1 n^2$. This contradicts Claim 1 and concludes the proof if $|S| \geq (0.9)d$.

Next suppose that $|S| < (0.9)d$. So for at least $(0.1)d$ edges $e = uyy' \in B_{XYY}$ we have a set $X_e \subset X$ such that

$$|X_e| \geq (\varepsilon_1 - o_\delta(1))(2n/3) > (\varepsilon_1/3)n$$

and $uyx \in \mathcal{H}$ for all $x \in X_e$ (also $uy'x \in \mathcal{H}$ but we wont use this).

Let $x, x' \in X_e$ and consider the triple $xx'y'$. We see that $e, uyx, uyx', xx'y'$ forms a potential copy of B_5. The number of choices for $(e, \{x, x'\})$ above is at least

$$(0.1)d \times \binom{(\varepsilon_1/3)n}{2} > (\varepsilon_1/5)n^2 \times (\varepsilon_1^2/20)n^2 = (\varepsilon_1^3/100)n^4.$$

If for at least half of these choices of $(e, \{x, x'\})$, we have $xx'y' \in \mathcal{H}$, then

$$\#B_5 > (\varepsilon_1^3/200)n^4 > \delta n^4,$$

a contradiction. So for at least half of the choices of $(e, \{x, x'\})$ above, $xx'y' \in M$. A given triple $xx'y' \in M$ is counted at most $|Y| < n/2$ times, so we obtain the contradiction

$$\frac{\varepsilon_1^3}{200}n^3 \le \frac{(0.1)d \times \binom{|X_e|}{2}}{n} < |M| = o_\delta(n^3).$$

This completes the proof of the subcase and the Theorem. $\square$

5 Counting Expansions of Cliques

In this section we will prove Theorems 7 and 8.

5.1 Asymptotic Counting

Theorem 7 follows from the following result. Recall that $l_{r+1} = \binom{r+2}{2}$ is the number of vertices of L_{r+1} and $c(n, L_{r+1}) = \Theta(n^{l_{r+1}-3})$.

Theorem 16. *For every $\varepsilon > 0$ there exists $\delta > 0$ and n_0 such that the following holds for $n > n_0$. Every n vertex 3-graph with $t_r^3(n) + 1$ edges contains either*

- *at least $\delta n^{l_{r+1}-1}$ copies of L_{r+1}, or*
- *an edge that lies in at least $c(n, L_{r+1})$ copies of L_{r+1}, or*
- *two edges that each lie in at least $(1-\varepsilon)c(n, L_{r+1})$ copies of L_{r+1} with none of these copies containing both edges.*

Proof of Theorem 7. Remove $q-1$ edges from $\mathcal{H}$ and apply Theorem 16. If we find $\delta n^{l_{r+1}-1}$ copies of L_{r+1}, then since $q < \delta n^2$, the number of copies is much larger than $q(1-\varepsilon)c(n, L_{r+1})$ and we are done. Consequently, we find an edge e_1 in at least $(1-\varepsilon)c(n, L_{r+1})$ copies of L_{r+1}. Now remove $q-2$ edges from $\mathcal{H} - e_1$ and repeat this argument to obtain e_2. In this way we obtain edges $e_1, \ldots, e_q$ as required.

Sharpness follows by the following construction: Take $T_r^3(n)$ with parts $V_1, \ldots, V_r$, pick any point $y \in V_2$, and add q edges of the form $xx'y$ with $x, x' \in V_1$. Each added edge lies in at most $(1+\varepsilon)c(n, L_{r+1})$ copies of L_{r+1}, and no two added edges lie in a common copy of L_{r+1}, since L_{r+1} has the property that for every two edges e, e' containing a common vertex v, there is another edge f containing a point from each of $e - \{v\}$ and $e' - \{v\}$ and $v \notin f$. Taking two edges containing y, we see that there is no edge that can play the role of f above. $\square$

We will need the following stability result proved by Pikhurko [24] (see also [17]).

