MATH 215 **Written Homework 2 Solution** **06/26/2009**

1. (**25 points total**) The assertion is $1 < n < 3$ implies $n^3 < 2n^2 + 15n$ for integers n.

a) Proof of the assertion by cases. There is but one case, $n = 1$. Since $1^3 = 1 < 17 = 2{\cdot}1^2 + 15{\cdot}1$ the assertion is true. (**5**)

b) Proof by "working backwards.

$$\begin{aligned} & n^3 < 2n^2 + 15n \\ & \quad \Leftarrow n(n^2 - 2n - 15) < 0 \\ & \quad \Leftarrow n(n-5)(n+3) < 0 \\ & \quad \Leftarrow (n-5)(n+3) < 0, n > 0 \\ & \quad \Leftarrow n - 5 < 0, n + 3 > 0, n > 0 \\ & \quad \Leftarrow 0 < n < 5 \\ & \quad \Leftarrow 1 < n < 3 \end{aligned}$$

for example. (**10**)

c) Note that $n^3 < 2n^2 + 15n$ is equivalent to $n^3 - 2n^2 - 15n < 0$ or $n(n + 3)(n - 5) < 0$. The latter holds if and only if all three factors are negative or one is and the other two are positive.

Case 1: All three factors are negative. This translates to $n < 0$, $n < -3$, and $n < 5$; the equivalent is $n < -3$. (**2**)

Case 2: $n < 0$ and the other two factors are positive. This translates to $n < 0$, $n > -3$, and $n > 5$. Since $5 < n < 0$ is not possible, no integers n satisfy this case. (**2**)

Case 3: $n+3 < 0$ and the other two factors are positive. This translates to $n > 0$, $n < -3$, and $n < 5$. Since $0 < n < -3$ is not possible, no integers satisfy this case. (**2**)

Case 4: $n-5 < 0$ and the other two factors are positive. This translates to $n > 0$, $n > -3$, and $n < 5$; the equivalent is $0 < n < 5$. (**2**)

Combining cases we see that $n^3 < 2n^2 + 15n$ if and only if $n < -3$ or $0 < n < 5$. (**2**)

2. (**20 points total**) Our assertion is $a^2 \geq 7a$ implies $a \leq 0$ or $a \geq 7$.

a) Proof by contradiction. Suppose the hypothesis is true and the conclusion is false; that is $a^2 \geq 7a$ is true and $a \leq 0$ or $a \geq 7$ is false. Then $a^2 \geq 7a$ and $0 < a < 7$. (**4**) Since $a > 0$ the latter implies $a^2 < 7a$ which contradicts the hypothesis. (**4**) Therefore if the hypothesis is true the conclusion must be also. (**2**)

b) Direct proof. Assume the hypothesis $a^2 \geq 7a$, or equivalently $a(a - 7) \geq 0$. We look at two cases.

Case 1: $a(a - 7) = 0$. Then $a = 0$ or $a = 7$. (**3**)

Case 2: $a(a-7) > 0$. Then $a > 0, a-7 > 0$, or equivalently $a > 7$, (**3**) or $a < 0, a-7 < 0$, or equivalently $a < 0$. (**3**)

Combining the results of both cases we have $a \leq 0$ or $7 \leq a$. (**1**)

3. (**20 points total**) Since $a^2 - 9a + 18 = (a-3)(a-6)$, the assertion is equivalent to $(a-3)(a-6) < 0$ implies $3 \leq a < 6$.

a) Proof by contradiction. Suppose $(a-3)(a-6) < 0$ is true and $3 \leq a < 6$ is false. Then $(a-3)(a-6) < 0$, and $a < 3$ or $6 \leq a$. (**3**) If $a < 3$ then $a-3$, $a-6 < 0$ which means $(a-3)(a-6) > 0$, a contradiction to the hypothesis. If $6 \leq a$ then $0 \leq a-6$ and $3 < a-3$ so $(a-3)(a-6) \geq 0$, a contradiction to the hypothesis. (**3**) Therefore if $(a-3)(a-6) < 0$ is true $3 \leq a < 6$ is also. (**2**)

b) Direct proof. Suppose $(a-3)(a-6) < 0$. Then one of the factors is positive and the other is negative.

Case 1: $a-3 > 0$ and $a-6 < 0$. Then $3 < a < 6$. (**3**)

Case 2: $a-3 < 0$ and $a-6 > 0$. Then $6 < a < 3$; therefore no real number satisfy this case. (**3**)

The combination of cases yields $3 < a < 6$ which implies $3 \leq a < 6$. (**2**)

c) The converse, $3 \leq a < 6$ implies $a^2 - 9a + 18 < 0$, is false. For when $a = 3$ the statement $3 \leq a < 6$ is true and the statement $a^2 - 9a + 18 < 0$, is false as $3^2 - 9 \cdot 3 + 18 = 0 \not< 0$. (**4**)

In the next two problems we assume the following about integers: (1) the sum of two even, or two odd, integers is even, (2) the sum of an odd integer and an even integer is odd, (3) the product of an even integer with any integer is even, and (4) the product of two odd integers is odd.

4. (**15 points total**) The natural case to consider are n even an n odd.

Case 1: n is even. Then $n^2 + \ell n = n(n+\ell)$ is even since it is the product of an even integer with an integer. (**8**)

Case 2: n is odd. Then $n + \ell$ is even since it is the sum of two odd integers; therefore $n^2 + \ell n = (n+\ell)n$ is even since it is the product of an even integer with an integer. (**7**)

5. (**20 points total**) We wish to show $n^2 + \ell n$ is even for $n \geq 1$ by induction. This is true for $n = 1$ since $1^2 + \ell \cdot 1 = 1 + \ell$ is even since it is the sum of two odd integers. (**5**)

Suppose $n \geq 1$ and $n^2 + \ell n$ is even. We will show that $(n+1)^2 + \ell(n+1)$ is even. (**5**) We have noted $\ell + 1$ is even. Therefore $2n + \ell + 1$ is even as it is the sum of two even integers which means

$$(n+1)^2 + \ell(n+1) = (n^2 + 2n + 1) + (\ell n + \ell) = (n^2 + \ell n) + (2n + \ell + 1)$$

is even as it is the sum of two even integers. (**8**) We have shown $n^2 + \ell n$ even implies $(n+1)^2 + \ell(n+1)$ is even. Therefore $n^2 + \ell n$ is even for all $n \geq 1$. (**2**)