Math 210: Chapter 14 Review

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Line Integrals (14.2)

Let C be a curve given by $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ for $a \leq t \leq b$.

Scalar Valued Function: $f(x, y, z)$

$$\int_C f ds = \int_a^b f(x(t), y(t), z(t))|\mathbf{r}'(t)|dt = \int_a^b f(x(t), y(t), z(t))\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}dt$$

Vector Field: $\mathbf{F} = \langle f, g, h\rangle$

$$\int_C \mathbf{F} \cdot \mathbf{T} ds = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt$$

Surface Integrals (14.6)

Let S be a surface in $\mathbb{R}^3$ given parametrically by $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$ for (u, v) in some region R in the uv-plane.

Let

$$\mathbf{t}_u = \frac{\partial}{\partial u}\mathbf{r} = \langle x_u, y_u, z_u\rangle$$

and

$$\mathbf{t}_v = \frac{\partial}{\partial v}\mathbf{r} = \langle x_v, y_v, z_v\rangle$$

Then $\mathbf{n} = \mathbf{t}_u \times \mathbf{t}_v$ is orthogonal to the surface S.

Scalar Valued Function: f(x,y,z)

Parameterized Surface

The *surface integral* of f over S is

$$\int\int_S f(x, y, z)dS = \int\int_R f(x(u, v), y(u, v), z(u, v))|\mathbf{t}_u \times \mathbf{t}_v|dA$$

Explicitly Defined Surface

For a surface S given explicitly by $z = g(x, y)$ for (x, y) in a region R in the xy-plane, the *surface integral* of f over S is

$$\int\int_S f(x,y,z)dS = \int\int_R f(x,y,g(x,y))\sqrt{z_x^2 + z_y^2 + 1^2}dA$$

Note that this follows from the parametric version, since $\mathbf{n} = \langle -z_x, -z_y, 1\rangle$ is orthogonal to the surface S.

Surface Area

When $f(x, y, z) = 1$,

$$\int\int_S f dS = \int\int_S dS$$

is the surface area of S.

Vector Field: $\mathbf{F} = \langle f, g, h\rangle$

Parameterized Surface

The *surface integral* of $\mathbf{F}$ over S is

$$\int\int_S \mathbf{F}\cdot\mathbf{n}dS = \int\int_R \mathbf{F}\cdot(\mathbf{t}_u \times \mathbf{t}_v)dA$$

Explicitly Defined Surface

For a surface S given explicitly by $z = g(x, y)$ for (x, y) in a region R in the xy-plane, the *s*urface integral of $\mathbf{F}$ is

$$\int\int_S \mathbf{F}\cdot\mathbf{n}dS = \int\int_R \mathbf{F}\cdot\langle -z_x, -z_y, 1\rangle dA = \int\int_R (-fz_x - gz_y + h)dA$$

Conservative Vector Fields (14.3)

A vector field $\mathbf{F}$ is *conservative* if we can find a potential function ϕ such that $\nabla\phi = \mathbf{F}$.

Checking if a Vector Field is Conservative

$\mathbf{F} = \langle f, g, h\rangle$ is conservative if

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$$
$$\frac{\partial f}{\partial z} = \frac{\partial h}{\partial x}$$
$$\frac{\partial g}{\partial z} = \frac{\partial h}{\partial y}$$

If $\mathbf{F} = \langle f, g\rangle$, we only need to check

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$$

Finding Potential Functions in $\mathbb{R}^3$

We want to find ϕ such that $\nabla\phi = \langle \phi_x, \phi_y, \phi_z \rangle = \langle f, g, h \rangle = \mathbf{F}$.

1. Integrate $\phi_x = f$ with respect to x to obtain $\phi = F(x, y, z) + c(y, z)$.
2. Compute $\phi_y = F_y(x, y, z) + c_y(y, z)$ and set it equal to g to solve for $c_y(y, z)$.
3. Integrate $c_y(y, z)$ with respect to y to obtain $c(y, z) = G(y, z) + d(z)$. So at this stage, $\phi = F(x, y, z) + G(y, z) + d(z)$.
4. Compute $\phi_z = F_z(x, y, z) + G_z(y, z) + d'(z)$ and set it equal to h to solve for $d'(z)$.
5. Integrate $d'(z)$ with respect to z to obtain $H(z)$. So $\phi = F(x, y, z) + G(y, z) + H(z)$.

