Rational Points of a Definable Set

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Overview

1. Background and Statement of the Theorem
2. r-parameterization
3. Diophantine Approximation
4. The Main Lemma
5. Proof of the Theorem

o-Minimality

Definition

An infinite structure $\langle M, <, \ldots \rangle$ which is totally ordered by $<$ is *o-minimal* if every definable subset of M is a finite union of points and intervals.

Example

$\overline{\mathbb{R}} := \langle \mathbb{R}, +, \cdot, -, 0, 1, < \rangle$

o-Minimality

Definition

An *o-minimal expansion* of an o-minimal $\mathcal{L}$-structure $\mathcal{M}$ is an $\mathcal{L}'$-structure $\mathcal{M}' = \langle \mathcal{M}, \ldots \rangle$ where $\mathcal{L}' \supset \mathcal{L}$, and $\mathcal{M}'$ is o-minimal.

Example

$\mathbb{R}_{exp} = \langle \overline{\mathbb{R}}, exp \restriction_{[0,1]} \rangle$ is an o-minimal expansion of $\overline{\mathbb{R}}$

Example (Non-Example)

$\mathbb{R}_{\sin} = \langle \overline{\mathbb{R}}, \sin \rangle$ is not an o-minimal expansion of $\overline{\mathbb{R}}$, since $\{x \mid \sin(x) = 0\}$ is definable but it is a discrete infinite set, so it is not o-minimal.

$\overline{\mathbb{R}}$ is o-Minimal

Definition

$X \subset \mathbb{R}^n$ is *semi-algebraic* if it is defined by a finite boolean combination of $f(x_1, \ldots, x_n) = 0$ and $g(x_1, \ldots, x_n) > 0$ for $f, g \in \mathbb{R}[x_1, \ldots, x_n]$.

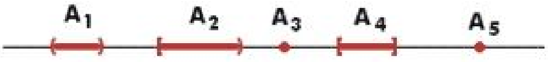
By quantifier elimination, semi-algebraic sets are exactly the definable sets in $\overline{\mathbb{R}}$.

So every definable $X \subset \mathbb{R}$ must be a finite union of intervals and points, since $f, g \in \mathbb{R}[x]$ have only finitely many roots (in $\mathbb{R}$).

Algebraic Transcendental Parts

Definition

For any $X \subset \mathbb{R}^n$, the *algebraic part* of X, X^{alg}, is the union of all infinite connected semi-algebraic subsets of X. $X^{tr} = X \setminus X^{alg}$ is the *transcendental part* of X.



Example

Let $X \subset \mathbb{R}$ be the definable set pictured above. Then $X^{alg} = A_1 \cup A_2 \cup A_4$, the infinite connected components, and $X^{tr} = A_3 \cup A_5$, the discrete points.

Note: By o-minimality, it is important that we include "infinite" in the definition of X^{alg}, otherwise we would have $X^{alg} = X$ for every definable $X \subset \mathbb{R}$.

Height

Definition

For $\frac{a}{b} \in \mathbb{Q}$, $a, b \in \mathbb{Z}$, $b > 0$ and $gcd(a, b) = 1$,

$$H(\frac{a}{b}) = \max\{|a|, |b|\}$$

is the *height* of $\frac{a}{b}$. For $q_1, \ldots, q_n \in \mathbb{Q}$, $H(q_1, \ldots, q_n) = \max_{1 \leq i \leq n} \{H(q_i)\}$.

Definition

For $X \subset \mathbb{R}^n$, $X(\mathbb{Q}, T) := \{\overline{q} \in X \cap \mathbb{Q}^n | H(\overline{q}) \leq T\}$.

Why do we care about height?

- Previously, for $S \subset \mathbb{R}^n$ and $t \in \mathbb{Z}$, we wanted to find the number of integer points in the *dilation* of S by t, that is, $\{(tx_1, \ldots, tx_n) | (x_1, \ldots, x_n) \in S\}$.
- This is equivalent to finding points in S of the form $(\frac{m_1}{t}, \ldots, \frac{m_n}{t})$ where $m_1, \ldots, m_n \in \mathbb{Z}$.
- Height is a generalization of this idea.

Goal

We're interested in establishing bounds on $|X^{tr}(\mathbb{Q}, T)|$ under natural geometric conditions on X and seeing how fast this set grows as we change T, with the guiding idea that transcendental sets should contain few rational points.

