REPRESENTATIONS PARAMETERIZED BY A PAIR OF CHARACTERS

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Abstract. Let U and A be algebras over a field k. We study algebra structures H on the underlying tensor product $U \otimes A$ of vector spaces which satisfy $(u \otimes a)(u' \otimes a') = uu' \otimes aa'$ if $a = 1$ or $u' = 1$. For a pair of characters $\rho \in \mathrm{Alg}(U, k)$ and $\chi \in \mathrm{Alg}(A, k)$ we define a left H-module $L(\rho, \chi)$. Under reasonable hypotheses the correspondence $(\rho, \chi) \mapsto L(\rho, \chi)$ determines a bijection between character pairs and the isomorphism classes of objects in a certain category ${}_H\underline{\mathcal{M}}$ of left H-modules. In many cases the finite-dimensional objects of ${}_H\underline{\mathcal{M}}$ are the finite-dimensional irreducible left H-modules.

We show that the finite-dimensional irreducible representations of a wide class of pointed Hopf algebras are parameterized by pairs of characters.

Introduction

This paper develops the theory of a type of modules for certain algebra structures H defined on tensor products which, in many cases, accounts for the finite-dimensional irreducible representations of H. The general results are applied to the study of irreducible modules of a certain class of pointed Hopf algebras over a field k. This class is very basic in light of the results of the program of Andruskiewitsch and the second author to determine the structure of pointed Hopf algebras with commutative coradicals [2, 3, 4]. The Hopf algebras of interest to us are quotients of certain two-cocycle twists $H = (U \otimes A)^\sigma$ of the tensor product of two pointed Hopf algebras U and A over k. As a vector space $H = U \otimes A$ and multiplication has the property

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$(u\otimes a)(u'\otimes a') = uu'\otimes aa'$ whenever $a = 1$ or $u' = 1$. The natural context for us to begin our study is in the category of algebra structures on the tensor product $U\otimes A$ of the underlying vector spaces of algebras U and A over k which satisfy the multiplication property.

One purpose of this paper is to show that the irreducible representations of the pointed Hopf algebras in our class are parameterized by pairs of characters and to construct them starting with the characters. For a pair of characters $\rho \in \mathrm{Alg}(U, k)$ and $\chi \in \mathrm{Alg}(A, k)$ we construct a left H-module $L(\rho, \chi)$ and a right H-module $R(\chi, \rho)$. They satisfy a duality relationship with respect to a certain H-balanced bilinear form Ψ.

We describe the modules $L(\rho, \chi)$ and $R(\chi, \rho)$ abstractly. The H-module $L(\rho, \chi)$ contains a codimension one left U-submodule and is generated as an H-module by a one-dimensional A-submodule. Let ${}_H\underline{\mathcal{M}}$ be the full subcategory of all left H-modules whose objects contain a left U-submodule of codimension one and contain a one-dimensional left A-submodule. Under mild conditions we show the correspondence $(\rho, \chi) \mapsto L(\rho, \chi)$ is one-to-one. One of our main results gives natural conditions under which this correspondence determines a bijection between the Cartesian product $\mathrm{Alg}(U, k)\times \mathrm{Alg}(A, k)$ and the isomorphism classes of ${}_H\underline{\mathcal{M}}$. We will need to know when finite-dimensional irreducible representations are one-dimensional.

This paper is organized as follows. In Section 1 we set our notation for algebras, coalgebras, Hopf algebras, and the like. Two-cocycles σ are reviewed and the Hopf algebra $H = (U\otimes A)^\sigma$ is described. We note that the Drinfeld double of a finite-dimensional Hopf algebra H over k can be viewed as $D(H) = (H^{*\,cop}\otimes H)^\sigma$ for some two-cocycle. See [6].

Let A and U be algebras over k. In Section 2 we develop a theory of algebra structures H on $U\otimes A$ which satisfy the multiplication property. We define the H-modules $L(\chi, \rho)$, $R(\rho, \chi)$ and study them as explicit constructions and also more abstractly. The H-modules $L(\chi, \rho)$ are objects of ${}_H\underline{\mathcal{M}}$, the category whose objects M are left H-modules which contain a codimension one left U-submodule N and are generated as an H-module by a one-dimensional left A-submodule km. We study this category and its refinement ${}_H\underline{\mathcal{M}}'$, whose objects are triples (M, N, km), and duality relations with their counterparts for right H-modules. There are natural H-balanced bilinear forms which provide duality relationships between the objects of ${}_H\underline{\mathcal{M}}'$ and $\underline{\mathcal{M}}'_H$.

Section 3 contains the main results for the modules $L(\chi, \rho)$ in the more abstract setting of the category ${}_H\underline{\mathcal{M}}$. We first consider conditions on the characters ρ and χ separately in connection with the correspondence $(\rho, \chi) \mapsto L(\rho, \chi)$. Our main theorem gives natural conditions under which this correspondence determines a bijection between $\mathrm{Alg}(U, k) \times \mathrm{Alg}(A, k)$ and the isomorphism classes of the objects of ${}_H\underline{\mathcal{M}}$. In this case each object of ${}_H\underline{\mathcal{M}}$ has a unique codimension one U-submodule and a unique one-dimensional left A-submodule. Under the hypothesis of the theorem, when the finite-dimensional irreducible representations of U and A are one-dimensional then the finite-dimensional irreducible left H-modules are the same as the finite-dimensional objects of ${}_H\underline{\mathcal{M}}$. We consider the case when U and A are Hopf algebras in Section 4.

In Section 5 we consider conditions under which the irreducible representations of an algebra are one-dimensional for applications to certain classes of pointed Hopf algebras. In Sections 6 and 7 we apply the results of the preceding sections to pointed Hopf algebras which arise in recent classification work [1],[3]. These Hopf algebras are quotients of two-cocycle twists $H = (U \otimes A)^\sigma$, where U and A have the form $B\#k[\Gamma]$, where Γ is an abelian group and B is a left $k[\Gamma]$-module algebra and a left $k[\Gamma]$-comodule coalgebra. In most cases $B = \mathfrak{B}(X)$ is a Nichols algebra of a finite-dimensional Yetter-Drinfeld module over the group algebra $k[\Gamma]$. See [2, Section 2] for a discussion of Nichols algebras and Yetter-Drinfeld modules in general.

Let $\mathcal{D} = \mathcal{D}(\Gamma, (g_i)_{1 \le i \le \theta}, (\chi_i)_{1 \le i \le \theta}, (a_{ij})_{1 \le i,j \le \theta})$ be a datum of finite Cartan type and suppose that λ is a family of linking parameters for $\mathcal{D}$ in the sense of [1]. We apply results of this paper to the infinite-dimensional Hopf algebras $U(\mathcal{D}, \lambda)$ in general, and also to the finite-dimensional versions $u(\mathcal{D}, \lambda)$ under some restrictions. The 2-cocycle σ is determined by the linking parameters.

Let $\widehat{\Gamma}$ be the k-valued characters of the abelian group Γ. The finite-dimensional irreducible left modules for $U(\mathcal{D}, \lambda)$ are parameterized by a subset of $\widehat{\Gamma}$ and the irreducible modules for $u(\mathcal{D}, \lambda)$ by $\widehat{\Gamma}$; these irreducible modules arise from $L(\rho, \chi)$'s defined for H where χ determines ρ.

The Hopf algebras $U(\mathcal{D}, \lambda)$ and $u(\mathcal{D}, \lambda)$ are described in [2] in a slightly different notation. There are many parallels between the Hopf algebras $U(\mathcal{D}, \lambda)$ and the quantized enveloping algebras $U_q(\mathfrak{g})$, where q is not a root of unity. Likewise there are many parallels between the Hopf algebras $u(\mathcal{D}, \lambda)$ and the Frobenius-Lusztig kernels.

Throughout k is a field and all vector spaces are over k. For vector spaces U and V we will drop the subscript k from $\mathrm{End}_k(V)$, $\mathrm{Hom}_k(U, V)$, and $U \otimes_k V$. We denote the identity map of V by $\mathrm{id}_V = \mathrm{id}$. For a nonempty subset S of the dual space V^* we let $S^\perp$ denote the subspace of

V consisting of the common zeros of the functionals in S. The "twist" map $\tau_{U,V} : U \otimes V \longrightarrow V \otimes U$ is defined by $\tau_{U,V}(u \otimes v) = v \otimes u$ for all $u \in U$ and $v \in V$. For $p \in U^*$ and $u \in U$ we denote the evaluation of p on u by $p(u)$ or $<p, u>$. Any one of [9, 12, 17] will serve as a Hopf algebra reference for this paper.

1. Preliminaries

For a group G we let $\widehat{G}$ denote the group of characters of G with values in k. $H = kG$ denotes the group algebra of G over k which is a Hopf algebra arising in most applications in this paper. We usually denote the antipode of a Hopf algebra over k by S.