Theorem 17. **(L_{r+1} Stability [24])** *Let $\mathcal{H}$ be a 3-graph with n vertices and $t_r^3(n) - o(n^3)$ edges that contains no copy of L_{r+1}. Then there is a partition of the vertex set of $\mathcal{H}$ into r parts so that the number of edges that intersect some part in at least two points is $o(n^3)$. In other words, $\mathcal{H}$ can be obtained from $T_r^3(n)$ by adding and deleting a set of $o(n^3)$ edges.*

Proof of Theorem 16. Given ε let $0 < \delta \ll \varepsilon$. Write $o_\delta(1)$ for any function that approaches zero as δ approaches zero and moreover, $o_\delta(1) \ll \varepsilon$. Let n be sufficiently large and let $\mathcal{H}$ be an n vertex 3-graph with $t_r^3(n) + 1$ edges. Write $\#L_{r+1}$ for the number of copies of L_{r+1} in $\mathcal{H}$.

If $\#L_{r+1} \geq \delta n^{l_{r+1}-1}$, then we are done so assume that $\#L_{r+1} < \delta n^{l_{r+1}-1}$. Then by the Removal lemma, there is a set of at most $o_\delta(n^3)$ edges of $\mathcal{H}$ whose removal results in a 3-graph $\mathcal{H}'$ with no copies of L_{r+1}. Since $|\mathcal{H}'| > t_r^3(n) - o_\delta(n^3)$, by Theorem 17, we conclude that there is an r-partition $V_1 \cup \cdots \cup V_r$ of $\mathcal{H}'$ (and also of $\mathcal{H}$) such that the number of edges that intersect some part in at least two points is $o_\delta(n^3)$. Now pick a partition $V_1 \cup \cdots \cup V_r$ of $\mathcal{H}$ that maximizes $h_1 + 2h_2 + 3h_3$, where h_i is the number of edges of $\mathcal{H}$ that intersect precisely i of the parts. The partition guaranteed by Theorem 17 satisfies $h_1 + 2h_2 = o_\delta(n^3)$, and hence for this particular partition $h_1 + 2h_2 + 3h_3 \geq 3|\mathcal{H}| - 2(h_1 + h_2) > 3t_r^3(n) - o_\delta(n^3)$. Since $h_1 + 2h_2 + 3h_3 \leq 3|\mathcal{H}| - (h_1 + h_2)$ we conclude that for the partition that maximizes

$h_1 + 2h_2 + 3h_3$ we have $h_1 + 2h_2 = o_\delta(n^3)$ and $h_3 \geq t_r^3(n) - o_\delta(n^3)$. A standard calculation also shows that for this partition each V_i has size $n/r \pm o_\delta(n)$.

Let $B = \mathcal{H} - \prod_{i=1}^r V_i$, let $G = \mathcal{H} - B$ and $M = \prod_{i=1}^r V_i - G$. Then $\mathcal{H} - B \cup M$ is r-partite so it has at most $t_r^3(n)$ edges. We conclude that

$$|M| < |B| = o_\delta(n^3),$$

in particular $|B| \geq 1$. We will now argue that we can improve this to $|B| \geq 2$. We may suppose that $n_i := |V_i|$ satisfy $n_1 \geq n_2 \geq \ldots \geq n_r$. Pick $e_1 \in B$. If $\mathcal{H} - e_1 \cong T_r^3(n)$, then clearly e_1 lies in at least $c(n, L_{r+1})$ copies of L_{r+1} and we are done. So assume that $\mathcal{H} - e_1 \ncong T_r^3(n)$. Suppose that $B \cap (\mathcal{H} - e_1) = \emptyset$. Then either $n_r \geq n_1 - 1$ and

$$t_r^3(n) = |\mathcal{H} - e_1| \leq \left(\sum_{S \in \binom{[r]}{3}} \prod_{i \in S} n_i \right) - 1 < t_r^3(n),$$

or $n_r < n_1 - 1$ and

$$t_r^3(n) = |\mathcal{H} - e_1| \leq \sum_{S \in \binom{[r]}{3}} \prod_{i \in S} n_i < t_r^3(n).$$

In either case we have a contradiction, so we may assume that $B \cap (\mathcal{H} - e_1) \neq \emptyset$. In other words, there exists $e_2 \neq e_1$ such that $e_2 \in B$ and therefore $|B| \geq 2$. We will now show that every $e \in B$ lies in at least $(1-\varepsilon)c(n, L_{r+1})$ copies of L_{r+1} in $\mathcal{H}$ and each copy uses a unique edge from B.

