MATH 215 **Final Examination Solution** **Radford** **08/08/08**

Name (print) ______________________________

(1) With the exception of part (a) of Problem 1, write your answers in the exam booklet provided. (2) *Return* this exam copy with your test booklet. (3) *You are expected to abide by the University's rules concerning academic honesty.*

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1. (**25 points**) Let P and Q be statements.

a) (**8**) Complete the following three truth tables (on this sheet if you wish):

P	Q	P implies (P implies Q)
T	T	
T	F	
F	T	
F	F	

P	Q	P and (not Q)
T	T	
T	F	
F	T	
F	F	

P	Q	(not P) or Q
T	T	
T	F	
F	T	
F	F	

Solution:

P	Q	P implies (P implies Q)
T	T	T
T	F	**F**
F	T	T
F	F	T

P	Q	P and (not Q)
T	T	F
T	F	**T**
F	T	F
F	F	F

P	Q	(not P) or Q
T	T	T
T	F	**F**
F	T	T
F	F	T

(b) (**11**) What are the logical relationships (implication, equivalence, negation) between the statements "P implies (P implies Q)", "P and (not Q)", and "(not P) or Q"? For convenience you may refer to these statements as A, B, and C respectively. Justify your answer in terms of the truth tables of parts (a).

Solution: A *and* C *are equivalent* since the last columns of their truth tables are identical (since their truth tables are the same in this case). Thus both imply each other.

A *and* B *are negations of each other* since the rows in the last columns of their truth tables have "opposite" values.

C *and* B *are negations of each other* for the same reason.

The only implications are between A and C. A does not imply B since there is a row in the truth table for A with a T in the last column and an F in the corresponding position in the truth table for B. For the same reason B does not imply A, B does not imply C and C does not imply B.

(c) (**6**) What is the negation of "P and Q" in terms of "not P", "not Q"? What is the negation of "P or Q" in terms of "not P", "not Q"?

Solution: "(not P) or (not Q)" and "(not P) and (not Q)" respectively.

2. (**30 points**) Consider the statement "$a^2 - 5a - 14 > 0$ implies $a \leq -2$ or $7 \leq a$".

(a) (**5**) What is the *converse* of the statement?

Solution: "$a \leq -2$ or $7 \leq a$ implies "$a^2 - 5a - 14 > 0$".

(b) (**5**) What is the *contrapositive* of the statement? Write it without "not".

Solution: "$-2 < a < 7$ implies "$a^2 - 5a - 14 \leq 0$".

(c) (**10**) Prove the statement by contradiction. Clearly state your assumptions.

Solution: Suppose that $a^2 - 5a - 14 > 0$ is true and $a \leq -2$ or $7 \leq a$ is false. Then $(a+2)(a-7) = a^2 - 5a - 14 > 0$ and $-2 < a < 7$. The latter implies $0 < a+2$ and $a-7 < 0$. Thus $(a+2)(a-7) < 0$, a contradiction. Therefore the statement is true.

(d) (**10**) Prove the *contrapositive* of the statement directly. Clearly state your assumptions.

Solution: Suppose that $-2 < a < 7$. Then $0 < a+2$ and $a - 7 < 0$. Thus $a^2 - 5a - 14 = (a+2)(a-7) < 0$ which implies $a^2 - 5a - 14 \leq 0$.

Base the proofs for parts (c) and (d) of Problem 2 on the axioms for the real number system $\mathbf{R}$ and for all for $a, b, c \in \mathbf{R}$: if $a, b > 0$ or $a, b < 0$ then $ab > 0$; if $a > 0$ and $b < 0$, or $a < 0$ and $b > 0$, then $ab < 0$; $a0 = 0 = 0a$; and if $a < b$ then $a - c < b - c$ and $c - a > c - b$.

3. (**15 points**) Let $f : \mathbf{R} \longrightarrow \mathbf{R}$ be a function and $a, L \in \mathbf{R}$. Then "$\lim_{x \longrightarrow a} f(x) = L$" can be expressed in terms of quantifiers by "$\forall \epsilon > 0, \exists \delta > 0, \forall x \in \mathbf{R}, 0 < |x-a| < \delta$ implies $|f(x)-L| < \epsilon$". (It is understood that $\epsilon, \delta \in \mathbf{R}$.)

(a) (**5**) Express the statement "$\lim_{x \longrightarrow a} f(x) = L$" in English with the quantifiers translated.

Solution: For all $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x \in \mathbf{R}$ the condition $0 < |x-a| < \delta$ implies $|f(x) - L| < \epsilon$.

