1. (**20 points total**) We are assuming (i) $A \cap B \subseteq A \cap C$ and (ii) $A \cup B \subseteq A \cup C$.

(a) We wish to show $B \subseteq C$. Let $b \in B$. Suppose that $b \in A$. Then $b \in A \cap B$ which means $b \in A \cap C$ by (i). Therefore $b \in C$. (**4**) Now suppose $b \notin A$. Since $b \in B$ it follows $b \in A \cup B$. Therefore $b \in A \cup C$ by (ii). Since $b \notin A$ necessarily $b \in C$. (**4**) In any event $b \in C$. We have shown $B \subseteq C$. (**4**)

(b) We are given $A \cap B = A \cap C$ and $A \cup B = A \cup C$. In particular $A \cap B \subseteq A \cap B$ and $A \cup B \subseteq A \cup C$. Thus $B \subseteq C$ by part (a). (**3**) The equations also imply $A \cap C \subseteq A \cap B$ and $A \cup C \subseteq A \cup B$. Therefore $C \subseteq B$ by part (a) as well. (**3**) We have shown $B = C$. (**2**)

2. (**20 points total**) The functions $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ satisfy $g \circ f = I_X$. The latter is equivalent to $g(f(x)) = x$ for all $x \in X$. (**4**) We use this equivalence in our proofs.

First of all we show f is injective. Suppose $x, x' \in X$ and $f(x) = f(x')$. (**3**) Then $x = x'$ since $x = g(f(x)) = g(f(x')) = x'$. (**3**) Therefore f is injective. (**2**)

Next we show that g is surjective. Suppose $x \in X$ and set $y = f(x)$. (**3**) Then $g(y) = g(f(x)) = x$. (**3**) Therefore f is surjective. (**2**)

3. (**20 points total**) $f : [1/2, \infty) \longrightarrow [-1/4, \infty)$ is defined by $f(x) = x^2 - x = (x - 1/2)^2 - 1/4$. We base arguments on facts derived about increasing functions and quadratics in class; that increasing functions are injective and the function $g : [0, \infty) \longrightarrow [0, \infty)$ given by $g(x) = x^2$ is bijective.

(a) For $x \geq 1/2$ observe that $x - 1/2 \geq 0$ and therefore $f(x)$ is increasing (as g is increasing). Therefore f is increasing and hence injective. (**6**)

(b) Let $y \in [-1/4, \infty)$ or equivalently $y \geq -1/4$. Then $y + 1/4 \geq 0$ so $\sqrt{y + 1/4}$ exists. Since the latter is non-negative $x = \sqrt{y + 1/4} + 1/2 \geq 1/2$ which means $x \in [1/2, \infty)$. (**4**) The calculation

$$\begin{aligned} f(x) &= f(\sqrt{y+1/4} + 1/2) \\ &= ((\sqrt{y+1/4} + 1/2) - 1/2)^2 - 1/4 \\ &= (\sqrt{y+1/4})^2 - 1/4 = (y + 1/4) - 1/4 \\ &= y \end{aligned}$$

shows that $y = f(x)$. (**4**) Therefore f is surjective.

Comment: Given y one discovers the solution x to $y = f(x)$ by working backwards. Here we omit those details and show that our x is indeed a solution.

(c) From part (b) the inverse $f^{-1} : [-1/4, \infty) \longrightarrow [1/2, \infty)$ is given by

$$f^{-1}(y) = \sqrt{y + /4} + 1/2 \ \ (\mathbf{4})$$

for all $y \in [-1/4, \infty)$, or in more standard notation, $f^{-1}(x) = \sqrt{x + 1/4} + 1/2$ for all $x \in [-1/4, \infty)$.

4. (**20 points total**) Recall $G_f^{op} = \{(y, x) \,|\, (x, y) \in G_f\}$ is the graph of a function if and only if (a) $\forall y \in Y,\ \exists x \in X,\ (y, x) \in G_f^{op}$ and (b) $(y, x), (y, x') \in G_f^{op}$ implies $x = x'$. This is given.

Therefore G_f^{op} is the graph of a function if and only if (a') $\forall y \in Y,\ \exists x \in X,\ (x, y) \in G_f$ and (b') $(x, y), (x', y) \in G_f$ implies $x = x'$. (**6**)

Note $(x, y) \in G_f$ if and only if $x \in X, y \in Y$, and $y = f(x)$. Thus (a') holds if and only if f is surjective (**7**) and (b') holds if and only if f is injective (**7**).

5. (**20 points total**) Here we show two functions are the same by using a modified truth table.

(a) From the table

$x \in A$	$x \in B$	$\chi_A(x)$	$\chi_B(x)$	$\chi_{A \cap B}(x)$	$\chi_A(x)\chi_B(x)$
T	T	1	1	1	1
T	F	1	0	0	0
F	T	0	1	0	0
F	F	0	0	0	0

(**6** for the table) we see that $\chi_{A \cap B} = \chi_A \chi_B$ since these functions agree in all cases as columns 5 and 6 of the table are identical (**4**).

(b) From the table

$x \in A$	$x \in B$	$\chi_A(x)$	$\chi_B(x)$	$\chi_{A \cup B}(x)$	$\chi_A(x) + \chi_B(x) - \chi_{A \cap B}(x)$
T	T	1	1	1	1
T	F	1	0	1	1
F	T	0	1	1	1
F	F	0	0	0	0

(**6** for the table) we see that $\chi_{A \cup B} = \chi_A + \chi_B - \chi_{A \cap B}$ since these functions agree in all cases as columns 5 and 6 of the table are identical (**4**).