Let $e = xyz \in B$. We may assume by symmetry that $x, y \in V_1$. Pick $(v_2, \ldots, v_r) \in V_2 \times \cdots \times V_r$ with $v_i \neq z$ for all i. For every pair of distinct vertices $\{a, b\}$ with $a \in \{v_2, \ldots, v_r\}$ and $b \in \{x, y, v_2, \ldots, v_r\}$ (there are $\binom{r-1}{2} + 2(r-1)$ such $\{a, b\}$), let v_{ab} be a vertex in a part different from a, b that is distinct from all other vertices being considered. The number of choices for the $(\binom{r+1}{2} + r - 2)$-tuple $(v_2, \ldots, v_r, \{v_{ab}\}_{a,b})$ is at least $(1 - o_\delta(1))c(n, L_{r+1})$. Moreover, the $\binom{r+1}{2}$ edges e and $\{abv_{ab}\}_{a,b}$ form a potential copy of L_{r+1} with $x, y, v_2, \ldots, v_r$ forming the original K_{r+1} whose edges have been expanded. At least $(\varepsilon/2)c(n, L_{r+1})$ of these potential copies of L_{r+1} have a triple from M, otherwise e would lie in at least $(1 - o_\delta(1) - \varepsilon/2)c(n, L_{r+1}) > (1-\varepsilon)c(n, L_{r+1})$ copies of L_{r+1} and we are done. Suppose that at least $(\varepsilon/4)c(n, L_{r+1})$ of these potential copies of L_{r+1} have the triple from M omitting e. Since each such triple from M is counted at most $n^{l_{r+1}-6}$ times, we obtain the contradiction $(\varepsilon/4)c(n, L_{r+1})/n^{l_{r+1}-6} \leq |M| < o_\delta(n^3)$. So at least $(\varepsilon/4)c(n, L_{r+1})$ of these potential copies of L_{r+1} have a triple from M containing x or y. Each such triple from M is counted at most $n^{l_{r+1}-5}$ times, so there are at least $(\varepsilon/4)c(n, L_{r+1})/n^{l_{r+1}-5} = \varepsilon' n^2$ triples from M containing x or y (for suitable $\varepsilon' > 0$ depending only on r). We may assume by symmetry that $d_M(x) > (\varepsilon'/2)n^2$.

We have shown above that for each $e \in B$, there is a vertex $x \in e$ that lies in the (unique) part that has at least two points from e, with $d_M(x) > (\varepsilon'/2)n^2$. Form a bipartite graph with parts B and M, where each $e \in B$ is adjacent to those f in M for which $e \cap f = \{x\}$ and x lies in the part that has at least two points of e. Then each vertex of B has degree at least $(\varepsilon'/2)n^2$. Since $|B| > |M|$ we conclude that there exists $f \in M$ adjacent to at least $(\varepsilon'/2)n^2$ different $e \in B$ in the way specified above. At least $(\varepsilon'/6)n^2$ of these $e \in B$ contain the same point $x \in f$. Assume wlog that $x \in V_1$.

For each $i \in [r]$ and $\varepsilon_1 = \varepsilon'/100$, define

$$A_i = \{y \in V_i : d_{\mathcal{H}}(xy) \geq \varepsilon_1 n\}.$$

Claim. $|A_i| < \varepsilon_1 n$ for some $i \in [r]$.

Proof of Claim. Suppose to the contrary that $|A_i| \geq \varepsilon_1 n$ for each i. Then the number of choices $(v_1, \ldots, v_r) \in A_1 \times \cdots \times A_r$ is at least $(\varepsilon_1 n)^r$. For every pair of distinct vertices $\{a, b\} \subset \{v_1, \ldots, v_r\}$, let $w_{ab} \neq x$ be a vertex in a part different from a, b (there are at least $(1 - 2/r)n > n/2r$ choices for w_{ab}). For every vertex $c \subset \{v_1, \ldots, v_r\}$, let w_c be a vertex such that $xcw_c \in \mathcal{H}$. By definition of A_i, we know that the number of such w_c is at least $\varepsilon_1 n$. Consequently, the number of choices for the $(l_{r+1} - 1)$-tuple of distinct vertices $(v_1, \ldots, v_r, \{w_{ab}\}_{a,b}, \{w_c\}_c)$ is at least

$$(\varepsilon_1 n)^r (n/2r)^{\binom{r}{2}} (\varepsilon_1 n)^r > (\varepsilon_1/r)^{r^2} n^{l_{r+1}-1} = \varepsilon_2 n^{l_{r+1}-1}.$$