Let (A, m, η) be an algebra over k, which we shall usually denote by A. Generally we represent algebraic objects defined on a vector space by their underlying vector space. We say that $a, b \in A$ *skew commute* if $ab = \omega ba$ for some non-zero $\omega \in k$. Note that (A, m^{op}, η) is an algebra over k, where $m^{op} = m \circ \tau_{A,A}$. We denote A with this algebra structure by A^{op} and we denote the category of left (respectively right) A-modules and module maps by ${}_A\mathcal{M}$ (respectively $\mathcal{M}_A$). If $\mathcal{C}$ is a category, by abuse of notation we will write $C \in \mathcal{C}$ to indicate that C is an object of $\mathcal{C}$.

Let M be a left A-module. Then M^* is a right A-module under the transpose action which is given by $(m^* \cdot a)(m) = m^*(a \cdot m)$ for all $m^* \in M^*$, $a \in A$, and $m \in M$. Likewise if M is a right A-module then M^* is a left A-module where $(a \cdot m^*)(m) = m^*(m \cdot a)$ for all $a \in A$, $m^* \in M^*$, and $m \in M$.

Let (C, Δ, ϵ) be a coalgebra over k, which we usually denote by C. Generally we use a variant on the Heyneman-Sweedler notation for the coproduct and write $\Delta(c) = c_{(1)} \otimes c_{(2)}$ to denote $\Delta(c) \in C \otimes C$ for $c \in C$. Note that $(C, \Delta^{cop}, \epsilon)$ is a coalgebra over k, where $\Delta^{cop} = \tau_{C,C} \circ \Delta$. We let C^{cop} denote the vector space C with this coalgebra structure. Observe that C is a C^*-bimodule with the actions defined by

$$c^* \rightharpoonup c = c_{(1)} <c^*, c_{(2)}> \quad \text{and} \quad c \leftharpoonup c^* = <c^*, c_{(1)}> c_{(2)}$$

for all $c^* \in C^*$ and $c \in C$.

Suppose that (M, δ) is a left C-comodule. For $m \in M$ we use the notation $\delta(m) = m_{(-1)} \otimes m_{(0)}$ to denote $\delta(m) \in C \otimes M$. If (M, δ) is a right C-comodule we denote $\delta(m) \in M \otimes C$ by $\delta(m) = m_{(0)} \otimes m_{(1)}$. Observe that our coproduct and comodule notations do not conflict.

Bilinear forms play an important role in this paper. We will think of them in terms of linear forms $\beta : U \otimes V \longrightarrow k$ and will often write $\beta(u, v)$ for $\beta(u \otimes v)$. Note that β determines linear maps $\beta_\ell : U \longrightarrow V^*$ and $\beta_r : V \longrightarrow U^*$ where $\beta_\ell(u)(v) = \beta(u, v) = \beta_r(v)(u)$ for all $u \in U$

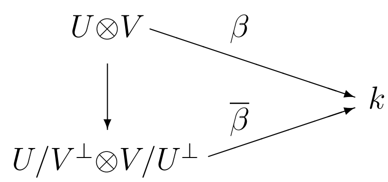
and $v \in V$. The form β is left (respectively right) non-singular if β_ℓ (respectively β_r) is one-one and β is non-singular if it is both left and right non-singular.

Suppose that A is an algebra over k, U is a right A-module, V is a left A-module, and $\beta : U\otimes V \longrightarrow k$ is a linear form. Then β is A-balanced if $\beta(u{\cdot}a, v) = \beta(u, a{\cdot}v)$ for all $u \in U$, $a \in A$, and $v \in V$.

For subspaces $X \subseteq U$ and $Y \subseteq V$ we define subspaces $X^\perp \subseteq V$ and $Y^\perp \subseteq U$ by

$$X^\perp = \{v \in V \mid \beta(X, v) = (0)\} \quad \text{and} \quad Y^\perp = \{u \in U \mid \beta(u, Y) = (0)\}.$$

Note that there is a form $\overline{\beta} : U/V^\perp \otimes V/U^\perp \longrightarrow k$ uniquely determined by the commutative diagram



where the vertical map is the tensor product of the projections. Observe that $V^\perp = \operatorname{Ker}\beta_\ell$, $U^\perp = \operatorname{Ker}\beta_r$, and that $\overline{\beta}$ is non-singular.

Let A be a bialgebra over k. A 2-cocycle for A is a convolution invertible linear form $\sigma : A\otimes A \longrightarrow k$ which satisfies

$$\sigma(x_{(1)}, y_{(1)})\sigma(x_{(2)}y_{(2)}, z) = \sigma(y_{(1)}, z_{(1)})\sigma(x, y_{(2)}z_{(2)})$$

for all $x, y, z \in A$. If σ is a 2-cocycle for A then A^σ is a bialgebra, where $A^\sigma = A$ as a coalgebra and multiplication $m^\sigma : A\otimes A \longrightarrow A$ is given by

$$m^\sigma(x\otimes y) = \sigma(x_{(1)}, y_{(1)})x_{(2)}y_{(2)}\sigma^{-1}(x_{(3)}, y_{(3)})$$

for all $x, y \in A$. See for example [6].

Let U and A be bialgebras over k and suppose that $\tau : U\otimes A \longrightarrow k$ is a linear form. Consider the axioms:

(A.1) $\tau(u, aa') = \tau(u_{(2)}, a)\tau(u_{(1)}, a')$ for all $u \in U$ and $a, a' \in A$;
(A.2) $\tau(1, a) = \epsilon(a)$ for all $a \in A$;
(A.3) $\tau(uu', a) = \tau(u, a_{(1)})\tau(u', a_{(2)})$ for all $u, u' \in U$ and $a \in A$;
(A.4) $\tau(u, 1) = \epsilon(u)$ for all $u \in U$.

We leave the reader with the exercise of establishing:

Lemma 1.1. *Let U and A be bialgebras over the field k and suppose $\tau : U\otimes A \longrightarrow k$ is a linear form. Then the following are equivalent:*

a) *(A.1)–(A.4) hold.*
b) *$\tau_\ell(U) \subseteq A^o$ and $\tau_\ell : U \longrightarrow A^{o\,cop} = A^{op\,o}$ is a bialgebra map.*
c) *$\tau_r(A) \subseteq U^o$ and $\tau_r : A \longrightarrow U^{o\,op}$ is a bialgebra map.*

□

Suppose that (A.1)–(A.4) hold, τ is convolution invertible, and define a linear form $\sigma : (U \otimes A) \otimes (U \otimes A) \longrightarrow k$ by $\sigma(u \otimes a, u' \otimes a') = \epsilon(a)\tau(u', a)\epsilon(a')$ for all $u, u' \in U$ and $a, a' \in A$. Then σ is a 2-cocycle. We denote the 2-cocycle twist bialgebra structure on the tensor product bialgebra $U \otimes A$ by $H = (U \otimes A)^\sigma$. Observe that

$$(u \otimes a)(u' \otimes a') = u\tau(u'_{(1)}, a_{(1)})u'_{(2)} \otimes a_{(2)}\tau^{-1}(u'_{(3)}, a_{(3)})a' \tag{1.1}$$

for all $u, u' \in U$ and $a, a' \in A$.

An easy, but important, exercise to do is the following.

Lemma 1.2. *Suppose that U, A are bialgebras over the field k and $\tau : U \otimes A \longrightarrow k$ satisfies (A.1)–(A.4). Then τ has a convolution inverse if*

a) *U is a Hopf algebra with antipode S, in which case $\tau^{-1}(u, a) = \tau(S(u), a)$ for all $u \in U$ and $a \in A$, or*
b) *A^{op} is a Hopf algebra with antipode T, in which case $\tau^{-1}(u, a) = \tau(u, T(a))$ for all $u \in U$ and $a \in A$.*

□

The quantum double provides an important example of a 2-cocycle twist bialgebra [6].

Example 1.3. *Let A be a finite-dimensional Hopf algebra with antipode s over k, let $U = A^{o\,cop}$, and let $\tau : U \otimes A \longrightarrow k$ be defined by $\tau(p, a) = p(a)$ for all $p \in U$ and $a \in A$. Then $\tau_\ell : U \longrightarrow A^{o\,cop}$ is the identity map and $(U \otimes A)^\sigma = D(A)$.*

Observe that finite-dimensionality was not necessary to define a bialgebra structure on $D(A) = A^{o\,cop} \otimes A$. For any Hopf algebra A with bijective antipode we let $D(A) = (U \otimes A)^\sigma$, where τ is defined as above.

