

Essay 2

Properties of infinite sets can defy intuition. The purpose of this essay is to explore some of these non-intuitive properties by studying a certain set of geometric objects.¹

Toys R Infinite is a store which sells various sets of blocks. One, called the DeLuX Set, consists of infinitely many cubes. The first cube has side length 1 foot, the second $1/2$ feet, the third $1/3$ feet, and in general the n^{th} cube has side length $1/n$ feet. Within the DeLuX set is a subset, called the Starter Set, which consists of the cubes of side length $1/2^n$ feet, where $n = 0, 1, 2, 3, \dots$

Give a convincing argument that it is possible to pack the DeLuX set into a box small enough to fit into a reasonably sized backpack and be carried around campus (we will not worry about weight). Show that if the blocks were stacked one on top of another then the universe as we know it would not be large enough to contain the stack. Show that if the blocks of the Starter Set, which is also an infinite set of blocks, were stacked on top of each other then the stack would fit comfortably into the backpack and find precisely how tall the stack would be.

Here are some other questions to discuss and resolve in your essay. Is it possible to spread the blocks of the DeLuX set on a small floor so that each cube has a face on the floor? If so, estimate how much floor space is required. Is the total surface area of the blocks in the DeLuX set finite? If so, estimate the square footage. You may want to treat the surface area question in terms of painting the blocks. Assume that the paint is special in that it has zero thickness and is sold by the square foot.

If the blocks are stacked on top of each other then the stack would be infinitely tall. Suppose that the blocks are stacked in a hypothetical corner and a thin stripe is painted along the edge of each block which lies in the corner. Then the painted stripe has infinite length. Is the amount of special paint required to paint such a stripe infinite? If not, estimate how many square feet it would take to paint this stripe.

Assume that your readers are familiar with college algebra (Math 121) and have no knowledge of calculus. You may remind your readers of formulas related to geometric series; however you are to refrain from discussing divergent series and definite integrals which give series estimates. Part of your job in writing this essay is to deal with some rather subtle ideas about infinite series using only elementary mathematics.

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¹This assignment is based on Joel Berman's 1999 adaptation of an exercise found in "Writing in the Learning and Teaching of Mathematics" by J. Meier and T. Rishel, MAA, 1998.