The Theorem

Theorem (Pila-Wilkie 2006)

For $X \subset \mathbb{R}^n$ *definable in an o-minimal expansion of* $\overline{\mathbb{R}}$*, for any* $\epsilon > 0$*,* $\exists c > 0$ *such that for all* $T \geq 1$*,*

$$\#X^{tr}(\mathbb{Q}, T) \leq cT^{\epsilon}.$$

Theorem (Uniform Pila-Wilkie)

Let $(X_a)_{a \in A}$ *be a family of subsets of* $\mathbb{R}^n$ *definable in an o-minimal expansion of* $\overline{\mathbb{R}}$*. For any* $\epsilon > 0$ *there is another definable family* $(Y_a)_{a \in A}$*, with* $Y_a \subset X_a^{alg}$*, and a constant* $c > 0$ *such that for all* $T \geq 1$*,*

$$\#(X_a \setminus Y_a)(\mathbb{Q}, T) < cT^{\epsilon}.$$

Note: $Y_a \subset X_a^{alg}$ means that $X_a \setminus Y_a \supseteq X_a^{tr}$, so the uniform version implies the original version.

r-Parameterization

Definition

Let $X \subset M^n$ be definable. A definable function $\phi : (0,1)^{\dim X} \to X$ is called a *partial parameterization* of X. A finite set S of partial parameterizations of X is called a *parameterization* of X if $\bigcup_{\phi \in S} Im(\phi) = X$.

Definition

A parameterization S of a definable set $X \subset M^n$ is called an *r-parameterization* if every $\phi \in S$ is $C^{(r)}$ and $\phi^{(\alpha)}$ is strongly bounded for $\alpha \in \mathbb{N}^{\dim X}$, $|\alpha| \leq r$ where $|\alpha|$ is the sum of the coordinates of α.

r-Parameterization

Theorem

For any $r \in \mathbb{N}$ and any strongly bounded definable set X, there exists an r-parameterization of X.

Corollary (Uniform Version)

Let $n, m, r \geq 1$ and suppose $X \subset (0,1)^n \times M^m$ is a definable family. Then there exists $N \in \mathbb{N}$ and, for each $\overline{y} \in M^m$, a set $S_{\overline{y}}$ of N functions, each mapping $(0,1)^{\dim X_{\overline{y}}} \to X_{\overline{y}}$ and each of class $C^{(r)}$ such that

(1) $\displaystyle\bigcup_{\phi \in S_{\overline{y}}} Im(\phi) = X_{\overline{y}}$, *and*

(2) $|\phi^{(\alpha)}(\overline{x})| \leq 1$ *for each* $\phi \in S_{\overline{y}}$, $\alpha \in \mathbb{N}^{\dim X_{\overline{y}}}$, $|\alpha| \leq r$ *for all* $\overline{x} \in (0,1)^{\dim X_{\overline{y}}}$.

Diophantine Approximation

From now on, by definable we mean definable in an o-minimal expansion of $\overline{\mathbb{R}}$.

Recall the following definitions:

Definition

A *hypersurface of degree d* in $\mathbb{R}^n$ is a set of the form $\{\overline{x} \in \mathbb{R}^n | f(\overline{x}) = 0\}$, $f \in \mathbb{R}[\overline{x}]$ non-zero, $\deg(f) = d$.

Definition

The *fiber dimension* of a definable family $Z \subset \mathbb{R}^n \times \mathbb{R}^m$ is the maximal dimension of a fiber of Z.

Diophantine Approximation

Theorem (Bombieri-Pila)

Let $k, n \in \mathbb{N}$, $k < n$. For each $d \in \mathbb{N}$, $d \geq 1$, there is $r = r(k, n, d)$, and positive constants $\epsilon = \epsilon(k, n, d)$ and $c = c(k, n, d)$ such that the following holds.
Let $\phi : (0, 1)^k \to \mathbb{R}^n$ be a function of class $C^{(r)}$ with $|\phi^{(\alpha)}(\overline{x})| \leq 1$. For $T \geq 1$, $(Im(\phi))(\mathbb{Q}, T)$ is contained in the union of at most cT^ϵ hypersurfaces of degree at most d. Furthermore, $\epsilon \to 0$ as $d \to \infty$.

Note: $\epsilon \to 0$ as $d \to \infty$, since as we get more complicated hypersurfaces (that is, of higher degree), we will not need as many.

The Main Lemma

Lemma (Main Lemma)

Let $X \subset (0,1)^n \times M^m$ be a definable family of fiber dimension $k < n$. For ease of notation, we let $A = \pi_2(Z)$.
Let $\epsilon > 0$ be given. There exists $d \in \mathbb{N}$ and $K > 0$ such that for any $a \in A$, $T \geq 1$, $X_a(\mathbb{Q}, T)$ is contained in the union of at most KT^ϵ hypersurfaces of degree d.