Suppose that U, $\overline{U}$ and A, $\overline{A}$ are algebras over k. Suppose further that $\tau : U \otimes A \longrightarrow k$ and $\overline{\tau} : \overline{U} \otimes \overline{A} \longrightarrow k$ are convolution invertible linear forms satisfying (A.1)–(A.4). Set $H = (U \otimes A)^\sigma$ and $\overline{H} = (\overline{U} \otimes \overline{A})^{\overline{\sigma}}$. Suppose that $f : U \longrightarrow \overline{U}$ and $g : A \longrightarrow \overline{A}$ are bialgebra maps such that $\overline{\tau}(f(u), g(a)) = \tau(u, a)$ for all $u \in U$ and $a \in A$. Then $f \otimes g : H \longrightarrow \overline{H}$ is a bialgebra map.

As a consequence $f : H \longrightarrow D(A)$ defined by $f(u \otimes a) = \tau_\ell(u) \otimes a$ for all $u \in U$ and $a \in A$ is a bialgebra map. In this paper we are interested in left modules over H. A good source is modules for the double in light of the map f.

2. Algebra Structures on the Vector Space $U{\otimes}A$, where U and A are algebras over k

Suppose that U and A are algebras over the field k. In this section we are interested in algebra structures $H = U{\otimes}A$ on the tensor product of their underlying vector spaces which satisfy

$$(u{\otimes}a)(u'{\otimes}a') = uu'{\otimes}aa' \qquad \text{whenever} \qquad a = 1 \text{ or } u' = 1. \tag{2.1}$$

For such an algebra the maps $U \longrightarrow H$ and $A \longrightarrow H$ given by $u \mapsto u{\otimes}1$ and $a \mapsto 1{\otimes}a$ respectively are algebra maps. As a consequence H is a left U-module and a right A-module by pullback action; thus

$$u{\cdot}(u'{\otimes}a) = (u{\otimes}1)(u'{\otimes}a) = uu'{\otimes}a$$

and

$$(u{\otimes}a){\cdot}a' = (u{\otimes}a)(1{\otimes}a') = u{\otimes}aa'$$

for all $u, u' \in U$ and $a, a' \in A$.

We list several examples of these algebras.

Example 2.1. *Let U and A be algebras over the field k. Then the tensor product algebra structure on $H = U{\otimes}A$ satisfies (2.1).*

Example 2.2. *Let U and A be algebras over the field k and $H = U{\otimes}A$ be an algebra structure on the tensor product of their underlying vector spaces which satisfies (2.1). Endow the vector space $A^{op}{\otimes}U^{op} = A{\otimes}U$ with the unique algebra structure which makes the twist map $\tau_{A,U} : A{\otimes}U \longrightarrow (U{\otimes}A)^{op}$ an algebra isomorphism. This algebra $H^{\widetilde{op}} = A^{op}{\otimes}U^{op}$ satisfies (2.1).*

Example 2.3. *Let A be a finite-dimensional Hopf algebra with antipode s over k and let $U = A^{*\,cop}$. As a vector space the Drinfeld double is $D(A) = U{\otimes}A$ and its product is determined by*

$$(p{\otimes}a)(q{\otimes}b) = p(a_{(1)}{\cdot}q{\cdot}s^{-1}(a_{(3)})){\otimes}a_{(2)}b$$

for all $p, q \in U$ and $a, b \in A$. Thus the underlying algebra structure of $D(A)$ satisfies (2.1).

The next example is the most important one for us. By virtue of Example 1.3 the preceding example is a special case of it.

Example 2.4. *Let U and A be bialgebras over the field k and suppose that $\tau : U{\otimes}A \longrightarrow k$ is a convolution invertible map such (A.1)–(A.4) are satisfied. Then $H = (U{\otimes}A)^{\sigma}$ defined in Section 1 is a bialgebra whose underlying algebra structure satisfies (2.1).*

Example 2.5. *Let H be a Hopf algebra with bijective antipode over k and suppose that $R \in {}^H_H\mathcal{Y}D$ is a bialgebra in the Yetter–Drinfeld category. Then the bi-product $A = R\#H$ satisfies (2.1). More generally smash products satisfy (2.1).*

A basic reference for the bi-product is [14]. See [2] for a discussion of the Yetter–Drinfeld category ${}^H_H\mathcal{Y}D$ and bi-products.

Suppose that U, $\overline{U}$, A, and $\overline{A}$ are algebras over k and that the vector spaces $U \otimes A$ and $\overline{U} \otimes \overline{A}$ have algebra structures which satisfy (2.1). A morphism $F : U \otimes A \longrightarrow \overline{U} \otimes \overline{A}$ of these algebras is a map of algebras which satisfies $F(U \otimes 1) \subseteq \overline{U} \otimes 1$ and $F(1 \otimes A) \subseteq 1 \otimes \overline{A}$. By virtue of (2.1) a morphism $F : U \otimes A \longrightarrow \overline{U} \otimes \overline{A}$ has the form $F = f \otimes g$, where $f : U \longrightarrow \overline{U}$ and $g : A \longrightarrow \overline{A}$ are algebra maps. Conversely, if $f : U \longrightarrow \overline{U}$ and $g : A \longrightarrow \overline{A}$ are algebra maps, and the function $f \otimes g : U \otimes A \longrightarrow \overline{U} \otimes \overline{A}$ is an algebra map, then $F = f \otimes g$ is a morphism.

Suppose that U' is a subalgebra of U and A' is a subalgebra of A such that $U' \otimes A'$ is a subalgebra of $U \otimes A$. Then $U' \otimes A'$ satisfies (2.1) and the tensor product of the inclusion maps $i_{U'} \otimes i_{A'} : U' \otimes A' \longrightarrow U \otimes A$ is a morphism.

Now suppose I is an ideal of U, that J is an ideal of A, and $K = I \otimes A + U \otimes J$ is an ideal of $U \otimes A$. Let $\pi_I : U \longrightarrow U/I$ and $\pi_J : A \longrightarrow A/J$ be the projections. Endow $(U/I) \otimes (A/J)$ with the algebra structure which makes the linear isomorphism $(U \otimes A)/K \longrightarrow (U/I) \otimes (A/J)$ given by $u \otimes a + K \mapsto (u + I) \otimes (a + J)$ an isomorphism of algebras. Then $(U/I) \otimes (A/J)$ satisfies (2.1) and the tensor product of projections $\pi_I \otimes \pi_J : U \otimes A \longrightarrow (U/I) \otimes (A/J)$ is a morphism.

The constructions of the preceding paragraph can be combined to give a first isomorphism theorem for the morphisms of this section. Let $F : U \otimes A \longrightarrow \overline{U} \otimes \overline{A}$ be a morphism and write $F = f \otimes g$, where $f : U \longrightarrow \overline{U}$ and $g : A \longrightarrow \overline{A}$ are algebra maps. Then $\mathrm{Im}\, f$ and $\mathrm{Im}\, g$ are subalgebras of U and A respectively and $\mathrm{Im}\, F = \mathrm{Im}\, f \otimes \mathrm{Im}\, g$ is a subalgebra of $\overline{U} \otimes \overline{A}$ which thus satisfies (2.1). Now Ker and $\mathrm{Ker}\, g$ are ideals of U and A respectively and $\mathrm{Ker}\, F = \mathrm{Ker}\, f \otimes A + U \otimes \mathrm{Ker}\, g$ is an ideal of $U \otimes A$. Identifying $(U \otimes A)/K$ and $(U/\mathrm{Ker}\, f) \otimes (A/\mathrm{Ker}\, g)$ as above, note there is a unique morphism

$$\overline{F} : (U/\mathrm{Ker}\, f) \otimes (A/\mathrm{Ker}\, g) \longrightarrow \mathrm{Im}\, F = \mathrm{Im}\, f \otimes \mathrm{Im}\, g$$

such that $\overline{F} \circ (\pi_{\mathrm{Ker}\, f} \otimes \pi_{\mathrm{Ker}\, g}) = F$.

2.1. **The Construction of** $\mathrm{L}(\rho, \chi)$ **and** $R(\chi, \rho)$**.** Let U and A be algebras over k and $H = U \otimes A$ be an algebra structure on the tensor product of their underlying vector spaces which satisfies (2.1). We will identify U and A with their images under the algebra maps $U \longrightarrow U \otimes A$

and $A \longrightarrow U\otimes A$ given by $u \mapsto u\otimes 1$ and $a \mapsto 1\otimes a$ respectively. In this section we construct special representations of H by induction on one-dimensional representations of A and of U determined by pairs (ρ, χ), where ρ is a character of U and χ is a character of A.