Moreover, the $\binom{r+1}{2}$ triples xcw_c, abw_{ab} over all choices of a, b, c form a potential copy of L_{r+1} with $x, v_1, \ldots, v_r$ forming the original K_{r+1} whose edges have been expanded. At least $(\varepsilon_2/2)n^{l_{r+1}-1}$ of these potential copies of L_{r+1} have a triple from M, otherwise $\#L_{r+1} \geq (\varepsilon_2/2)n^{l_{r+1}-1} > \delta n^{l_{r+1}-1}$ and we are done. Each such triple from M omits x and is therefore counted at most $n^{l_{r+1}-4}$ times (since x is fixed and L_{r+1} has l_{r+1} vertices) so we obtain the contradiction $(\varepsilon_2/2)n^{l_{r+1}-1}/n^{l_{r+1}-4} \leq |M| < o_\delta(n^3)$. This completes the proof of the Claim.

Let $B(x)$ be the set of edges of B containing x with at least two vertices in V_1. Then we had earlier shown that $|B(x)| \geq (\varepsilon'/6)n^2 > 10\varepsilon_1 n^2$.

Let $H(x)$ be the set of pairs $\{y, z\}$ such that $xyz \in B(x)$, so one of $y, z \in V_1$ and $|H(x)| = |B(x)|$. Now $|A_1| \geq \varepsilon_1 n$ for otherwise we obtain the contradiction

$$|B(x)| \leq \sum_{v \in V_1} d_{H(x)}(v) = \sum_{v \in A_1} d_{H(x)}(v) + \sum_{v \in V_1 - A_1} d_{H(x)}(v) \leq (\varepsilon_1 n)n + (n/2)(\varepsilon_1 n) < 2\varepsilon_1 n^2.$$

The Claim implies that one of $|A_2|, \ldots, |A_r|$ is less than $\varepsilon_1 n$. By symmetry, we may assume that $|A_r| < \varepsilon_1 n$. The number of edges in $\mathcal{H}$ containing x and some vertex of V_r is at most $|A_r|n + |V_r|\varepsilon_1 n < 2\varepsilon_1 n^2$. Hence the number of edges in $B(x)$ that have no vertex in V_r is at least $|B(x)| - 2\varepsilon_1 n^2 > 8\varepsilon_1 n^2$.

Now let us contemplate moving x from V_1 to V_r. The edges of $\mathcal{H}$ containing x whose contribution to $\sum_i ih_i$ decreases (by at most one) must have a vertex in V_r, and their number is at most $2\varepsilon_1 n^2$. The edges in $B(x)$ that have no vertex in V_r give an increased contribution to $\sum_i ih_i$ (each edge contributes an increase of exactly one), and their number is at least $8\varepsilon_1 n^2$. All other edges containing x (i.e. those with $r - 1$ vertices in $V_2 \cup \cdots \cup V_{r-1}$) do not change their contribution to $\sum_i ih_i$. The net contribution to $\sum_i ih_i$ therefore increases by at least $6\varepsilon_1 n^2 > 0$, thus contradicting the choice of the partition and completing the proof. $\square$

5.2 Exact Counting

In this subsection we will use Theorem 16 to prove Theorem 8.

Proof of Theorem 8. Given $q \geq 1$, let $0 < \varepsilon \leq 1/(q + 1)$. Then $2(1 - \varepsilon)c(n, L_{r+1}) \geq c(n, L_{r+1})$. Let δ and n_0 be the outputs of Theorem 7 with input ε. Choose $n > n_0$ such that it also satisfies $\delta n^{l_{r+1}-1} > q \times c(n, L_{r+1})$ (this is a triviality since $c(n, L_{r+1}) = O(n^{l_{r+1}-3})$).

Suppose that $\mathcal{H}$ is an n vertex 3-graph with $t_r^3(n) + q$ edges. Write $\#L_{r+1}$ for the number of copies of L_{r+1} in $\mathcal{H}$. Let us prove by induction on q that $\#L_{r+1} \geq q \times c(n, L_{r+1})$. If $q = 1$, then Theorem 16 and the definitions of ε, δ, n imply that

$$\#L_{r+1} \geq \min\{\delta n^{l_{r+1}-1},\ c(n, L_{r+1}),\ 2(1 - \varepsilon)c(n, L_{r+1})\} \geq c(n, L_{r+1}).$$

Let us assume that $q > 1$ and the result holds for $q - 1$.

