The Pila-Wilkie Theorem
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Abstract

We will go over the proof of the main theorem in Pila and Wilkie's 2006 paper *The Rational Points of a Definable Set*.

It says that for $X \subset \mathbb{R}^n$ which is definable in an o-minimal expansion of $\overline{\mathbb{R}}$, there are (in a suitable sense) very few rational points of X which do not lie on some connected semialgebraic subset of X of positive dimension.

1 Definitions and Statement of the Theorem

Definition 1. For any $X \subset \mathbb{R}^n$, the *algebraic part* of X, X^{alg}, is the union of all infinite connected semi algebraic subsets of X. $X^{tr} = X \setminus X^{alg}$ is the *transcendental part* of X.

Note: By o-minimality, $X \subset \mathbb{R}$ definable, the "infinite" part of this definition matters, since if we also include the finite components (points), we would always get $X^{alg} = X$.

Recall: $X \subset \mathbb{R}^n$ is *semi-algebraic* if it is a finite boolean combination of $f(x_1, \ldots, x_n) = 0$ and $g(x_1, \ldots, x_n) > 0$ for $f, g \in \mathbb{R}[x_1, \ldots, x_n]$.
Here are some examples:

1. If $X \subset \mathbb{R}$ is semi-algebraic, since $\mathbb{R}$ is o-minimal and X is definable, $X = I_1 \cup \ldots \cup I_j \cup Y_1 \cup Y_2$ where $I_1, \ldots, I_j$ are open intervals, $Y_1 \subset \partial(I_1) \cup \ldots \cup \partial(I_j)$, and Y_2 is a finite set of points, disjoint from $\overline{I_1 \cup \ldots \cup I_j \cup Y_1}$. Then $X^{alg} = I_1 \cup \ldots I_j \cup Y_1$.
2. If $X \subset \mathbb{R}^n$ is open, $X^{alg} = X$.
3. $X = \{(x, y, z) \in \mathbb{R}^3 | x > 0, x^y = z\}$, then $X^{alg} = \{(x, q, z) | x, z \in \mathbb{R}, q \in \mathbb{Q}, x > 0, x^q = z\}$. Note that this is a countably infinite union of algebraic curves.

Definition 2. For $\frac{a}{b} \in \mathbb{Q}$, $a, b \in \mathbb{Z}$, $b > 0$ and $gcd(a, b) = 1$, $H(\frac{a}{b}) = \max\{|a|, |b|\}$ is the *height* of $\frac{a}{b}$. For $q_1, \ldots, q_n \in \mathbb{Q}$, $H(q_1, \ldots, q_n) = \max_{1 \le i \le n}\{H(q_i)\}$.

Definition 3. For $X \subset \mathbb{R}^n$, $X(\mathbb{Q}, T) := \{\overline{q} \in X \cap \mathbb{Q}^n | H(\overline{q}) \le T\}$.

Previously, we were interested in finding, for $X \subset \mathbb{R}^n$ and $t \in \mathbb{Z}$, the number of integer points in $\{(tx_1, \ldots, tx_n) : (x_1, \ldots, x_n) \in X\}$ (this was called the *dilation* of X by t). There were some different bounds established for when X was the graph of some $f : [0,1] \to \mathbb{R}$ (obviously $\#(tX \cap \mathbb{Z}) \leq t+1$ here) using various smoothness conditions on f.

Looking for the integer points in tX is like looking for the points of the form $\frac{m}{t}$ for $m \in \mathbb{Z}$ in X, so we generalize this using this notion of height.

We're interested in establishing bounds on $|X^{tr}(\mathbb{Q}, T)|$ under natural geometric conditions on X and seeing how fast it grows as we change T with the guiding idea that transcendental sets should contain few rational points.

Recall: $\overline{\mathbb{R}} := \langle \mathbb{R}; +, \cdot, -, 0, 1, < \rangle$. Structures like $\mathbb{R}_{rexp} = \langle \overline{\mathbb{R}}, exp \upharpoonright_{(0,1)} \rangle$ are o-minimal expansions of $\overline{\mathbb{R}}$.