Suppose that $\rho \in \mathrm{Alg}(U, k)$ and $\chi \in \mathrm{Alg}(A, k)$. We give k the left A-module structure $(k, \cdot_\chi)$ where

$$a{\cdot_\chi}1 = \chi(a)1 \qquad \text{for all } a \in A.$$

Recall that $H = U\otimes A$ is a right A-module via $(u\otimes a){\cdot}a' = u\otimes aa'$ for all $u \in U$ and $a, a' \in A$. We identify the left H-module $H\otimes_A k$ and U by the linear isomorphism $H\otimes_A k \longrightarrow U$ given by $(u\otimes a)_A\otimes 1 \mapsto u\chi(a)$ for all $u \in U$ and $a \in A$ and denote the resulting right H-module structure on U by $(U, \cdot_\chi)$. We write U_χ for U with this module action implicitly understood. Note that

$$(u\otimes a){\cdot_\chi}u' = u\big((\mathrm{I}_U\otimes\chi)((1\otimes a)(u'\otimes 1))\big)\,, \tag{2.2}$$

and consequently

$$u{\cdot_\chi}u' = uu' \qquad \text{and} \qquad a{\cdot_\chi}1 = \chi(a)1, \tag{2.3}$$

for all $u, u' \in U$ and $a \in A$. Thus 1 generates U_χ as a left H-module and $k1$ is a one-dimensional left A-submodule of U_χ.

Let $I(\rho, \chi)$ be the sum of all the left H-submodules of U_χ contained in $\mathrm{Ker}\,\rho$, let

$$L(\rho, \chi) = U_\chi/I(\rho, \chi)$$

be the resulting quotient left H-module, and let $\pi_{(\rho,\chi)} : U_\chi \longrightarrow L(\rho, \chi)$ be the projection. Using (2.3) we see that $M = L(\rho, \chi)$ is a cyclic left H-module generated by $\pi_{(\rho,\chi)}(1)$, that $k\pi_{(\rho,\chi)}(1)$ is a one-dimensional left A-submodule of M, and that $N = \pi_{(\rho,\chi)}(\mathrm{Ker}\,\rho)$ is a codimension one left U-submodule of M with the property that the only left H-submodule of M contained in N is (0).

In a similar manner we define a right H-module structure on A. Regard k as the right U-module $(k, \cdot_\rho)$ where

$$1_\rho{\cdot}u = \rho(u)1 \qquad \text{for all } u \in U.$$

Recall that $H = U\otimes A$ is a left U-module via $u{\cdot}(u'\otimes a) = uu'\otimes a$ for all $u, u' \in U$ and $a \in A$. We identify the right H-module $k\otimes_U H$ and the vector space A by the linear isomorphism $k\otimes_U H \longrightarrow A$ given by $1\otimes_U(u\otimes a) \mapsto \rho(u)a$ for all $u \in U$ and $a \in A$ and we denote the resulting module structure on A by $(A, \cdot_\rho)$. Observe that

$$a{\cdot_\rho}(u\otimes a') = \big((\rho\otimes\mathrm{I}_A)((1\otimes a)(u\otimes 1))\big)a', \tag{2.4}$$

and thus

$$a{\cdot_\rho}a' = aa' \qquad \text{and} \qquad 1{\cdot_\rho}u = \rho(u)1, \tag{2.5}$$

for all $a, a' \in A$ and $u \in U$. As a consequence 1 generates A_ρ as a right H-module and $k1$ is a one-dimensional left U-submodule of A_ρ.

Let $J(\chi, \rho)$ be the sum of all the right H-submodules of A_ρ contained in $\operatorname{Ker}\chi$, let

$$R(\chi, \rho) = A_\rho / J(\chi, \rho)$$

be the quotient right H-module, and $\pi_{(\chi,\rho)} : A_\rho \longrightarrow R(\chi, \rho)$ be the projection. Using (2.5) we see that $M = R(\chi, \rho)$ is a cyclic right H-module generated by $\pi_{(\chi,\rho)}(1)$, that $k\pi_{(\chi,\rho)}(1)$ is a one-dimensional right U-submodule of M, and that $N = \pi_{(\chi,\rho)}(\operatorname{Ker}\chi)$ is a codimension one left A-submodule of M with the property that the only right H-submodule of M contained in N is (0).

2.2. **A Bilinear Form Arising from Character Pairs (ρ, χ).** We continue with the notation and assumptions of the preceding section. Let $\chi \in \operatorname{Alg}(A, k)$, let $\rho \in \operatorname{Alg}(U, k)$, and let $\Psi : A \otimes U \longrightarrow k$ be the linear form defined by

$$\Psi(a, u) = (\rho \otimes \chi)\big((1 \otimes a)(u \otimes 1)\big)$$

for all $a \in A$ and $u \in U$. The $L(\rho, \chi)$ and $R(\chi, \rho)$ constructions of the preceding section are related in fundamental ways through this form.

Regard U^* as a right H-module with the transpose action determined by U_χ and likewise regard A^* as left H-module with the transpose action determined by A_ρ. Our notations for these actions are given in the equations

$$(u^*{\cdot_\chi}h)(u) = u^*(h{\cdot_\chi}u) \qquad \text{and} \qquad (h{\cdot_\rho}a^*)(a) = a^*(a{\cdot_\rho}h)$$

for all $u^* \in U^*$, $h \in H$, $u \in U$, $a^* \in A^*$, and $a \in A$.

Proposition 2.6. *Let U and A be algebras over the field k and let $H = U \otimes A$ be an algebra structure on the tensor product of their underlying vector spaces which satisfies (2.1). Let $\chi \in \operatorname{Alg}(A, k)$, let $\rho \in \operatorname{Alg}(U, k)$, and let $\Psi : A \otimes U \longrightarrow k$ be the bilinear form defined above. Then:*

a) $\Psi(a, u) = \rho((1 \otimes a){\cdot_\chi}u) = \chi(a{\cdot_\rho}(u \otimes 1))$ *for all $a \in A$ and $u \in U$.*
b) $\Psi(a{\cdot_\rho}h, u) = \Psi(a, h{\cdot_\chi}u)$ *for all $a \in A$, $h \in H$, and $u \in U$; that is Ψ is H-balanced.*
c) $A^\perp = I(\rho, \chi)$ *and thus* $U/A^\perp = L(\rho, \chi)$. *Furthermore there is an isomorphism of left H-modules $L(\rho, \chi) \longrightarrow U{\cdot_\rho}\chi$ given by $u + I(\rho, \chi) \mapsto \Psi_r(u) = u{\cdot_\rho}\chi$ for all $u \in U$.*

d) *$U^\perp = J(\chi,\rho)$ and thus $A/U^\perp = R(\chi,\rho)$. Furthermore there is an isomorphism of right H-modules $R(\chi,\rho) \longrightarrow \rho{\cdot_\chi}A$ given by $a + J(\chi,\rho) \mapsto \Psi_\ell(a) = \rho{\cdot_\chi}a$ for all $a \in A$.*

e) *There is a non-singular H-balanced bilinear form*

$$\overline{\Psi} : R(\chi,\rho)\otimes L(\rho,\chi) \longrightarrow k$$

determined by the commutative diagram

$$\begin{array}{ccc} A\otimes U & \xrightarrow{\quad\Psi\quad} & k \\ \big\downarrow & \nearrow_{\overline{\Psi}} & \\ R(\chi,\rho)\otimes L(\rho,\chi) & & \end{array}$$

where the vertical arrow is the tensor product of projections.

Proof: Part a) follows by the definition of the form and of the module actions. To show part b) we use several standard isomorphisms involving tensor products. Note that k is a left and right A-module via $a{\cdot_\chi}1 = 1{\cdot_\chi}a = \chi(a)1$ for all $a \in A$ and k is also a left and right U-module via $u{\cdot_\rho}1 = 1{\cdot_\rho}u = \rho(u)1$ for all $u \in U$. Consider the composites $f : A\otimes U \longrightarrow k\otimes_U(H\otimes_A k)$ and $g : k\otimes_U(H\otimes_A k) \longrightarrow k$ defined by

$$\begin{aligned} A\otimes U \simeq (k\otimes_U H)\otimes(H\otimes_A k) &\longrightarrow (k\otimes_U H)\otimes_H(H\otimes_A k) \\ &\simeq k\otimes_U(H\otimes_H(H\otimes_A k)) \simeq k\otimes_U(H\otimes_A k) \end{aligned}$$

and

$$k\otimes_U(H\otimes_A k) \xrightarrow{\mathrm{I}_k\otimes((\chi\otimes\rho)\otimes\mathrm{I}_k)} k\otimes_U(k\otimes_A k) \xrightarrow{\mathrm{I}_k\otimes m} k\otimes_U k \xrightarrow{m} k$$

respectively, where m is multiplication. Since

$$(g\circ f)(a\otimes u) = g(1\otimes_U((1\otimes a)(u\otimes 1)_A\otimes 1)) = (\rho\otimes\chi)((1\otimes a)(u\otimes 1))$$

for all $a \in A$ and $u \in U$ we conclude that $\Psi = g\circ f$. From the second isomorphism in the definition of f we see that $f((a{\cdot_\rho}h)\otimes u) = f(a\otimes(h{\cdot_\chi}u))$ for all $a \in A$ and $u \in U$. Part b) now follows.