(b) (**10**) Express the statement "$\lim_{x \longrightarrow a} f(x) \neq L$", that is "not ($\lim_{x \longrightarrow a} f(x) = L$)", in terms of quantifiers without the use of "not".

Solution: "$\exists \epsilon > 0, \forall \delta > 0, \exists x \in \mathbf{R}, 0 < |x - a| < \delta$ and $|f(x) - L| \geq \epsilon$"

4. (**25 points**) Show, by induction, that the sum of the squares of first n odd integers is given by the formula $1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \dfrac{n(2n-1)(2n+1)}{3}$ for all $n \geq 2$ ($n = 2$ is the base case).

Solution: Suppose $n = 2$. Then $1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = 1^2 + 3^2 = 10$ and

$$\frac{n(2n-1)(2n+1)}{3} = \frac{2(4-1)(4+1)}{3} = \frac{30}{3} = 10.$$

Therefore the equation holds in the base case $n = 2$.

Suppose $n \geq 2$ and the equation holds. Then

$$\begin{aligned}
&1^2 + 3^2 + 5^2 + \cdots + (2(n+1)-1)^2 \\
&= (1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2) + (2(n+1)-1)^2 \\
&= \frac{n(2n-1)(2n+1)}{3} + (2n+1)^2 \\
&= \frac{n(2n-1)(2n+1) + 3(2n+1)^2}{3} \\
&= \frac{(2n+1)[n(2n-1) + 3(2n+1)]}{3} \\
&= \frac{(2n+1)[2n^2 + 5n + 3]}{3} \\
&= \frac{(2n+1)(n+1)(2n+3)}{3} \\
&= \frac{(n+1)(2(n+1)-1)(2(n+1)+1)}{3}.
\end{aligned}$$

Thus the formula holds for $n+1$. By induction the formula holds for all $n \geq 2$.

5. (**30 points**) Let X, Y, and Z be sets.

(a) (**8**) Give the conditional definitions of $X{-}Y$ and $X{\times}Y$.

Solution: $X{-}Y = \{x \,|\, x \in X \text{ and } x \notin Y\}$ and $X{\times}Y = \{(x,y) \,|\, x \in X, y \in Y\}$.

(b) (**5**) In terms of X and Y, what does it mean for $x \notin X{-}Y$?

Solution: $x \notin X$ or $x \in Y$.

(c) (**10**) Show that $X{-}(Y{-}Z) \subseteq (X-Y){\cup}Z$.

Solution: We need only show that an element of $X{-}(Y{-}Z)$ is an element of $(X-Y){\cup}Z$.

Let $x \in X{-}(Y{-}Z)$. Then $x \in X$ and $x \notin Y{-}Z$. Therefore $x \in X$ and ($x \notin Y$ or $x \in Z$). Consequently $x \in X$ and $x \notin Y$, in which case $x \in X{-}Y$, or $x \in X$ and $x \in Z$, in which case $x \in Z$. In either case $x \in (X{-}Y){\cup}Z$.

(d) (**7**) Find subsets X, Y, and Z of $\{1,2,3\}$ such that $(X{-}Y){\cup}Z \subseteq X{-}(Y{-}Z)$ is false. Compute the two preceding sets for your example.

Solution: There are many possibilities. Take $X = \{1\}$, $Y = \{2\}$, and $Z = \{3\}$ for example. Then

$$X{-}(Y{-}Z) = \{1\}{-}(\{2\}{-}\{3\}) = \{1\}{-}\{2\} = \{1\}$$

and

$$(X-Y)\cup Z = (\{1\}-\{2\})\cup\{3\} = \{1\}\cup\{3\} = \{1,3\}.$$

6. (**25 points**) Suppose that $f : X \longrightarrow Y$ and $g : Y :\longrightarrow Z$ are functions.

(a) (**5**) Define, using quantifiers, what it means for f to be an *injection*.

Solution: Either

"$\forall x, x' \in X, f(x) = f(x')$ implies $x = x'$."

or

"$\forall x, x' \in X, x \neq x'$ implies $f(x) \neq f(x')$."

(b) (**5**) Define, using quantifiers, what it means for f to be a *surjection*.

Solution: "$\forall y \in Y, \exists x \in X, y = f(x)$."

(c) (**5**) Suppose that $g\circ f$ is a surjection. Show that g is a surjection.

Solution: Let $z \in Z$. Since $g\circ f$ is a surjection there is an $x \in X$ such that $g\circ f(x) = z$. Let $y = f(x)$. Then $y \in Y$ and $g(y) = g(f(x)) = (g\circ f)(x) = z$. Therefore g is a surjection.

