MATH 215 # Hour Exam I **Radford 09/28/05**

(and solution)

Name (print) ________________________________

(1) There are *four questions* on this exam. (2) *Return* this exam copy with your test booklet. (3) *You are expected to abide by the University's rules concerning academic honesty.* (4) *Base the proofs* for problems 2c) and 4 on the axioms for the real number system and:

a) the product of two positive real numbers is positive,

b) the product of two negative real numbers is positive,

c) the product of a positive real number and a negative real number is negative,

d) the product of zero and any real number is zero, and

e) if a, b, c are real numbers and $a < b$ then $a - c < b - c$ and $c - a > c - b$.

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1. (25 pts.) Let P and Q be statements.

   a) Write out the truth table for the statement "(P implies Q) implies P". Include a column for "P implies Q" in your table.

   Solution:

P	Q	P implies Q	(P implies Q) implies P
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	F

   (**8 points**)

   b) Write out the truth table for the statement "P implies (Q implies P)". Include a column for "Q implies P" in your table.

   Solution:

P	Q	Q implies P	P implies (Q implies P)
T	T	T	T
T	F	T	T
F	T	F	T
F	F	T	T

(**8 points**)

c) Are the statements of part a) and part b) logically equivalent? Justify your answer in terms of the truth tables of parts a) and b).

Solution: The statements are *not* logically equivalent since the last columns of the truth tables are not the same. (**5 points**)

d) Does the statement of part a) imply the statement of part b)?

Solution: *Yes* since whenever the first statement is true the second is also true. (**4 points**)

2. (30 pts.) Consider the statement "$(a-4)(a-7) \geq 0$ implies $a \leq 4$ or $7 \leq a$".

a) What is the converse of the statement?

Solution: The converse is "$a \leq 4$ or $7 \leq a$ implies $(a-4)(a-7) \geq 0$". (**10 points**)

b) What is the contrapositive of the statement? Write it without "not".

Solution: The contrapositive literally is "(not ($a \leq 4$ or $7 \leq a$) implies (not $((a-4)(a-7) \geq 0)$)" which without "not" is written "$4 < a < 7$ implies $(a-4)(a-7) < 0$". (**10 points**)

c) Prove the statement by contradiction.

Solution: Suppose that the hypothesis $(a-4)(a-7) \geq 0$ is true but the conclusion $a \leq 4$ or $7 \leq a$ is false. Then $4 < a < 7$. Since $4 < a$ and $a < 7$ we have $0 < a-4$ and $a-7 < 0$. Therefore $(a-4)(a-7)$ is the product of a positive real number and a negative real number which means that it is negative. Since $(a-4)(a-7) \geq 0$ this is a contradiction. Therefore the statement is true. (**10 points**)

3. (25 pts.) Prove by induction that the sum of the odd integers

$$1 + 3 + 5 + \cdots + (2n+1) = (n+1)^2$$

for all $n \geq 2$.

Solution: If $n = 2$ not that $2n+1 = 5$. In this case the left hand side of the formula is $1+3+5$ and the right hand side is $(2+1)^2 = 3^2 = 9$. Therefore the formula is true when $n = 2$. (**5 points**)

Suppose $n \geq 2$ and the formula is true. Then

$$\begin{aligned}
1 + 3 + 5 + \cdots + (2(n+1)+1) &= (1 + 3 + 5 + \cdots + (2n+1) + (2(n+1)+1) \\
&= (n+1)^2 + (2n+3) \\
&= n^2 + 2n + 1 + 2n + 3
\end{aligned}$$

$$
\begin{aligned}
&= n^2 + 4n + 4 \\
&= (n+2)^2 \\
&= ((n+1)+1)^2
\end{aligned}
$$

which means that the formula is true for $n+1$. By induction the formula is true for all $n \geq 2$. (**induction hypothesis, use of formula for** n**, and algebra; 5, 5, and 10 points respectively**)

4. (20 pts.) Show directly that $a \leq 5$ or $8 < a$ implies $(a-5)(a-8) \geq 0$.

   Solution: There are two cases to consider.

   Case 1: $a \leq 5$. If $a = 5$ then $(a-5)(a-8) = 0(a-8) = 0$. (**4 points**) If $a \neq 5$ then $a < 5$ which means $a - 8 < a - 5 < 0$. In this situation $(a-5)(a-8)$ is the product of two negative numbers and is thus positive. In any event $(a-5)(a-8) \geq 0$. (**8 points**)

   Case 2: $8 < a$. Here $0 < a - 5 < a - 8$ which means that $(a-5)(a-8)$ is the product of two positive real numbers and is therefore positive. Thus $(a-5)(a-8) > 0$ which means that $(a-5)(a-8) \geq 0$. (**8 points**)