We next show part c). Let $u \in U$. Since $\Psi_r(u)(1) = \rho(u)$ by part a) it follows that $\operatorname{Ker}\Psi_r \subseteq \operatorname{Ker}\rho$. Now $\operatorname{Ker}\Psi_r$ is a left H-submodule of U_χ by part b). Suppose that L is a left H-submodule of U_χ and $L \subseteq \operatorname{Ker}\rho$. Then by part a) again $\Psi(A,L) \subseteq \rho((1\otimes A){\cdot_\chi}L) \subseteq \rho(L) = (0)$. Therefore $L \subseteq \operatorname{Ker}\Psi_r$. We have shown that $A^\perp = \operatorname{Ker}\Psi_r = I(\rho,\chi)$.

To complete the proof of part c) we note that $\Psi_r : U_\chi \longrightarrow A^*$ is a map of left H-modules by part b). Let $u \in U$. Since $\Psi_r(u) = \Psi_r(u{\cdot_\chi}1) = u{\cdot_\rho}\Psi_r(1) = u{\cdot_\rho}\chi$, the isomorphism of left H-modules

$$L(\rho,\chi) = U_\chi/A^\perp = U/\operatorname{Ker}\Psi_r \simeq U{\cdot_\rho}\chi$$

is given by $u+I(\rho,\chi)\mapsto u\cdot_\rho\chi$. We have established part c). The proof of part d) is similar. Part e) follows from parts b)–d). $\square$

By virtue of part c) the quotient $L(\rho,\chi)$ can be realized as submodule.

2.3. **The Connection Between Morphisms of the Algebra Structures and the Constructions** $\mathrm{L}(\rho,\chi)$**,** $R(\chi,\rho)$**.** Let U, $\overline{U}$, A, and $\overline{A}$ be algebras over k such that $H=U\otimes A$ and $\overline{H}=\overline{U}\otimes\overline{A}$ are algebras which satisfy (2.1). Suppose that $F:U\otimes A\longrightarrow\overline{U}\otimes\overline{A}$ is a morphism and let $f:U\longrightarrow\overline{U}$ and $g:A\longrightarrow\overline{A}$ be the unique algebra maps which satisfy $F=f\otimes g$.

Now suppose that $\overline{\rho}\in\mathrm{Alg}(\overline{U},k)$ and $\overline{\chi}\in\mathrm{Alg}(\overline{A},k)$. Then $\rho=\overline{\rho}\circ f\in\mathrm{Alg}(U,k)$ and $\chi=\overline{\chi}\circ g\in\mathrm{Alg}(A,k)$. In this section we examine the connection between $L(\rho,\chi)$ and $L(\overline{\rho},\overline{\chi})$ and also the relationship between $R(\chi,\rho)$ and $R(\overline{\rho},\overline{\chi})$.

First we consider U_χ and $\overline{U}_{\overline{\chi}}$. Let $u,u'\in U$ and $a\in A$. By the calculation

$$\begin{aligned}
f((u\otimes a)\cdot_\chi u') &= f(u\big((I_U\otimes\chi)((1\otimes a)(u'\otimes 1))\big)) \\
&= f(u\big((I_U\otimes(\overline{\chi}\circ g))((1\otimes a)(u'\otimes 1))\big)) \\
&= f(u)\big((I_U\otimes\overline{\chi})\,\big((f\otimes g)((1\otimes a)(u'\otimes 1))\big)\big) \\
&= f(u)\big((I_U\otimes\overline{\chi})\,\big(F((1\otimes a)(u'\otimes 1))\big)\big) \\
&= f(u)\big((I_U\otimes\overline{\chi})\,\big(F(1\otimes a)F(u'\otimes 1)\big)\big) \\
&= f(u)\big((I_U\otimes\overline{\chi})\,\big((1\otimes f(a))(g(u')\otimes 1)\big)\big) \\
&= F(u\otimes a)\cdot_{\overline{\chi}} f(u')
\end{aligned}$$

we see that

$$f((u\otimes a)\cdot_\chi u')=F(u\otimes a)\cdot_{\overline{\chi}} f(u'). \tag{2.6}$$

Thus f is F-linear and the preceding equation has a simple interpretation in terms of module maps. Regard $\overline{U}_{\overline{\chi}}$ as a left H-module by pullback along F. Then $f:U_\chi\longrightarrow\overline{U}_{\overline{\chi}}$ is a map of left H-modules.

Since $\overline{\rho}\circ f=\rho$, we have $f(\mathrm{Ker}\,\rho)\subseteq\mathrm{Ker}\,\overline{\rho}$. Thus $f(I(\rho,\chi))\subseteq I(\overline{\rho},\overline{\chi})$ as f is F-linear. Consequently f gives rise to an F-linear map $L(f):L(\rho,\chi)\longrightarrow L(\overline{\rho},\overline{\chi})$.

Proposition 2.7. *Suppose U, $\overline{U}$, A, and $\overline{A}$ are algebras over k. Let $H=U\otimes A$ and $\overline{H}=\overline{U}\otimes\overline{A}$ be algebra structures on the tensor product of underlying vector spaces which satisfy (2.1). Suppose further that $F:U\otimes A\longrightarrow\overline{U}\otimes\overline{A}$ is a morphism and let $f:U\longrightarrow\overline{U}$ and $g:A\longrightarrow\overline{A}$ be the unique algebra maps which satisfy $F=f\otimes g$. Then:*

a) $f : U_\chi \longrightarrow \overline{U}_{\overline{\chi}}$ *is* F*-linear.*
b) *There is a unique* F*-linear map* $L(f) : L(\rho, \chi) \longrightarrow L(\overline{\rho}, \overline{\chi})$ *which makes the diagram*

$$\begin{array}{ccc} L(\rho,\chi) & \xrightarrow{L(f)} & L(\overline{\rho},\overline{\chi}) \\ \uparrow & & \uparrow \\ U_\chi & \xrightarrow{f} & \overline{U}_{\overline{\chi}} \end{array}$$

commute, where the vertical arrows are the projection maps.
c) *Suppose that* f *is onto. Then* $L(f)$ *is an isomorphism.*

PROOF: We have established parts a) and b) in the discussion preceding the statement of the proposition. It remains to show part c).

We first observe that $\overline{\rho} \circ f = \rho$ implies $\operatorname{Ker} \rho = f^{-1}(\operatorname{Ker} \overline{\rho})$. Therefore $f^{-1}(I(\overline{\rho}, \overline{\chi}))$ is a left H-submodule of U_χ contained in $\operatorname{Ker} \rho$ which implies $f^{-1}(I(\overline{\rho}, \overline{\chi})) \subseteq I(\rho, \chi)$. We have seen that $f(I(\rho, \chi)) \subseteq I(\overline{\rho}, \overline{\chi})$ in any event. Therefore $f^{-1}(I(\overline{\rho}, \overline{\chi})) = I(\rho, \chi)$ which implies that $L(f)$ is an isomorphism. □

Suppose that F is onto. Then the hypothesis of part c) is met. Regard $I(\overline{\rho}, \overline{\chi})$ as a left H-module by pullback along F. Then the H-submodules and $\overline{H}$-submodules of $L(\overline{\rho}, \overline{\chi})$ are the same and the H-module $L(\rho, \chi)$ is understood in terms of the left $\overline{H}$-module $L(\overline{\rho}, \overline{\chi})$ and the algebra map F.

There is an analog of the preceding proposition for $R(\chi, \rho)$ and $R(\overline{\chi}, \overline{\rho})$. One can show that

$$g(a \cdot_\rho (u \otimes a')) = g(a) \cdot_{\overline{\rho}} F(u \otimes a') \tag{2.7}$$

for all $a, a' \in A$ and $u \in U$ by mimicking the calculation which establishes (2.6). By modifying the proof of the preceding proposition one can easily show:

Proposition 2.8. *Suppose* U*,* $\overline{U}$*,* A*, and* $\overline{A}$ *are algebras over* k*. Let* $H = U \otimes A$ *and* $\overline{H} = \overline{U} \otimes \overline{A}$ *be algebra structures on the tensor product of underlying vector spaces which satisfy (2.1). Suppose further that* $F : U \otimes A \longrightarrow \overline{U} \otimes \overline{A}$ *is a morphism and let* $f : U \longrightarrow \overline{U}$ *and* $g : A \longrightarrow \overline{A}$ *be algebra maps which satisfy* $F = f \otimes g$*. Then:*

a) $g : A_\rho \longrightarrow \overline{A}_{\overline{\rho}}$ *is* F*-linear.*
b) *There is a unique* F*-linear map* $R(g) : R(\chi, \rho) \longrightarrow R(\overline{\chi}, \overline{\rho})$ *which makes the diagram*

$$\begin{array}{ccc} R(\chi,\rho) & \xrightarrow{R(g)} & R(\overline{\chi},\overline{\rho}) \\ \uparrow & & \uparrow \\ A_\rho & \xrightarrow{g} & \overline{A}_{\overline{\rho}} \end{array}$$

commute, where the vertical arrows are the projection maps.

