MATH 121 **Hour Exam I Review** **Radford** **09/25/02**

Below are some typical homework problems discussed in the context of the the material covered in the course to date. *This is not a complete review.* A solution to the sample Exam 1 Form A is included also.

4.3 45: $p(x) = 2x^5 + 5x^4 - 11x^3 + 4x^2 = x^2(2x^3 + 5x^2 - 11x + 4)$ has root $x = 0$ since $x = x - 0$ divides $p(x)$. The other roots of $p(x)$ are the roots of $q(x) = 2x^3 + 5x^2 - 11x + 4$. Since $q(x)$ has non-zero constant term and *integer coefficients*, the *rational* roots of $q(x)$ aree among $\pm 1/1$, $\pm 1/2$, $\pm 2/2$, $\pm 2/2$, $\pm 4/2$, $\pm 4/2$. Since $q(1) = 0$ it follows that $x = 1$ is a root of $q(x)$. Therefore $x - 1$ divides $q(x)$. By long division, dividing $q(x)$ by $x - 1$ gives quotient $2x^2 + 7x - 4$, and remainder 0 (as it should). Thus

$$p(x) = 2x^5 + 5x^4 - 11x^3 + 4x^2 = x^2(2x^3 + 5x^2 - 11x + 4) = x^2(x - 1)(2x^2 + 7x - 4).$$

The roots of $p(x)$, other than $x = 0, 1$ are the roots of $2x^2 + 7x - 4$. To find them we may use the quadatic formula, or again search among $\pm 1/1$, $\pm 1/2$, $\pm 2/2$, $\pm 2/2$, $\pm 4/2$, $\pm 4/2$. Either method gives $x = -4, 1/2$ as roots. Note that $2x^2 + 7x - 4 = (x - (-4))(2x - 1)$. Thus the roots of $p(x)$ are $x = 0, 1, -4, 1/2$ and a complete factorization of $p(x)$ is

$$2x^5 + 5x^4 - 11x^3 + 4x^2 = x^2(x - 1)(x + 4)(2x - 1).$$

The root $x = 0$ has multiplicity 2 and the others have multiplicity 1.

4.3 27: The polynomial $p(x) = x^4 + x^3 - 19x^2 + 32x - 12$ has *integer* coeficients and non-zero constant term. Thus the *rational* roots of $p(x)$ are among $\pm 1/1$,$\pm 2/1$, $\pm 3/1$, $\pm 6/1$. Since $p(2) = 0$ it follows that $x = 2$ is a root of $p(x)$. Dividing $x - 2$ into $p(x)$ gives a quotient of $x^3 + 3x^2 - 13x + 6$, and remainder 0 (as it should). Therefore

$$p(x) = x^4 + x^3 - 19x^2 + 32x - 12 = (x - 2)(x^3 + 3x^2 - 13x + 6).$$

Therefore the roots of $p(x)$, other than $x = 2$, are the roots of $q(x) = x^3 + 3x^2 - 13x + 6$. To find them, let us try to write $q(x)$ as a product of linear factors. Note that the rational roots of $q(x)$ are among $\pm 1/1$, $\pm 2/1$, $\pm 3/1$, $\pm 6/1$. Since $q(2) = 0$ it follows that 2 is a root of $q(x)$. Check that

$$p(x) = (x - 2)(x^3 + 3x^2 - 13x + 6) = (x - 2)\left((x - 2)(x^2 + 5x - 3)\right).$$

Thus the roots of $p(x)$ are $x = 2$ and the roots of $x^2 + 5x - 3$. By the quadratic formula the roots of the latter are $x = \dfrac{-5 \pm \sqrt{37}}{2}$. Thus the (real) roots of $p(x)$ are 2, $\dfrac{-5 + \sqrt{37}}{2}$, $\dfrac{-5 + \sqrt{37}}{2}$. A factorization of $p(x)$ into linearfactors is

$$p(x) = x^4 + x^3 - 19x^2 + 32x - 12 = (x - 2)^2(x - \frac{-5 + \sqrt{37}}{2})(x - \frac{-5 - \sqrt{37}}{2}).$$

4.3 7: The roots of $f(x) = \frac{1}{12}x^3 - \frac{1}{12}x^2 - \frac{2}{3}x + 1$ are the same as the roots of $12f(x) = x^3 - x^2 - 8x + 12$, which has *integer* coefficients and non-zero constant term. The *ratrional* roots of $12f(x)$ are among $\pm 1/1$, $\pm 2/1$, $\pm 3/1$, $\pm 3/1$, $\pm 12/1$. Since $12f(2) = 0$ it folllowsthat $x = 2$ is a root of $12f(x)$. The reader is left with the verification of

$$12f(x) = x^3 - x^2 - 8x + 12 = (x-2)(x^2 + x - 6).$$

The roots of $x^2 + x - 6$ are $x = -3, 2$ by the quadratic formula, or by noting $x^2 = x - 6 = (x-2)(x+3)$. Thus the roots of $12f(x)$, and thus the roots of $f(x)$, are $x = 2, -3$. Note that $12f(x) = (x-2)(x-2)(x+3)$ so $f(x) = \frac{1}{12}(x-2)^2(x+3)$. The root $x = 2$ has multilpicity 2 and the root $x = -3$ has multiplicity 1.