Let e_1 be an edge of $\mathcal{H}$ that lies in the maximum number of copies of L_{r+1}, say that it lies in $c_1(n)$ copies. If $c_1(n) \geq c(n, L_{r+1})$, then let $\mathcal{H}_1 = \mathcal{H} - e_1$. By induction, $\mathcal{H}_1$ has at least

$(q-1)c(n, L_{r+1})$ copies of L_{r+1}. These copies are distinct from those containing e_1 so we obtain

$$\#L_{r+1} \geq c_1(n) + (q-1)c(n, L_{r+1}) \geq qc(n, L_{r+1})$$

and we are done.

We may therefore assume that $c_1(n) < c(n, L_{r+1})$. Let e_2 be an edge of $\mathcal{H}_1$ that lies in the maximum number $c_2(n)$ of copies of L_{r+1} in $\mathcal{H}_1$. Since $\mathcal{H}_1 \subset \mathcal{H}$, clearly $c_2(n) \leq c_1(n)$. Let $\mathcal{H}_2 = \mathcal{H}_1 - e_2$ and continue this process to obtain $e_1, \ldots, e_{q-1}$. For each $i \leq q-1$, Theorem 16 implies that $\#L_{r+1} \geq \delta n^{l_{r+1}-1} > qc(n, L_{r+1})$ or $c_i(n) \geq (1-\varepsilon)c(n, L_{r+1})$. In the former case we are done, so we may assume that

$$(1-\varepsilon)c(n, L_{r+1}) \leq c_{q-1}(n) \leq \cdots \leq c_1(n) < c(n, L_{r+1}).$$

Consider $\mathcal{H}_{q-1} = \mathcal{H} - e_1 - e_2 \ldots - e_{q-1}$. Then

$$|\mathcal{H}_{q-1}| = |\mathcal{H}| - (q-1) = t_r^3(n) + q - (q-1) = t_r^3(n) + 1.$$

Since $c_{q-1}(n) < c(n, L_{r+1})$, Theorem 16 implies that $\mathcal{H}_{q-1}$ has at least $2(1-\varepsilon)c(n, L_{r+1})$ copies of L_{r+1}. Altogether we have

$$\#L_{r+1} \geq 2(1-\varepsilon)c(n, L_{r+1}) + \sum_{i=1}^{q-1} c_i(n) \geq (1-\varepsilon)(q+1)c(n, L_{r+1}) \geq qc(n, L_{r+1})$$

where the last equality follows from $\varepsilon \leq 1/(q+1)$. This completes the proof. □

6 Concluding Remarks

• We have given counting results for every triple system for which a stability result is known except for one family which is derived from the expanded cliques. This was studied in [20], and included the triple system $\{123, 145, 167, 357\}$ which is the smallest non-3-partite linear (every two edges share at most one vertex) 3-graph. It appears that our approach will give appropriate counting results for this problem as well and we did not feel motivated to carry out the details.

• Our results suggest that whenever one can obtain stability and exact results for an extremal problem, one can also obtain counting results. However, in each case the argument is different. It would be interesting to unify this approach (at least for certain classes) so one does not have to use new methods for each F. We formulate this as a conjecture. Say that a 3-graph F is stable if $\mathrm{ex}(n, F)$ is achieved uniquely by the n vertex 3-graph $\mathcal{H}(n)$ for sufficiently large n, and every n vertex 3-graph with $(1-o(1))\mathrm{ex}(n, F)$ edges and no copy of F can be obtained from $\mathcal{H}(n)$ by changing at most $o(n^3)$ edges.

Conjecture 18. *Let F be a non 3-partite stable 3-graph. For every positive integer q, the following holds for sufficiently large n: Every n vertex 3-graph with $ex(n, F)+q$ edges contains at least $qc(n, F)$ copies of F.*

- We have not been able to prove exact counting results for F_5 and B_5. The reason for this is that we need to use the minimum degree condition in the proof and we don't know how to get around this technical difficulty.

- All our theorems find $\alpha(1-o(1))n^{\beta}$ copies of F on an edge, or δn^{γ} copies of F altogether, for suitable $\alpha, \beta, \gamma, \delta$. However, in each case our proofs give δn^{γ} copies of F on a single vertex.