(d) (**10**) Let $A, B \subseteq Y$. Show that $\overleftarrow{f}(A\cup B) = \overleftarrow{f}(A)\cup\overleftarrow{f}(B)$. [For all $C \subseteq Y$ recall that $\overleftarrow{f}(C) = \{x \in X \mid f(x) \in C\}$.]

Solution: We need only show that an element $\overleftarrow{f}(A\cup B)$ is an element of $\overleftarrow{f}(A)\cup\overleftarrow{f}(B)$ and vice versa.

Let $x \in \overleftarrow{f}(A\cup B)$. Then $f(x) \in A\cup B$. Therefore $f(x) \in A$, in which case $x \in \overleftarrow{f}(A)$, or $f(x) \in B$, in which case $x \in \overleftarrow{f}(B)$. In either case $x \in \overleftarrow{f}(A)\cup\overleftarrow{f}(B)$.

Let $x \in \overleftarrow{f}(A)\cup\overleftarrow{f}(B)$. Then $x \in \overleftarrow{f}(A)$, in which case $f(x) \in A$, or $x \in \overleftarrow{f}(B)$, in which case $f(x) \in B$. In either case $f(x) \in A\cup B$ and therefore $x \in \overleftarrow{f}(A\cup B)$.

7. (**30 points**) *In this problem all binomial symbols must be computed.* A committee of 11 is to be formed from a group of 14 people.

(a) (**5**) How many such committees are there?

Solution: $\binom{14}{11} = \dfrac{14\cdot 13\cdot 12}{3\cdot 2\cdot 1} = 14\cdot 13\cdot 2 = 364.$

(b) (**5**) Suppose a certain 2 individuals from this group are to be *excluded*. How many such committees are there?

Solution: $\binom{14-2}{11} = \binom{12}{11} = \dfrac{12}{1} = 12.$

(c) (**5**) Suppose that a certain 5 individuals from this group are to be *included*. How many such committees are there?

Solution: $\binom{14-5}{11-5} = \binom{9}{6} = \frac{9 \cdot 8 \cdot 7}{3 \cdot 2 \cdot 1} = 3 \cdot 4 \cdot 7 = 84.$

Suppose A and B are different individuals in this group. Let X be the set of committees which include A and let Y the set of committees which include B.

(d) (**7**) Use the Principle of Inclusion-Exclusion to calculate the number of elements of $X \cup Y$, the set of committees which *include* A or B.

Solution:

$$\begin{aligned} |X \cup Y| &= |X| + |Y| - |X \cap Y| \\ &= \binom{14-1}{11-1} + \binom{14-1}{11-1} - \binom{14-2}{11-2} \\ &= \binom{13}{10} + \binom{13}{10} - \binom{12}{9} \\ &= 286 + 286 - 220 \\ &= 352. \end{aligned}$$

(e) (**8**) Let U be the set of all committees of 11 which can be formed from the group of 14 people. Express the set of committees which *exclude* both A and B in terms of X and Y and use De Morgan's Law to compute the number of such committees.

Solution: This set of committees is $X^c \cap Y^c$ and thus

$$|X^c \cap Y^c| = |(X \cup Y)^c| = |U| - |(X \cup Y)| = 364 - 352 = 12.$$

8. (**20 points**) Give the definition of

(a) (**3**) finite set,

Solution: A set A is finite if $A = \emptyset$ or if there is a bijection $f : \mathbf{N}_n \longrightarrow A$ for some $n \geq 1$.

(b) (**3**) denumerable set,

Solution: A set A is denumerable if there is a bijection $f : \mathbf{Z}^+ \longrightarrow A$.

(c) (**3**) countable set,

Solution: A set is countable if it is either finite or denumerable.

and (d) (**11**) use the Euclidean Algorithm to find the greatest common divisor of 372 and 58.

Solution:

$$\begin{aligned}
\mathbf{372} &= \mathbf{58}\cdot 6 + \mathbf{24} \\
\mathbf{58} &= \mathbf{24}\cdot 2 + \mathbf{10} \\
\mathbf{24} &= \mathbf{10}\cdot 2 + \mathbf{4} \\
\mathbf{10} &= \mathbf{4}\cdot 2 + \mathbf{2} \\
\mathbf{4} &= \mathbf{2}\cdot 2 + 0;
\end{aligned}$$

therefore the greatest common divisor of 372 and 58 is 2.