4.2 15: Dividing $x^2 + 3x - 1$ into $x^3 + 2x^2 - 5x - 6$ gives a quotient of $x - 1$ and remander of $x - 7$. since the remainder is non-zero, $x^2 + 3x - 1$ is not a factor of $x^3 + 2x^2 - 5x - 6$.

4.2 21: $h(x) = x^3 + x^2 - 8x - 8$. To see which of $2\sqrt{2}, \sqrt{2}, 1, -1$ are roots of $h(x)$ we can simply see which of $h(2\sqrt{2})$, $h(2\sqrt{2})$, $h(1)$, and $h(-1)$ is zero. Another way. Note that $h(1) = 0$; thus $x = 1$ is a root of $h(x)$. Therefore $x - 1$ divides $h(x)$, or $x - 1$ is a factor of $h(x)$. The reader can check that $h(x) = (x-1)(x^2 - 8)$; thus $x = 1, \pm\sqrt{2}$ are the roots of $h(x)$. The solution to the problem is $x = 1, \sqrt{2}$.

4.2 53: $x + 2 = x - (-2)$ is a factor of $p(x) = x^3 + 3x^2 + kx - 2$ if and only if -2 is a root of $p(x)$, that is $0 = p(-2) = (-2)^3 + 3(-2)^2 + k(-2) - 2$. The solutions to the last equation are $k = 1$.

4.2 51: Since $f(x)$ has roots $0, 5, 8$ it follows that $x - 0$, $x - 5$, and $x - 8$ are factors of $f(x)$. Thus $(x-0)(x-5)(x-8)$ divides $f(x)$. Now this product and $f(x)$ both have degree 3. Therefore $f(x) = a(x-0)(x-5)(x-8)$ for some non-zero number a. Since

$$17 = f(10) = (10-0)(10-5)(10-8) = a100$$

we conclude that $a = 17/100$. Therefore

$$f(x) = \frac{17}{100}(x-0)(x-5)(x-8) = \frac{17}{100}x(x-5)(x-8) = \frac{17}{100}(x^3 - 13x^2 + 40).$$

4.2 57: For every real number $c^4 + c^2 + 1 \geq 10$ since $c^4, c^2 \geq 0$. Thus no real number c is a root of $x^4 + x^2 + 1$ which means that $x - c$ is not a factor of $x^4 + x^2 + 1$ for all real numbers c.

4.1 35: Let $f(x)$ be the quadratic which describes the parabola. Since $x = -4$ and $x = 4$ are roots of the quadratic, $f(x) = a(x - (-4))(x - 4) = a(x+4)(x-4)$ for some real number a. Now

$$\frac{1}{6} = f(0) = a(4 + \frac{1}{6})(4 - \frac{1}{6}) = a(\frac{24 \cdot 23}{36}$$

, so $a = 1/92$. Thus a) $f(x) = \frac{1}{92}(x+4)(x-4)$. Part b) is a matter of solving $-\frac{1}{12} = \frac{1}{92}(x-4)(x+4)$ or equivalently $x^2 - 16 = 92/3$, that is $x^2 = 50$. Thus $x = \pm\sqrt{50}$.

4.1 41: Some background. Consider the function $f(x) = ax^2 + bx + c$, where a is not zero. Then the grapfh of $y = f(x)$ is a parabola. By completing the square

$$f(x) = ax^2 + bx + c = a(x^2 + (\frac{b}{a})x) + c = a(x + \frac{b}{2a})^2 + (c - \frac{b^2}{4a^2}).$$

Case 1: $a > 0$. The graph of $y = f(x)$ opens upward. The function has a unique *minimum* at $x = -b/2a$, and the minimum value of of $f(x)$ is $c - \frac{b^2}{4a})$.
Case 1: $a < 0$. The graph of $y = f(x)$ opens downward. The function has a unique *maximum* at $x = -b/2a$, and the maximum value of $f(x)$ is $c - \frac{b^2}{4a})$.