Facts About Conservative Vector Fields

For a curve C with initial point A and terminal point B, if $\mathbf{F}$ is a conservative vector field with potential function ϕ, by the Fundamental Theorem of Line Integrals,

$$\int_C \mathbf{F} \cdot \mathbf{T} ds = \int_C \mathbf{F} \cdot d\mathbf{r} = \phi(B) - \phi(A)$$

If C is a closed curve,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 0$$

If $\mathbf{F}$ is conservative,

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = 0$$

Circulation and Curl

2 Dimensions

Let C be a curve in $\mathbb{R}^2$ given by $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ for $a \leq t \leq b$. Let $\mathbf{F} = \langle f, g \rangle$ be a vector field.

The *curl* (14.2) of $\mathbf{F}$ is

$$\text{curl } \mathbf{F} = \frac{\partial}{\partial x} g - \frac{\partial}{\partial y} f$$

.

The *circulation* (14.2) of $\mathbf{F}$ is the sum of the components of the vectors of $\mathbf{F}$ in the direction of C, over C. It is calculated as follows:

$$\int_C \mathbf{F} \cdot \mathbf{T} ds = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt$$

For a closed curve C, by the circulation form of Green's Theorem (14.4)

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int \int_R \text{curl } \mathbf{F} dA = \int \int_R \frac{\partial}{\partial x} g - \frac{\partial}{\partial y} f dA$$

where R is the region in xy-plane enclosed by C.

3 Dimensions

Let C be a curve in $\mathbb{R}^3$ given by $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ for $a \leq t \leq b$. Let $\mathbf{F} = f, g, h\rangle$ be a vector field.

The *curl* (14.5) of $\mathbf{F}$ is

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix} = (\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z})\mathbf{i} + (\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x})\mathbf{j} + (\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y})\mathbf{k}$$

By Stokes' Theorem, the circulation of $\mathbf{F}$ over C is

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS$$

where $\mathbf{n}$ is the unit vector normal to an oriented surface S in $\mathbb{R}^3$ whose boundary is C, and whose orientation is consistent with C (via the right hand rule).

Flux and Divergence

2 Dimensions

Let C be a curve in $\mathbb{R}^2$ given by $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ for $a \leq t \leq b$. Let $\mathbf{F} = \langle f, g\rangle$ be a vector field.

The *divergence* (14.2) of $\mathbf{F}$ is

$$\text{div } \mathbf{F} = \frac{\partial}{\partial x} f + \frac{\partial}{\partial y} g$$

.

The *flux* (14.2) of $\mathbf{F}$ is the sum of the components of the vectors of $\mathbf{F}$ orthogonal to the direction of C, over C. It is calculated as follows:

$$\int_C \mathbf{F} \cdot \mathbf{n} ds \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \langle y'(t), -x'(t)\rangle dt$$

since $\mathbf{n} = \langle y', -x'\rangle$ is orthogonal to $\mathbf{r}' = \langle x', y'\rangle$ (which can be seen using a dot product).

For a closed curve C, by the flux form of Green's Theorem (14.4)

$$\int_C \mathbf{F} \cdot \mathbf{n} ds = \int_C f dy - g dz = \int\int_R \text{div } \mathbf{F} dA = \int\int_R \frac{\partial}{\partial x} f + \frac{\partial}{\partial y} g dA$$

where R is the region in xy-plane enclosed by C.

3 Dimensions

Let S be an oriented surface in $\mathbb{R}^3$. Let $F = \langle f, g, h\rangle$ be a vector field.

The *divergence* (14.5) of $\mathbf{F}$ is

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial}{\partial x} f + \frac{\partial}{\partial y} g + \frac{\partial}{\partial z} h$$

The *flux* of $\mathbf{F}$ is the sum of the components of the vectors of $\mathbf{F}$ orthogonal to the direction of S, over S. It is calculated by

$$\int\int_S \mathbf{F}\cdot\mathbf{n}dS$$

Note that if S is given explicitly via $z = g(x, y)$ for (x, y) in some region R in $\mathbb{R}^2$, then to calculate the upward flux, we use

$$\int\int_R \mathbf{F}(x, y, g(x, y))\cdot\langle -z_x, -z_y, 1\rangle dA$$

and to calculate the downward flux, we use

$$\int\int_R \mathbf{F}(x, y, g(x, y))\cdot\langle z_x, z_y, -1\rangle dA$$

Extra Credit: By the Divergence Theorem (14.8),

$$\int\int_S \mathbf{F}\cdot\mathbf{n}dS = \int\int\int_D \text{div }\mathbf{F}dV = \int\int\int_D \nabla\cdot\mathbf{F}dV$$

where D is the region in $\mathbb{R}^3$ bounded by S, and $\mathbf{n}$ is the outward unit normal vector.