Theorem 1 (Pila-Wilkie 2006)**.** *For $X \subset \mathbb{R}^n$ definable in an o-minimal expansion of $\overline{\mathbb{R}}$, for any $\epsilon > 0$, $\exists c(\epsilon) > 0$ such that for all $T \geq 1$, $\#X^{tr}(\mathbb{Q}, T) \leq c(\epsilon) T^{\epsilon}$.*

Theorem 2 (Uniform Pila-Wilkie)**.** *Let $(X_a)_{a \in A}$ be a family of subsets of $\mathbb{R}^n$ definable in an o-minimal expansion of $\overline{\mathbb{R}}$. For any $\epsilon > 0$ there is another definable family $(Y_a)_{a \in A}$, $Y_a \subset X_a^{alg}$ and a constant $c(\epsilon) > 0$ such that for all $T \geq 1$,*

$$\#(X_a \setminus Y_a)(\mathbb{Q}, T) < c(\epsilon) T^{\epsilon}.$$

2 r-parameterization

Let $\mathcal{S}$ be an o-minimal expansion of a real closed field $\mathcal{M}$. Ultimately, we're interested in o-minimal expansions of $\mathbb{R}$, but these results hold for all real closed fields.

When we say definable, we mean definable in $\mathcal{S}$.

Definition 4. $a \in M$ is *strongly bounded* if there is some $c \in \mathbb{N}$ such that $|a| \leq c$. $a = (a_1, \ldots, a_n) \in M^n$ is *strongly bounded* if each a_i is, and $A \subset M^n$ is *strongly bounded* if there is a fixed finite bound for all coordinates of all elements of A. A definable function is *strongly bounded* if its graph is.

Definition 5. Let $X \subset M^n$ be definable. A definable function $\phi : (0,1)^l \to X$ where $l = \dim X$ is called a *partial parameterization* of X. A finite set S of partial parameterizations of X is called a *parameterization* of X if $\bigcup_{\phi \in S} Im(\phi) = X$.

Definition 6. A parameterization S of a definable set $X \subset M^n$ is called an *r-parameterization* if every $\phi \in S$ is $C^{(r)}$ and $\phi^{(\alpha)}$ is strongly bounded for $\alpha \in \mathbb{N}^{\dim X}$, $|\alpha| \leq r$ where $|\alpha|$ is the sum of the coordinates of α.

Theorem 3. *For any $r \in \mathbb{N}$ and any strongly bounded definable set X, there exists an r-parameterization of X.*

Corollary 4. *Let $m, r \geq 1$, $X \subset (0,1)^m$ definable. Then there exists a finite set S of functions each mapping $(0,1)^{\dim X} \to X$ and of class $C^{(r)}$ such that $\bigcup_{\phi \in S} Im(\phi) = X$ and $|\phi^{(\alpha)}(\overline{x})| \leq 1$ for $\phi \in S$, $\alpha \in \mathbb{N}^{\dim X}$, $|\alpha| \leq r$ for all $\overline{x} \in (0,1)^{\dim X}$.*

Proof. Let S^* be an r-parameterization of X, as given by the parameterization theorem. So everything holds, except that we have $|\phi^{\alpha}(\overline{x})| \leq c$ for some finite c (not necessarily 1). Cover $(0,1)^{\dim X}$ with $(2c)^{\dim X}$ cubes of side $\frac{1}{c}$. For each such cube K, let $\lambda_K : (0,1)^{\dim X} \to K$ be the obvious linear bijection. $S = \{\phi \circ \lambda_K | \phi \in S^*, K \text{ a cube}\}$ works. □

Corollary 5. *[Uniform Version] Let $n, m, r \geq 1$ and suppose $X \subset (0,1)^n \times M^m$ is a definable family. Then there exists $N \in \mathbb{N}$ and, for each $\overline{y} \in M^m$, a set $S_{\overline{y}}$ of N functions, each mapping $(0,1)^{\dim X_{\overline{y}}} \to X_{\overline{y}}$ and each of class $C^{(r)}$ such that*

(1) $\displaystyle\bigcup_{\phi \in S_{\overline{y}}} Im(\phi) = X_{\overline{y}}$, *and*

(2) $|\phi^{(\alpha)}(\overline{x})| \leq 1$ *for each* $\phi \in S_{\overline{y}}$, $\alpha \in \mathbb{N}^{\dim X_{\overline{y}}}$, $|\alpha| \leq r$ *for all* $\overline{x} \in (0,1)^{\dim X_{\overline{y}}}$.

Proof. Suppose not.