$h(t) = -16t^2 + v_0 t + s_0$, where $h(t)$ is the position of the object at time t, v_0 is its initial velocity, and h_0 is its initial position. Thus $v_0 = 80$, $h_0 = 96$, so

$$h(t) = -16t^2 + 80t + 96 = at^2 + bt + c.$$

The object reaches its maximum height at time $t = -\frac{b}{2a} = -\frac{80}{-32} = -\frac{5}{2}$ seconds and its height at that time is $c - \frac{b^2}{4a} = 96 - \frac{25}{4} = \frac{449}{4}$ feet.

There are related questions. When does the object hit the ground? Solve $h(t) = 0$. When is the ojject at height 33 feet? Solve $h(t) = 33$.

Comment: If the oject were thrown *downward* then $h(t) = -16t^2 - 80t + 96$, that is $v_0 = -80$. Pay attention to the sign convention for velocity.

3.3 29: The box has a square base of side length x and height h. Let V be the volume of the box and S be the surface area. Here are some general formulas.

Case 1: The box has a top.

$V = x^2h$ and $S = 2x^2 + 4xh$. Thus we may express V in terms of x and S by $V = x^2h = x^2\left(\frac{S - 2x^2}{4x}\right) = \frac{Sx - 2x^3}{4}$. Likewise we may express S in terms of x and V by $S = 2x^2 + 4xh = 2x^2 + 4x\left(\frac{V}{x^2}\right) = 2x^2 + \frac{4V}{x}$.

The problem at hand is to minimize surface area, given $V = 867$. Thus we need to minimize $S(x) = 2x^2 + \frac{3648}{x}$, $x > 0$. A calculator gives $x \approx 333$.

Suppose that we change the problem as let $S = 867$. Then $V(x) = \frac{867x - 2x^3}{4}$, $0 \leq x\sqrt{867}/2$. Observe that $V(0) = 0 = V(\sqrt{867}/2)$. The problem here is to maximize volume. A calculator gives $x \approx 444444$.

Case 2: The box has no top.

The formulas for volume and surface area are now $V = x^2h$ and $S = x^2 + 4xh$. Thus $V = x^2h = x^2\left(\dfrac{S-x^2}{4x}\right) = \dfrac{Sx - x^3}{4}$ and $S = x^2 + 4xh = x^2 + 4x\left(\dfrac{V}{x^2}\right) = x^2 + \dfrac{4V}{x}$.

page 110 33: Let y be the lengths of the base and top and let x be the length of the three vertical peices. Then $120 = 2y + 3x$. The area of the pen is therefore

$$A = xy = x\left(\frac{120-3x}{2}\right) = -\frac{3}{2}x^2 + 60x.$$

To see how to maximize the area $A = A(x)$, see **4.1** 41 above.

SOLUTION TO THE SAMPLE EXAM 1 FORM A

1. $\sqrt{x-1}$ is meaningful if and only if $x \geq 1$, and $x-3$ is not zero if and only if x is not 3. Therefore the domain of $f(x)$ is $[1,\infty)\cap\{x \mid x \neq 3\} = [1,3)\cup(3,\infty)$.

2. (a) Slope $= \dfrac{(-3)-3}{2-(-2)} = -\dfrac{3}{2}$. Thus $y-(-3) = -\dfrac{3}{2}(x-2)$ is one answer. (b) The slope of $x - 2y = 5$, or $y = \dfrac{1}{2}x - \dfrac{5}{2}$, is $1/2$. Therefore $y - (-3) = \dfrac{1}{2}(x-2)$ is one answer.

3. $y = f(x) = x^3 - 3x^2 - 6x + 9$ crosses the x-axis at $x \approx -2.054, 1.111, 3.944$. Decreasing on $[1.111, 3.944]$. Local maximum occurs at $x \approx -.759$. Here $y \approx 11.389$.

4. Translate the graph of $f(x)$ 3 units to the right and then translate the graph of the resulting function 3 units down.

5. Write $f(x) = (2x^2 - 9x + 13)^5 = (g(x))^5 = h(g(x))$, where $g(x) = 2x^2 - 9x + 13$ and $h(x) = x^5$.

6. $y = f(x) = \dfrac{2x-3}{x+1}$ solves y in terms of x. We wish to solve x in terms of y.

$$y = \frac{2x-3}{x+1}, \qquad y(x+1) = 2x-3, \qquad x(y-2) = -y-3, \qquad x = \frac{y+3}{-y+2}.$$

Therefore $f^{-1}(x) = \dfrac{x+3}{-x+2}$.

7. $v_0 = 56$ and $h_0 = 800$. Therefore $h(t) = -16t^2 + 56t + 800$. We want to find t when $h(t) = 0$, or $-16t^2 + 56t + 800 = 0$, or $-2t^2 + 7t + 100 = 0$. By the quadratic formula $t = \dfrac{-7 - \sqrt{7^2 - 4(-2)(100)}}{2(-2)} = \dfrac{7+\sqrt{849}}{4} \approx 9.034$ seconds.