- Our results for F_5 appear to be weaker than the other results. In particular, we only allow $q < \delta n$ unlike in the other cases where we allow $q < \delta n^2$. However, this cannot be improved further. Indeed, for any $\varepsilon > 0$ (take $\varepsilon = 1/2$ for example) and all n, there exists an n vertex 3-graph $\mathcal{H}$ with $t^3(n) + \varepsilon n$ edges and the following two properties:

(1) for every edge $e \in \mathcal{H}$, the number of copies of F_5 containing e is less than $(3-\varepsilon)(n/3)^2$

(2) the number of copies of F_5 in $\mathcal{H}$ is less then εn^3.

To see this, let $T^3(n)$ have parts X, Y, Z and construct $\mathcal{H}$ as follows. Pick $(x, y) \in X \times Y$, delete $\varepsilon n/3$ edges of the form xyz with $z \in Z$, and add $4\varepsilon n/3$ edges of the form $x_i xy$ with $x_i \in X$. Then $|\mathcal{H}| = t^3(n) + \varepsilon n$. A copy of F_5 in $\mathcal{H}$ must contain an edge $e_i = x_i xy$, and the number of copies containing e_i is at most $(3-\varepsilon)(n/3)^2$. Therefore the total number of copies of F_5 in $\mathcal{H}$ is at most $(4\varepsilon n/3)(3-\varepsilon)(n/3)^2 < \varepsilon n^3$.

- Our results for L_r can be extended to the k-uniform case without too much difficulty. We describe some of the details below. For $r > k \geq 2$, Let L_r^k be the k-graph obtained from the complete graph K_r by enlarging each edge with a set of $k-2$ new vertices. These sets of new vertices are disjoint for each edge, so L_r^k has $r + (k-2)\binom{r}{2}$ vertices and $\binom{r}{2}$ edges. Write $T_r^k(n)$ for the complete r-partite k-graph with the maximum number of edges. So $T_r^k(n)$ has vertex partition $V_1 \cup \cdots \cup V_r$, where $n_i := |V_i| = \lfloor (n+i-1)/r \rfloor$, and all k-sets with at most one point in each V_i. Define

$$t_r^k(n) := |T_r^k(n)| = \sum_{S \in \binom{[r]}{k}} \prod_{i \in S} n_i.$$

Every set of $r+1$ vertices in $T_r^k(n)$ contains two vertices in the same part, and these two vertices lie in no edge. Consequently, $L_{r+1}^k \not\subset T_r^k(n)$. The author [17] conjectured, and Pikhurko [24] proved, that among all n vertex k-graphs containing no copy of L_{r+1}^k ($r \geq k \geq 2$ fixed, n sufficiently large), the unique one with the maximum number of edges is $T_r^k(n)$. Define $c_{r+1}^k(n)$ to be the minimum number of copies of L_{r+1}^k in a k-graph obtained from $T_r^k(n)$ by adding one edge. The following theorem can be proved by extending the ideas of [24] and Theorem 7's proof in the obvious way.

Theorem 19. *Fix $r \geq k \geq 3$. For every $\varepsilon > 0$ there exists $\delta > 0$ and n_0 such that the following holds for $n > n_0$. Let $\mathcal{H}$ be a k-graph with $t_r^k(n) + q$ edges where $q < \delta n^{k-1}$. Then the number of copies of L_{r+1}^k in $\mathcal{H}$ is at least $q(1-\varepsilon)c_{r+1}^k(n)$. The expression q is sharp for $1 \leq q < \delta n^{k-1}$. Moreover, if the number of copies is less than $\delta n^{r+(k-2)\binom{r+1}{2}}$, then there is a collection of q distinct edges that each lie in $(1-\varepsilon)c_{r+1}^k(n)$ copies of L_{r+1}^k with no two of these edges accounting for the same copy of L_{r+1}^k.*

The exact result for this situation can also be proved using the same methods.

Alon and Pikhurko [1] proved that $\mathrm{ex}(n, L^k(G)) = t_r^k(n)$ (for $n > n_0$) where $L^k(G)$ is the k-graph obtained from an r-color critical graph G by expanding each edge of G by a new set of $k-2$ vertices. In [13] we had proved the corresponding counting result for $L^2(G)$ and those ideas combined with the ones in this paper can be used to give similar results for $L^k(G)$.

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