Let $\Gamma_N(\overline{v})$ be the set of formulas saying that for each N sized set of functions satisfying (2), the union of the images of those functions is not equal to $X_{\overline{v}}$ (that is, (1) fails). Let $\Gamma = \displaystyle\bigcup_{N \in \mathbb{N}} \Gamma_N(\overline{v})$. Then, for a finite subset of Γ, there is some N such that it is contained in $\Gamma_1 \cup \ldots \cup \Gamma_N$. Thus, by assumption, there must be some $\overline{v}$ which witnesses it (or else this would be the N required for the corollary to hold). Hence, since $\Gamma(\overline{v})$ is finitely satisfiable, it is satisfiable. Let $\mathcal{N} \succeq \mathcal{M}$ be a saturated elementary extension (and thus, $\mathcal{N}$ is a real closed field) and let $\overline{v} \in N^m$ witness $\Gamma(\overline{v})$. But then consider $X_{\overline{v}}$ in $\mathcal{N}$. By the previous corollary, there should be some N such that there is a set of N functions from $(0,1)^{\dim X_{\overline{v}}} \to X_{\overline{v}}$ satisfying (1) and (2). $\Rightarrow\Leftarrow$ □

Note that in the previous proof, it was important that we had proved the parameterization results for arbitrary real closed fields and o-minimal expansions, not just $\overline{\mathbb{R}}$.

3 Diophantine Approximation

We now restrict our attention to $\mathcal{S}$ an o-minimal expansion of $\overline{\mathbb{R}}$. By definable we still mean definable in $\mathcal{S}$.

The following is a result of Bombieri and Pila. First we require a few definitions.

Definition 7. A *hypersurface of degree d* in $\mathbb{R}^n$ is a set of the form $\{\overline{x} \in \mathbb{R}^n | f(\overline{x}) = 0\}$, $f \in \mathbb{R}[\overline{x}]$ non-zero, $\deg(f) = d$.

Definition 8. The *fiber dimension* of a definable family $Z \subset \mathbb{R}^n \times \mathbb{R}^m$ is the maximal dimension of a fiber of Z.

Theorem 6 (Bombieri-Pila)**.** *Let $k, n \in \mathbb{N}$, $k < n$. For each $d \in \mathbb{N}$, $d \geq 1$, there is $r = r(k,n,d)$, and positive constants $\epsilon = \epsilon(k,n,d)$ and $c = c(k,n,d)$ such that the following holds.*

If $\phi : (0,1)^k \to \mathbb{R}^n$ is a function of class $C^{(r)}$ with $|\phi^{(\alpha)}(\overline{x})| \leq 1$ for $\overline{x} \in (0,1)^k$, $\alpha \in \mathbb{N}^k$, $|\alpha| \leq r$. Let $X = \phi((0,1)^k) = Im(\phi)$, and let $T \geq 1$. Then $X(\mathbb{Q}, T)$ is contained in the union of at most cT^{ϵ} hypersurfaces of degree at most d. Furthermore, $\epsilon \to 0$ as $d \to \infty$.

Note that $\epsilon \to 0$ as $d \to \infty$, since as we get more complicated hypersurfaces (that is, of higher degree), we don't need as many.

4 Main Lemma

Lemma 7 (Main Lemma)**.** *Let $X \subset (0,1)^n \times M^m$ be a definable family of fiber dimension $k < n$. For ease of notation, we let $A = \pi_2(Z)$ and let X_a for $a \in A$ denote the fibers of X. Let $\epsilon > 0$ be given. There exists $d = d(\epsilon, k, n) \in \mathbb{N}$ and $K = K(X, \epsilon) > 0$ such that for any $a \in A$, $T \geq 1$, $X_a(\mathbb{Q}, T)$ is contained in the union of at most KT^ϵ hypersurfaces of degree d.*

Proof. Let $\epsilon > 0$ be given. Choose d from the previous theorem such that $\epsilon(k, n, d) \leq \epsilon$ and set $r = r(k, n, d)$ from the previous theorem. By Corollary **??**, there exists N such that for every $a \in A$, there is an r-parameterization of X_a, call it S_a, of at most N maps $\phi(0,1)^k \to \mathbb{R}^n$ of class $C^{(r)}$ with $|\phi^{(\alpha)}(\overline{x})| \leq 1$ for $\alpha \in \mathbb{N}^k$, $|\alpha| \leq r$. So for each $\phi \in S_a$, by the previous proposition, $Im(\phi)(\mathbb{Q}, T)$ is contained in at most $c(k, n, d)T^{\epsilon(k,n,d)}$ hypersurfaces of degree at most d. So let $K = N \cdot c$. Hence, since $\epsilon(k, n, d) < \epsilon$, $X_a(\mathbb{Q}, T)$ is contained in at most KT^ϵ hypersurfaces of degree at most d.