c) *Suppose that g is onto. Then $R(g)$ is an isomorphism.*

□

We conclude this section by noting the relationship between the linear forms $\Psi : A \otimes U \longrightarrow k$ and $\overline{\Psi} : \overline{U} \otimes \overline{A} \longrightarrow k$ defined by

$$\Psi(a,u) = (\rho \otimes \chi)((1 \otimes a)(u \otimes 1)) \quad \text{and} \quad \overline{\Psi}(\overline{a},\overline{u}) = (\overline{\rho} \otimes \overline{\chi})((1 \otimes \overline{a})(\overline{u} \otimes 1))$$

for all $a \in A$, $u \in U$, $\overline{a} \in \overline{A}$, and $\overline{u} \in \overline{U}$. Since $F = f \otimes g$ is an algebra map, the calculation

$$\begin{aligned} (f \otimes g)((1 \otimes a)(u \otimes 1)) &= ((f \otimes g)(1 \otimes a))((f \otimes g)(u \otimes 1)) \\ &= (1 \otimes g(a))(f(u) \otimes 1) \end{aligned}$$

shows that

$$\Psi(a,u) = \overline{\Psi}(g(a), f(u)) = \overline{\Psi}(F(a), F(u))$$

for all $a \in A$ and $u \in U$. In the last expression we regard $a, u \in H$ by the identifications $a = 1 \otimes a$ and $u = u \otimes 1$.

2.4. **The Categories ${}_H\underline{\mathcal{M}}$ and $\underline{\mathcal{M}}_H$ and Duality.** Let $H = U \otimes A$ be an algebra structure defined on the tensor product of their underlying vector spaces and suppose that (2.1) holds for H. Let ${}_H\underline{\mathcal{M}}$ be the category whose objects M are left H-modules which are generated by a one-dimensional left A-submodule km and have a codimension-one left U-submodule N with the property that (0) is the only left H-submodule of M contained in N. We take maps of left H-modules to be our morphisms. The category $\underline{\mathcal{M}}_H$ is defined in the same manner with "right" replacing "left" and with the roles of U and A reversed.

For M as described above observe that

$$(2.8) \qquad \mathrm{ann}_A(km) = \mathrm{Ker}\,\chi \qquad \text{and} \qquad \mathrm{ann}_U(M/N) = \mathrm{Ker}\,\rho$$

for some characters $\chi \in \mathrm{Alg}(A,k)$ and $\rho \in \mathrm{Alg}(U,k)$. For any pair of characters $\chi \in \mathrm{Alg}\,(A,k)$ and $\rho \in \mathrm{Alg}(U,k)$ observe that $M = L(\rho,\chi)$ is an object of ${}_H\mathcal{M}$ which satisfies (2.8).

Conversely, suppose that M is an object of ${}_H\underline{\mathcal{M}}$ which satisfies (2.8). Then the rule $H \otimes_A k \longrightarrow M$ given by $h \otimes_A 1 \mapsto h{\cdot}m$ is a well-defined map of left H-modules and the composite $f : U \simeq H \otimes_A k \longrightarrow M$, which is given by $f(u) = u{\cdot}m$ for all $u \in U$, has kernel $\mathrm{I}(\chi, \rho)$. Therefore f lifts to an isomorphism of left H-modules $L(\rho, \chi) \simeq M$. Observe that $f^{-1}(N) = \operatorname{Ker}\rho$. Therefore N is the only codimension one left U-submodule N' of M such that $\mathrm{ann}_U(M/N') = \operatorname{Ker}\rho$.

To discuss duality we need to specify a particular one-dimensional left A-submodule and a particular codimension-one U-submodule of each object M of ${}_H\underline{\mathcal{M}}$. Let ${}_H\underline{\mathcal{M}}'$ be the category whose objects are triples (M, km, N), where M is a left H-module, km is a left A-submodule of M which generates M as a left H-module, and N is a codimension-one left U-submodule of M such that (0) is the only left H-submodule of M contained in N. A morphism $f : (M, km, N) \longrightarrow (M', km', N')$ of ${}_H\underline{\mathcal{M}}'$ is a map of left H-modules which satisfies $f(km) \subseteq km'$ and $f(N) \subseteq N'$. We define a category $\underline{\mathcal{M}}'_H$ in the same manner replacing "left" by "right".

There is a natural contravariant functor ${}_H\underline{\mathcal{M}}' \longrightarrow \underline{\mathcal{M}}'_H$. To describe it we start in a slightly more general context.

Consider a triple (M, km, N), where M is a cyclic left H-module generated by m, where N is a codimension one left U-submodule of M, and (2.8) is satisfied for some $\chi \in \mathrm{Alg}(A, k)$ and $\rho \in \mathrm{Alg}(U, k)$. Thus we are not requiring that the only left H-submodule of M contained in N is (0). We regard $M^* \in \mathcal{M}_H$ by the transpose action on $M \in {}_H\mathcal{M}$. Since N is a subspace of M of codimension one and $m \notin N$ there is a non-zero $m^\bullet \in M^*$ uniquely determined by $m^\bullet(N) = (0)$ and $m^\bullet(m) = 1$.

Consider the right H-submodule $M^\bullet = m^\bullet{\cdot}H$ of M^*. Now $\operatorname{Ker}\rho = \mathrm{ann}_U(M/N)$ implies that $(\operatorname{Ker}\rho){\cdot}M \subseteq N$. From the calculation

$$(m^\bullet{\cdot}(\operatorname{Ker}\rho))(M) = m^\bullet((\operatorname{Ker}\rho){\cdot}M) \subseteq m^\bullet(N) = (0)$$

we conclude that $m^\bullet{\cdot}(\operatorname{Ker}\rho) = (0)$. Therefore $m^\bullet{\cdot}u = \rho(u)m^\bullet$ for all $u \in U$. The calculation

$$(m^\perp{\cdot}A)(m) = m^\perp(A{\cdot}m) = m^\perp(km) = (0)$$

shows that $m^\perp{\cdot}A \subseteq m^\perp$. Therefore $N^\bullet = m^\perp \cap M^\bullet$ is a right A-submodule of $M^\bullet$. Since

$$(0) \neq m^\bullet(M) = m^\bullet(H{\cdot}m) = (m^\bullet{\cdot}H)(m) = M^\bullet(m)$$

it follows that $M^\bullet \not\subseteq m^\perp$. This means that the right A-submodule $N^\bullet$ is a codimension one subspace of $M^\bullet$. Since

$$(M^\bullet{\cdot}(\operatorname{Ker}\chi))(m) = M^\bullet((\operatorname{Ker}\chi){\cdot}m) = M^\bullet(0) = (0)$$

we conclude that $M^\bullet{\cdot}(\operatorname{Ker}\chi) \subseteq m^\perp \cap M^\bullet = N^\bullet$. We have shown that

$$\mathrm{ann}_U(km^\bullet) = \operatorname{Ker}\rho \qquad \text{and} \qquad \mathrm{ann}_A(M^\bullet/N^\bullet) = \operatorname{Ker}\chi. \tag{2.9}$$

We next show that the only right H-submodule of $M^\bullet$ contained in $N^\bullet$ is (0). Let L be a right H-submodule of $M^\bullet$ contained in $N^\bullet$. Then $L(M) = L(H{\cdot}m) = (L{\cdot}H)(m) \subseteq m^\perp(m) = (0)$ implies that $L = (0)$. We have shown that $(M^\bullet, km^\bullet, N^\bullet) \in \underline{\mathcal{M}}'_H$; in particular $M^\bullet \simeq R(\chi, \rho)$.

Consider the bilinear form $\beta : M^\bullet \otimes M \longrightarrow k$ given by $\beta(p, n) = p(n)$ for all $p \in M^\bullet$ and $n \in M$. Note that β is right non-singular. By definition of the transpose module action $\beta(p{\cdot}h, n) = \beta(p, h{\cdot}n)$ for all $p \in M^\bullet$, $h \in H$, and $n \in M$; that is β is H-balanced. We observe that $(M^\bullet)^\perp$ is the largest H-submodule of M contained in N. For let $n \in N$. Then $H{\cdot}n \subseteq N$ if and only if $(0) = m^\bullet(H{\cdot}n) = (m^\bullet{\cdot}H)(n) = M^\bullet(n)$; that is $H{\cdot}n \subseteq N$ if and only if $M^\bullet(n) = (0)$. Thus $M/(M^\bullet)^\perp \simeq L(\rho, \chi)$ and β induces an H-balanced bilinear form $\overline{\beta} : M^\bullet \otimes (M/(M^\bullet)^\perp) \longrightarrow k$. Compare with Proposition 2.6.

