Exponential and Logarithmic Functions

MATH 121 **Radford 10/11/02**

The purpose of these notes is to summarize *some* of the basic points of Sections 5.2–5.3A.

1 The general exponential function

Let a be a positive number. Then a^x is meaningful for all real numbers x. Let P be a positive number also. Then the function

$$f(x) = Pa^x$$

is called an *exponential function*. The tables below summarize important properties of $f(x) = a^x$.

$a > 1$:	*Domain* of Pa^x is $(-\infty, \infty)$
	Range of Pa^x is $(0, \infty)$
	Pa^x is *increasing* on $(-\infty, \infty)$
	Rate of increase r, where $a = 1 + r$

In the case $a > 1$ the function $f(x)$ is referred to as an *exponential growth function*.

$a = 1$:	*Domain* of Pa^x is $(-\infty, \infty)$
	Range of Pa^x is the one element set $\{P\}$
	Pa^x is *constant* on $(-\infty, \infty)$

We ignore the case $a = 1$.

$1 > a > 0$:	*Domain* of Pa^x is $(-\infty, \infty)$
	Range of Pa^x is $(0, \infty)$
	Pa^x is *decreasing* on $(-\infty, \infty)$
	Rate of decrease r, where $a = 1 - r$

In the case $1 > a > 0$ the function $f(x)$ is referred to as an *exponential decay function*.

In terms of percentages, the growth rate is $100r\%$. Observe that the *initial value* $f(0) = P$ and that the *base* $a = f(1)/f(0)$. Exponential functions are usually written $f(x) = Pb^{kx}$ where P and b are positive and k is some real number. The base is $a = b^k$.

Let b be *any* fixed positive number. Since the range of b^x is all positive numbers, $a = b^k$ for some real number k. Thus $a^x = (b^k)^x = b^{kx}$ means that $f(x) = Pa^x = Pb^{kx}$. Therefore *any* exponential function can be written $f(x) = Pb^{kx}$, where $P > 0$ and k is some real number.

2 The exponential function $f(x) = Pe^{kx}$

There is a number, denoted by e, which plays a special role in mathematics. The exponential function e^x is one of your calculator functions. Since $2 < e < 3$, $1 < e$ and therefore the exponential function e^x is increasing. By the last paragraph of the preceding section *any* exponential function can be written

$$f(x) = Pe^{kx}$$

for some $P > 0$ and real number k.

Since $e^0 = 1$ and e^x is an increasing function, we conclude that $e^k < 1$ when $k < 0$ and $e^k > 1$ when $k > 0$. Therefore:

$f(x) = Pe^{kx}$ is an exponential *growth* function when $k > 0$;
$f(x) = Pe^{kx}$ is an exponential *decay* function when $k < 0$.

3 Compound interest

Suppose P dollars is invested at an interest rate of $100r\%$ per time period (say a year). After one time period the value of the investment is $P + Pr = P(1+r)$ dollars. Thus after two time periods the value is $(P(1+r))(1+r) = P(1+r)^2$ dollars, after three time periods the value is $(P(1+r)^2)(1+r) = P(1+r)^3$ dollars, ..., after n time periods the value is $P(1+r)^n$ dollars. Thus the value of the initial investment at time period x is calculated by the exponential function

$$f(x) = P(1+r)^x;$$

the initial amount is $f(0) = P$ and the base is $a = 1 + r$.

What is meant by *invested at an interest rate of* $100r\%$ *compounded* n *times per year*? Terminology: $n = 1$, annually; $n = 2$, semi-annually; $n = 4$ quarterly; $n = 52$, weekly, ... The time period is $(\frac{1}{n})^{th}$ of a year and the interest rate (for the time period) is defined to be $\frac{r}{n}$. Thus after ℓ time periods the value of the investment is $P(1+\frac{r}{n})^{\ell}$. Therefore the value of the investment after x years (that is nx time periods) is

$$f(x) = P(1+\frac{r}{n})^{nx} = P((1+\frac{r}{n})^n)^x. \qquad (1)$$

A natural question to ask at this point is what happens if n becomes larger and larger? The question boils down to the behavior of $(1+\frac{r}{n})^n$ as n becomes large. For reasons beyond the scope of this course, this expression tends to e^r. Thus the right hand expression in (1) tends to Pe^{rx} as n becomes large. We are led to the notion of compounding continuously.

The value at time x years of P dollars invested at the annual interest rate $100r\%$ *compounded continuously* is given by $f(x) = Pe^{rx}$.

4 The basic logarithmic functions

Logarithms and exponentials are inverse functions. Specifically, let a be a positive real number which is *not* equal to 1. The function a^x is *one-one* which means that it has a function inverse. Its inverse is denoted $\log_a x$.

The tables below summarize important properties of $f(x) = \log_a x$. See the corresponding tables for Pa^x; here $P = 1$.

$a > 1$:	*Domain* of $\log_a x$ is $(0, \infty)$
	Range of $\log_a x$ is $(-\infty, \infty)$
	$\log_a x$ is *increasing* on $(0, \infty)$

When $a = 1$ the function $a^x = 1$ is constant and therefore has no inverse.

$1 > a > 0$:	*Domain* of $\log_a x$ is $(0, \infty)$
	Range of $\log_a x$ is $(-\infty, \infty)$
	$\log_a x$ is *decreasing* on $(0, \infty)$

By definition of inverse function we have

$$a^x = y \qquad \text{if and only if} \qquad \log_a y = x.$$

Thus

$$a^{\log_a x} = x \qquad \text{and} \qquad \log_a a^x = x.$$

Properties of logarithms mirror those of exponentials. Here is a comparison table.

$a^0 = 1$	$\log_a 1 = 0$
$a^1 = a$	$\log_a a = 1$
$a^{x+y} = a^x a^y$	$\log_a x + \log_a y = \log_a (xy)$
$a^{x-y} = a^x / a^y$	$\log_a x - \log_a y = \log_a (x/y)$
$a^{xy} = (a^x)^y$	$y \log_a x = \log_a (x^y)$

When $a = 10$ we write $\log x = \log_{10} x$ and refer to $\log x$ as a *common logarithm*. When $a = e$ we write $\ln x = \log_e x$ and refer to $\ln x$ as a *natural logarithm*.