Proof of the Main Lemma

Proof.

- Given $\epsilon > 0$, let d be such that $\epsilon(k, n, d)$ from the Bombieri-Pila Theorem is $< \epsilon$, and let $r = r(k, n, d)$.
- There exists N, and for each $a \in A$, S_a which is an r-parameterization of X_a with $|S_a| \leq N$.
- By the Bombieri-Pila Theorem, $(Im(\phi))(\mathbb{Q}, T)$ is contained in at most cT^ϵ hypersurfaces of degree at most d, for some c.
- Let $K = N \cdot c$.
- $X_a(\mathbb{Q}, T)$ is in at most KT^ϵ hypersurfaces of degree $\leq d$.

□

The Theorem

Theorem (Uniform Pila-Wilkie)

Let $(X_a)_{a\in A}$ *be a family of subsets of* $\mathbb{R}^n$ *definable in an o-minimal expansion of* $\overline{\mathbb{R}}$*. For any* $\epsilon > 0$ *there is another definable family* $(Y_a)_{a\in A}$*,* $Y_a \subset X_a^{alg}$ *and a constant* $c > 0$ *such that for all* $T \geq 1$*,*

$$\#(X_a \setminus Y_a)(\mathbb{Q}, T) < cT^{\epsilon}.$$

Proof of the Theorem

Note: $x \mapsto -x$ and $x \mapsto x^{-1}$ preserves height and infinite connected components, so without loss of generality, we may assume that $X \subset [0,1]^n \times \mathbb{R}^m$.

Lemma

If the theorem holds for definable families of the form $X \subset (0,1)^n \times \mathbb{R}^m$, then it holds for definable families of the form $X \subset [0,1]^n \times \mathbb{R}^m$.

So, without loss of generality, we may further assume that $X \subset (0,1)^n \times \mathbb{R}^m$.

Proof of the Theorem

Let $A = \pi_2(X)$. We proceed by induction on $k = \max_{a \in A} \dim X_a$.

k=0

- X_a is finite by o-minimality.
- By uniform bounding, there is $N \in \mathbb{N}$ such that $|X_a| < N$ for all $a \in A$.
- Let $Y_a := \emptyset \subset X_a^{alg}$. Let $c = N$.
- For $T \geq 1$, $\#(X_a \setminus Y_a)(\mathbb{Q}, T) \leq |X_a| \leq N \leq cT^{\epsilon}$.

Proof of the Theorem

$0 < k < n$

- Let d be as in the Main Lemma for $\frac{\epsilon}{2}$.
- Use $b \in \mathbb{R}^j$ to index hypersurfaces of degree $\leq d$ (for some sufficiently large j), call these H_b.
- Define $Y \subset (0,1)^n \times (\mathbb{R}^m \times \mathbb{R}^j)$ by $Y_{ab} = X_a \cap H_b$.
- $\dim Y_{ab} < \dim X_a \leq k$ for all a, b, since H_b is a hypersurface.
- By IH, there is $Z \subset (0,1)^n \times (\mathbb{R}^m \times \mathbb{R}^j)$, $Z_{ab} \subset Y_{ab}^{alg}$, $\#(Y_{ab} \setminus Z_{ab})(\mathbb{Q}, T) < cT^{\frac{\epsilon}{2}}$
- Let $Z_a = \bigcup_{b \in \mathbb{R}^j} Z_{ab}$.
- By the Main Lemma, for every $a \in A$ there is $B'_a \subset \mathbb{R}^j$, $|B'_a| \leq KT^{\frac{\epsilon}{2}}$ such that $X_a(\mathbb{Q}, T)$ is contained in $\bigcup_{b \in B'_a} H_b$.
- Hence, $\#(X_a \setminus Z_a)(\mathbb{Q}, T) \leq cT^{\frac{\epsilon}{2}} KT^{\frac{\epsilon}{2}} = CT^{\epsilon}$ for $C = cK$.

Proof of the Theorem

We must handle this case separately, since the Main Lemma requires $k < n$.

$k = n$

- For $y \in A$, let Z_y be the set of interior points of X_y. This is definable.
- Each $x \in Z_y$ is in an open neighborhood contained in $Z_y \subset X_y$, and thus, an infinite connected component. So $Z_y \subset X_y^{alg}$.
- For $y \in A$ with $\dim X_y = n$, the interior of $X_y \neq \emptyset$ since $X_y \subset (0,1)^n$, so $\dim X_y \setminus Z_y \leq n - 1$.
- The family $X \setminus Z$ has fiber dimension $\leq n - 1$, so use IH to get the result.

Thank You!