Proposition 2.9. *Let U and A be algebras over the field k, let $H = U \otimes A$ be an algebra structure on the tensor product of their underlying vector spaces which satisfies (2.1), and let $(M, km, N) \in {}_H\underline{\mathcal{M}}'$.*

a) *Let $\chi \in \mathrm{Alg}(A, k)$ and $\rho \in \mathrm{Alg}(U, k)$ satisfy*

$$\mathrm{ann}_A(km) = \mathrm{Ker}\,\chi \qquad \textit{and} \qquad \mathrm{ann}_U(M/N) = \mathrm{Ker}\,\rho.$$

Then $(M^\bullet, km^\bullet, N^\bullet) \in \underline{\mathcal{M}}'_H$ and satisfies

$$\mathrm{ann}_U(km^\bullet) = \mathrm{Ker}\,\rho \qquad \textit{and} \qquad \mathrm{ann}_A(M^\bullet/N^\bullet) = \mathrm{Ker}\,\chi.$$

b) *Suppose that $f : (M, km, N) \longrightarrow (M', km', N')$ is a morphism in ${}_H\underline{\mathcal{M}}'$. Then $f^*(M'^\bullet) \subseteq M^\bullet$ and the restriction $f^r = f^*|M'^\bullet$ is a morphism $f^r : (M'^\bullet, m'^\bullet, N'^\bullet) \longrightarrow (M^\bullet, m^\bullet, N^\bullet)$ in $\underline{\mathcal{M}}'_H$.*

c) *Suppose that $N^\bullet$ is the only codimension-one right A-submodule of $M^\bullet$. Then km is the only one-dimensional left A-submodule of M.*

Proof: We have shown part a). Part b) is left as an easy exercise. We establish part c).

Suppose that $n \in M$ and $A{\cdot}n = kn$. Then $n^\perp \cap M^\bullet = M^\bullet$, in which case $M^\bullet(n) = (0)$, or $n^\perp \cap M^\bullet$ is a codimension one right A-submodule of $M^\bullet$, in which case $n^\perp \cap M^\bullet = m^\perp \cap M^\bullet$ by assumption. In any event $N^\bullet(n) = (0)$.

Let $C = \{n \in M \,|\, N^\bullet(n) = (0)\}$. Observe that $m \in C$. The rule $f : C \longrightarrow (M^\bullet/N^\bullet)^*$ given by $f(n)(p + N^\bullet) = p(n)$ is describes a well-defined linear function. We show that f is one-one. Suppose that $n \in C$ and $f(n) = 0$. Then

$$(0) = f(n)(M^\bullet) = M^\bullet(n) = (m^\bullet{\cdot}H)(n) = m^\bullet(H{\cdot}n)$$

implies that $H{\cdot}n \subseteq \operatorname{Ker} m^\bullet = N^\bullet$. But $N^\bullet$ contains no left H-submodules other than (0). Therefore $n = 0$. We have shown that f is one-one.

Since $m \in C$, f is one-one, and $\operatorname{Im} f$ is at most one-dimensional, it follows that $C = km$. This concludes our proof. □

The "right" counterpart of the preceding proposition holds by virtue of Example 2.2.

3. The Main Results for ${}_H\underline{\mathcal{M}}$

We begin by describing the type of algebra of fundamental importance in Sections 6 and 7. These algebras A (and U also) are generated by a subgroup Γ of the group of units of A and a indexed set of elements $\{a_i\}_{i\in I}$. There is an indexed set of characters $\{\chi_i\}_{i\in I} \subseteq \widehat{\Gamma}$ such that $ga_ig^{-1} = \chi_i(g)a_i$ for all $g \in \Gamma$ and $i \in I$. Let A' be the subalgebra of A generated by Γ.

Suppose that $\chi_i \neq 1$ for all $i \in I$. Then $\rho(a_i) = 0$ for all $\rho \in \operatorname{Alg}(A, k)$ and $i \in I$. Thus the restriction map $\operatorname{Alg}(A, k) \longrightarrow \operatorname{Alg}(A', k)$ is one-one. In important applications U and A below will have this description and thus the restriction map is one-one.

The theorem of this section is derived from two results.

Lemma 3.1. *Let U and A be algebras over the field k and let $H = U{\otimes}A$ be an algebra structure on the tensor product of the underlying vector spaces of U and A which satisfies (2.1). Suppose that U' is a subalgebra of U such that:*

a) *The restriction map $\operatorname{Alg}(U, k) \longrightarrow \operatorname{Alg}(U', k)$ is one-one*
b) *and $(u{\otimes}a){\cdot}_\chi u' = \chi(a)uu'$ for all $u \in U$, $a \in A$, $\chi \in \operatorname{Alg}(A, k)$, and $u' \in U'$.*

Let $\rho, \rho' \in \operatorname{Alg}(U, k)$ and $\chi, \chi' \in \operatorname{Alg}(A, k)$. If $L(\rho, \chi) \simeq L(\rho', \chi')$ as left U-modules then $\rho = \rho'$. In particular there is a unique codimension one left U-module of $L(\rho, \chi)$ which contains $I(\rho, \chi)$.

Proof: Suppose that $L(\rho, \chi) \simeq L(\rho', \chi')$ as left U-modules and consider the composite of left U-modules $f : U \longrightarrow L(\rho, \chi) \simeq L(\rho', \chi') = M$. Since the latter contains a codimension one left U-submodule N with $\operatorname{ann}_U(M/N) = \operatorname{Ker} \rho'$, it follows that $I(\rho, \chi) \subseteq f^{-1}(N) = \operatorname{Ker} \rho'$. We have shown that $I(\rho, \chi) \subseteq \operatorname{Ker}\rho'$. The calculation $\rho((u{\otimes}a){\cdot}_\chi u') = \rho(\chi(a)uu') = \chi(a)\rho(u)\rho(u')$ for all $u \in U$, $a \in A$, and $u' \in U'$ shows that $H{\cdot}_\chi(\operatorname{Ker} \rho{\cap}U') \subseteq \operatorname{Ker}\rho$. Therefore

$$(\operatorname{Ker} \rho){\cap}U' \subseteq I(\rho, \chi) \subseteq \operatorname{Ker}\rho'$$

which implies that $(\operatorname{Ker} \rho){\cap}U' = (\operatorname{Ker} \rho'){\cap}U'$ since both intersections are codimension one subspaces of U'. The preceding equation implies

$\rho|U' = \rho'|U'$ from which $\rho = \rho'$ follows by assumption. The last statement in the conclusion of the lemma is evident. □

Lemma 3.2. *Let U and A be algebras over the field k and let $H = U \otimes A$ be an algebra structure on the tensor product of the underlying vector spaces of U and A which satisfies (2.1). Suppose that A' is a subalgebra of A such that:*

a) *The restriction map $\mathrm{Alg}(A,k) \longrightarrow \mathrm{Alg}(A',k)$ is one-one and*
b) *$a' \cdot_\rho (u \otimes a) = \rho(u)a'a$ for all $a' \in A'$, $\rho \in \mathrm{Alg}(U,k)$, $u \in U$, and $a \in A$.*

Let $\rho \in \mathrm{Alg}(U,k)$ and $\chi, \chi' \in \mathrm{Alg}\,(A,k)$. If $L(\rho,\chi) \simeq L(\rho,\chi')$ as left H-modules then $\chi = \chi'$.

Proof: Regard k as a left U-module via $u \cdot 1 = \rho(u)$ for all $u \in U$. Consider the composite $\rho' : U_\chi \longrightarrow k$ of left U-module maps given by $U_\chi \longrightarrow L(\rho,\chi) \xrightarrow{f} L(\rho,\chi') \xrightarrow{\overline{\rho}} k$, where the first map is the projection, f is an isomorphism of left H-modules, and the third $\overline{\rho}$ is the map is given by $u + I(\rho,\chi') \mapsto \rho(u)$ for all $u \in U$. Since ρ' is a left U-module map we have $\rho'(u) = \rho'(u1) = u \cdot \rho'(1) = \rho(u)\rho'(1)$. Therefore $\rho' = \rho'(1)\rho$.