□

5 Proof of the Theorem

Note that rational points of height at most T are stable under maps of the form $x \mapsto \pm x^{\pm 1}$, since $H(\frac{a}{b}) = \max(|a|, |b|) = H(-\frac{a}{b}) = H(\frac{b}{a})$. These maps also preserve X^{alg}, so we may assume $X \subset [0,1]^n \times \mathbb{R}^m$.

Lemma 8. *If the theorem holds for definable families of the form $X \subset (0,1)^n \times \mathbb{R}^m$, then it holds for definable families of the form $X \subset [0,1]^n \times \mathbb{R}^m$.*

Proof. Let $X \subset [0,1]^n \times \mathbb{R}^m$ be definable. Let $\epsilon > 0$ be given.

For $\alpha \subsetneqq \{1, \ldots, n\}$, and $\gamma \in \{0,1\}^{|\alpha|}$, let $X_{a,\alpha,\gamma}$ be X_a intersected with the set of n-tuples with γ in the positions specified by α, and elements of $(0,1)$ in the other positions.

For example, if $n = 5$, $\alpha = \{2, 4, 5\}$, $\gamma = (0, 0, 1)$, then if X_a is just $[0,1]^5$ for some a, an element of $X_{a,\alpha,\gamma}$ could be something like $(\frac{1}{2}, 0, \frac{1}{4}, 0, 1)$.

So $X_a = \bigcup_{\alpha \subsetneqq \{1,\ldots,n\}} (\bigcup_{\gamma \in \{0,1\}^{|\alpha|}} X_{a,\alpha,\gamma}) \cup (X_a \cap (\{0,1\})^n)$.

For each $\alpha \subsetneqq \{1, \ldots, n\}$ and $\gamma \in \{0,1\}^{|\alpha|}$, consider the family $X_{\alpha,\gamma}$ where $(X_{\alpha,\gamma})_a = X_{a,\alpha,\gamma}$. We can view this as a definable family in $(0,1)^{n-|\alpha|} \times \mathbb{R}^m$, so by assumption, there is a definable family $Y_{\alpha,\gamma}$ with $Y_{a,\alpha,\gamma} \subset X^{alg}_{a,\alpha,\gamma}$ and $c_{\alpha,\gamma}$ such that for all T, $\#(X_{a,\alpha,\gamma} \setminus Y_{a,\alpha,\gamma})(\mathbb{Q}, T) \leq c_{\alpha,\gamma}T^\epsilon$. Note that when we are talking about the heights here, we are talking about the heights of the coordinates not equal to 0 or 1. But by including these coordinates, we do not change whether or not the tuple is counted: If the original coordinates were not rational, the new tuple is not either. If it is, then since $T \geq 1$, including 0's and 1's will not change whether or not the tuple's height is $\leq T$.

So, let $C = \sum_{\alpha \subsetneqq \{1,\ldots,n\}} \sum_{\gamma \in \{0,1\}^{|\alpha|}} c_{\alpha,\gamma} + 2^n$.

Let $Y_a = \bigcup_{\alpha \subsetneqq \{1,\ldots,n\}} (\bigcup_{\gamma \in \{0,1\}^{|\alpha|}} Y_{a,\alpha,\gamma})$. This is definable, and $Y_a \subset X^{alg}_a$, since of $x \in Y_a$, x is in an infinite connected component of $X_{a,\alpha,\gamma} \subset X_a$. Thus, for $a \in A$, $(X_a \setminus Y_a)(\mathbb{Q}, T) \subset$

$\bigcup_{\alpha \subsetneq \{1,\ldots,n\}} (\bigcup_{\gamma \in \{0,1\}^{|\alpha|}} (X_{a,\alpha,\gamma} \setminus Y_{a,\alpha,\gamma})(\mathbb{Q},T)) \cup (X_a \cap (\{0,1\})^n)$, and thus $\#(X_a \setminus Y_a)(\mathbb{Q},T) \leq \sum_{\alpha \subsetneq \{1,\ldots,n\}} \sum_{\gamma \in \{0,1\}^{|\alpha|}} c_{\alpha,\gamma} T^\epsilon + 2^n < CT^\epsilon$, as required.

