MATHEMATICS 586: Homework 2
University of Illinois at Chicago (Professor Nicholls)
Spring 2024

Due Friday, February 23 by 2pm.

1. Wilmott, Howison, & Dewynne, Chapter 4, # 1.
2. Wilmott, Howison, & Dewynne, Chapter 5, # 2.
3. Wilmott, Howison, & Dewynne, Chapter 5, # 3.
4. Consider the function $f(x) = e^x$.

   (a) Verify computationally that both the "forward difference" approximation

   $$(\delta_+ f)(x; h) := \frac{f(x+h) - f(x)}{h}$$

   and the "backward difference" approximation

   $$(\delta_- f)(x; h) := \frac{f(x) - f(x-h)}{h}$$

   simulate $f'(x)$ to order one in h at $x = 1$. In particular, pick several (four or five) values of $h = h_j$, compute $(\delta_+ f)(1; h_j)$ and $(\delta_- f)(1; h_j)$, compute the errors

   $$\varepsilon_+(h_j) = \frac{|(\delta_+ f)(1; h) - f'(1)|}{|f'(1)|}, \quad \varepsilon_-(h_j) = \frac{|(\delta_- f)(1; h) - f'(1)|}{|f'(1)|},$$

   and then do a least squares fit to the error relation $\varepsilon = C h^r$; you should see $r \approx 1$. To do this fit, use the fact that

   $$\log(\varepsilon) = \log(C) + r \log(h),$$

   so that you can do a *linear* least squares fit to the data $\{\log(h_j), \log(\varepsilon_j)\}$.

   (b) Verify computationally that the "centered difference" approximation

   $$(\delta_0 f)(x; h) := \frac{f(x+h) - f(x-h)}{2h}$$

   simulates $f'(x)$ to order two in h at $x = 1$.

5. Implement the Jacobi, Gauss–Seidel, and Successive Over–Relaxation (SOR) methods for solving tridiagonal systems of linear equations, $M\vec{x} = \vec{b}$. Use the stopping criteria:

   $$\max\left\{ \|\vec{x}^k - \vec{x}^{k-1}\|, \|A\vec{x}^k - \vec{b}\| \right\} < \tau.$$

   Consider the Crank–Nicholson matrix

   $$M = \begin{pmatrix} 1 & 0 & & \dots & & 0 \\ -\mu/2 & (1+\mu) & -\mu/2 & 0 & \dots & 0 \\ & & \ddots & \ddots & & \\ 0 & \dots & 0 & -\mu/2 & (1+\mu) & -\mu/2 \\ 0 & & \dots & & 0 & 1 \end{pmatrix} \in \mathbf{R}^{m \times m},$$

with $\mu = 0.25$, and the vectors $\vec{x} = (\pi, \ldots, \pi)^T \in \mathbf{R}^m$ and $\vec{b} = (\pi, \ldots, \pi)^T \in \mathbf{R}^m$ (Note that these choices give us the solution $\vec{x} = M^{-1}\vec{b}$).

(a) Solve the system $M\vec{x} = \vec{b}$ using these three algorithms with $\tau = 10^{-10}$ and $\omega = 1.5$ (for SOR) for $m = 10, 10^2, 10^3, 10^4$. Report the relative errors. How many iterations did each method require?

(b) For

$$\omega_j = j/50, \quad j = 1, \ldots, 99$$

solve $M\vec{x} = \vec{b}$ ($m = 10^4$, $\tau = 10^{-10}$) using SOR. Report the number of iterations required for each ω_j. For which value(s) of ω_j is the number of iterations minimized?

6. The θ–scheme for the heat equation is

$$-\theta\mu v_{j+1}^{n+1} + (1+2\theta\mu)v_j^{n+1} - \theta\mu v_{j-1}^{n+1} = (1-\theta)\mu v_{j+1}^n + (1-2(1-\theta)\mu)v_j^n + (1-\theta)\mu v_{j-1}^n,$$

where $\mu = (\Delta t)/(\Delta x)^2 > 0$, $0 \leq \theta \leq 1$. (Notice that if $\theta = 1/2$ you get Crank–Nicholson).

(a) If $1/2 \leq \theta \leq 1$, show that this scheme is unconditionally stable.

(b) If $0 < \theta < 1/2$, show that this scheme is stable if

$$\mu \leq \frac{1}{2(1-2\theta)}.$$

7. Consider the heat equation:

$$\begin{aligned} &\partial_t u = \partial_x^2 u \\ &u(x,0) = \sin(2x) \\ &u(0,t) = u(\pi,t) = 0. \end{aligned}$$

(a) Implement the Crank–Nicholson Finite Difference scheme to approximate the solution to the heat equation above. Keeping the ratio $\lambda = (\Delta t)/(\Delta x)$ *fixed* at 1.0, choose four or five $(\Delta t, \Delta x)$ pairs to demonstrate the convergence of your code at $T = 1$ by measuring

$$\varepsilon(\Delta t, \Delta x) := \frac{\left\| v^N - u_{exact}(\cdot, T) \right\|_{\Delta x}}{\left\| u_{exact}(\cdot, T) \right\|_{\Delta x}}.$$

(b) Identify the order of convergence of this method. Why do you get this answer?