□

So we may assume $X \subset (0,1)^n \times \mathbb{R}^m$ is definable in some o-minimal expansion of $\overline{\mathbb{R}}$. As before, let $A = \pi_2(X)$, and let $X_a \subset (0,1)^n$ denote the fibers in X for $a \in A$.

Let $k = \max_{a \in A} \dim X_a$. We proceed by induction on k. Let $\epsilon > 0$ be given.

$k = 0$: Then X_a is finite by o-minimality, so by uniform bounding, there is N such that $|X_a| < N$ for all $a \in A$. Let $Y_a = \emptyset$, so $Y_a \subset X_a^{alg}$ for all $a \in A$. Let $c(\epsilon) = N$. Then, for any $T \geq 1$, $\#(X_a \setminus Y_a)(\mathbb{Q},T) \leq |X_a| \leq N \leq c(\epsilon)T^\epsilon$.

$0 < k < n$: Let $d(\frac{\epsilon}{2})$ be as in the Main Lemma. Let j be the number of coefficients of a polynomial of degree $\leq d$ in n variables, so each $b \in \mathbb{R}^j$ corresponds to exactly one such polynomial, and thus, one hypersurface of degree $\leq d$, call it H_b. Consider the family $Y \subset (0,1)^n \times (\mathbb{R}^m \times \mathbb{R}^j)$ where $Y_{ab} = X_a \cap H_b$. This is also definable. Since H_b is a hypersurface, $\dim Y_{ab} < \dim X_a \leq k$ for all a, b. So by IH, there is a definable family $Z \subset (0,1)^n \times (\mathbb{R}^m \times \mathbb{R}^j)$ with $Z_{ab} \subset Y_{ab}^{alg}$ such that $\#(Y_{ab} \setminus Z_{ab})(\mathbb{Q},T) < c' T^{\frac{\epsilon}{2}}$ for all $T \geq 1$ (for some $c' > 0$ depending on $\frac{\epsilon}{2}$).

Let $Z_a = \bigcup_{b \in \mathbb{R}^j} Z_{ab}$. Note that this is still definable since if $\Phi(x,a,b)$ defines Z, $x \in Z_a \Leftrightarrow \exists b \Phi(x,a,b)$. $Z_a \subset \bigcup_{b \in \mathbb{R}^j} Y_{ab}^{alg} \subset \bigcup_{b \in \mathbb{R}^j} (X_a \cap H_b)^{alg} \subset X_a^{alg}$, since if x is in an infinite connected componenet of $X_a \cap H_b$ for some hypersurface H_b, this is contained in an infinite connected component of X_a.

Let $K = K(\frac{\epsilon}{2}, X)$ be from the Main Lemma. So $\forall a \in A$, $X_a(\mathbb{Q},T)$ is contained in $KT^{\frac{\epsilon}{2}}$ many hypersurfaces of degree $\leq d$, and since it is a subset of this, $(X_a \setminus Z_a)(\mathbb{Q},T)$ is.

Let $B' \subset \mathbb{R}^j$ be such that $|B'| \leq KT^{\frac{\epsilon}{2}}$ and $X_a(\mathbb{Q},T)$ is contained $\bigcup_{b \in B'} H_b$. Let $y \in (X_a \setminus Z_a)(\mathbb{Q},T)$ and let $b \in B'$ be such that $y \in H_b$. So $y \in (X_a \cap H_b) \setminus Z_a \subset (X_a \cap H_b) \setminus Z_{ab} = Y_{ab} \setminus Z_{ab}$. So since $y \in \mathbb{Q}^n$ and has height $\leq T$, $y \in (Y_{ab} \setminus Z_{ab})(\mathbb{Q},T)$.

Hence, $\#(X_a \setminus Z_a)(\mathbb{Q},T) \leq cT^{\frac{\epsilon}{2}} KT^{\frac{\epsilon}{2}} = CT^\epsilon$, where $C = cK$ is our required constant.

$k = n$: For $y \in \mathbb{R}^m$, let Z_y be the set of C^1-smooth points of X_y. Note that this is a definable family and that if $x \in Z_y$, then there is an open neighborhood in $\mathbb{R}^m$ and thus, an infinite connected component of X_y containing x. So $Z_y \subset X_y^{alg}$. The dimension of $X_y \setminus Z_y$ is at most $\dim X_y - 1$, so we may use IH on the family $X \setminus Z$ to get the result.