Let $u \in U$ satisfy $f(1 + I(\rho,\chi)) = u + I(\rho,\chi')$ and let $a \in A$. Using the definition of ρ', the fact that f is a map of left A-modules, and part a) of Proposition 2.6, we see that

$$\begin{aligned}
\rho'((1 \otimes a) \cdot_\chi (1 + I(\rho,\chi)) &= \rho(f((1 \otimes a) \cdot_\chi (1 + I(\rho,\chi))) \\
&= \overline{\rho}((1 \otimes a) \cdot_{\chi'} f(1 + I(\rho,\chi))) \\
&= \overline{\rho}((1 \otimes a) \cdot_{\chi'} (u + I(\rho,\chi'))) \\
&= \rho((1 \otimes a) \cdot_{\chi'} u) \\
&= \chi'(a \cdot_\rho (u \otimes 1)).
\end{aligned}$$

Since $\rho' = \rho'(1)\rho$ we calculate on the other hand that

$$\begin{aligned}
\rho'((1 \otimes a) \cdot_\chi (1 + I(\rho,\chi)) &= \rho'(1)\rho((1 \otimes a) \cdot_\chi (1 + I(\rho,\chi)) \\
&= \rho'(1)\rho((1 \otimes a) \cdot_\chi 1) \\
&= \rho'(1)\chi(a).
\end{aligned}$$

We have shown that $\chi'(a \cdot_\rho (u \otimes 1)) = \rho'(1)\chi(a)$ for all $a \in A$. Now suppose that $a' \in A'$. By virtue of the preceding equation

$$\chi'(a')\rho(u) = \chi'(a'\rho(u)\chi'(1)) = \chi'(a' \cdot_\rho (u \otimes 1)) = \rho'(1)\chi(a');$$

the second equation follows by assumption. Therefore $\rho'(u) = 1$ and $\chi(a') = \chi'(a')$ for all $a' \in A'$. By assumption $\chi = \chi'$. □

For a category $\mathcal{M}$ we denote the isomorphism classes of objects in $\mathcal{M}$ by $[\mathcal{M}]$ and for an object $M \in \mathcal{M}$ we let $[M]$ be the isomorphism class of M. As a consequence of the two preceding lemmas:

Theorem 3.3. *Let U and A be algebras over the field k and let $H = U \otimes A$ be an algebra structure on the tensor product of the underlying vector spaces of U and A which satisfies (2.1). Suppose that:*

a) *U' is a subalgebra of U, $(u \otimes a) \cdot_\chi u' = \chi(a) u u'$ for all $u \in U$, $a \in A$, $\chi \in \mathrm{Alg}(A, k)$, and $u' \in U'$, and the restriction map $\mathrm{Alg}(U, k) \longrightarrow \mathrm{Alg}(U', k)$ is one-one;*
b) *A' is a subalgebra of A, $a' \cdot_\rho (u \otimes a) = \rho(u) a' a$ for all $a' \in A'$, $\rho \in \mathrm{Alg}(U, k)$, $u \in U$, and $a \in A$, and the restriction map $\mathrm{Alg}(A, k) \longrightarrow \mathrm{Alg}(A', k)$ is one-one.*

Then $\mathrm{Alg}(U, k) \times \mathrm{Alg}(A, k) \longrightarrow [{}_H\underline{\mathcal{M}}]$ given by $(\rho, \chi) \mapsto [L(\rho, \chi)]$ is bijective. $\square$

Observe that the hypothesis of the theorem holds for $H^{\tilde{op}}$ as well, where A'^{op} in $H^{\tilde{op}}$ plays the role of U' in H and U'^{op} in $H^{\tilde{op}}$ plays the role of A' in H. Therefore:

Corollary 3.4. *Under the hypothesis of the preceding theorem, the function $\mathrm{Alg}(A, k) \times \mathrm{Alg}(U, k) \longrightarrow [\underline{\mathcal{M}}_H]$ given by $(\chi, \rho) \mapsto [R(\chi, \rho)]$ is bijective.* $\square$

The theorem has interesting consequences for objects of the category ${}_H\underline{\mathcal{M}}$.

Corollary 3.5. *Let U and A be algebras over the field k and let $H = U \otimes A$ be an algebra structure on the tensor product of the underlying vector spaces of U and A which satisfies (2.1). Assume that the hypothesis of Theorem 3.3 holds. Then:*

a) *Every object of ${}_H\underline{\mathcal{M}}$ has a unique one-dimensional A-submodule and a unique codimension one U-submodule.*
b) *Suppose that $f : M \longrightarrow M'$ is a left H-module map, where M and M' are objects of ${}_H\underline{\mathcal{M}}$. Then $f = 0$ or f is an isomorphism.*

Proof: We first show part a). We have noted that the hypothesis of the theorem apples to $H^{\tilde{op}}$. In light of Lemma 3.1 and Theorem 3.3 we need only show that the object $M \in \underline{\mathcal{M}}$ has a 1-dimensional A-submodule. Let $M \in {}_H\underline{\mathcal{M}}$ and let $(M, km, N) \in {}_H\underline{\mathcal{M}}$ be derived from M. Let $\rho \in \mathrm{Alg}(U, k)$ and $\chi \in \mathrm{Alg}(A, k)$ satisfy (2.8). Regard the right H-module $M^\bullet$ as a left $H^{\tilde{op}}$-module by pullback along the algebra map $H^{\tilde{op}} \longrightarrow H^{op}$ given by $a \otimes u \mapsto u \otimes a$ for all $a \in A$ and $u \in U$. Then $M^\bullet \simeq L^{\tilde{op}}(\chi, \rho)$ as left $H^{\tilde{op}}$-modules, where $L^{\tilde{op}}(\chi, \rho)$ is the counterpart of $L(\rho,\chi)$ for H. By Lemma 3.1 and Theorem 3.3 there is only one codimension left $H^{\tilde{op}}$-module, or equivalently right H-module, in $M^\bullet$. Therefore M has a unique one-dimensional left A-module by part c) of Proposition 2.9.

Part b) follows from part a). We first note that since f is a map of left H-modules it is also a map of left A-modules and left U-modules.

Let km be a one-dimensional left A-submodule of M which generates M as a left H-module. Since f is a map of left H-modules $f(M) = f(H{\cdot}m) = H{\cdot}f(m)$. Since f is a map of left A-modules $kf(m)$ is a left A-submodule of M'.

Suppose that $f \neq 0$. Then $f(m) \neq 0$. Therefore $kf(m)$ is a one-dimensional left A-submodule of M'. Now M' is generated as a left H-module by some one-dimensional left A-submodule of M'. By uniqueness this submodule must be $kf(m)$. Thus f is onto. It remains to show that f is one-one.

Now M' contains a codimension one left U-submodule N'. Since f is onto and a map of left U-modules $f^{-1}(N')$ is a codimension one left U-submodule of M. Now M has a codimension one left U-submodule which contains no left H-submodule of M other than (0). By uniqueness this submodule must be $f^{-1}(N')$. Since $\operatorname{Ker} f \subseteq f^{-1}(N')$ and is an H-submodule of M it follows that $\operatorname{Ker} f = 0$. We have shown that f is one-one. □

For an algebra A we denote the set of isomorphism classes of finite-dimensional irreducible left A-modules by $\operatorname{Irr}(A)$. As a result of the preceding corollary:

Corollary 3.6. *Let U and A be algebras over the field k and let $H = U \otimes A$ be an algebra structure on the tensor product of the underlying vector spaces of U and A which satisfies (2.1). Assume that the hypothesis of Theorem 3.3 holds and also that the irreducible left U-modules and irreducible left A-modules are one-dimensional. Then:*

a) *The finite-dimensional irreducible left H-modules are the same as the finite-dimensional objects of ${}_H\underline{\mathcal{M}}$.*
b) *Suppose that U and A are finite-dimensional. Then the function $\operatorname{Alg}(U,k)\times \operatorname{Alg}(A,k) \to \operatorname{Irr}(H)$ given by $(\chi,\rho) \mapsto [L(\rho,\chi)]$ is bijective.*

□

4. When U and A are Bialgebras

Let U and A be bialgebras over the field k, suppose $\tau : U\otimes A \longrightarrow k$ is convolution invertible and satisfies (A.1)–(A.4), and let $H = (U\otimes A)^\sigma$. In this section we apply the major ideas of the preceding section to H.

Suppose that $\rho \in G(U^o) = \mathrm{Alg}(U, k)$ and $\chi \in G(A^o) = \mathrm{Alg}(A, k)$. Using (1.1) we see that (2.2) in this case is

$$(u\otimes a){\cdot_\chi}u' = u\tau(u'_{(1)}, a_{(1)})u'_{(2)}\chi(a_{(2)})\tau^{-1}(u'_{(3)}, a_{(3)}) \tag{4.1}$$

for all $u, u' \in U$ and $a \in A$ and (2.4) in this case is

$$a{\cdot_\rho}(u\otimes a') = \tau(u_{(1)}, a_{(1)})\rho(u_{(2)})a_{(2)}\tau^{-1}(u_{(3)}, a_{(3)})a' \tag{4.2}$$

for all $a, a' \in A$ and $u \in U$. Observe that

$$\Psi = (\tau(\rho\otimes\chi)\tau^{-1})\circ\tau_{A,U} \tag{4.3}$$

is just conjugation of $\rho\otimes\chi$ by τ in the dual algebra $(U\otimes A)^*$ preceded by the twist map.

Now let $U' = kG(U)$, $A' = kG(A)$, and suppose that $G(U^o)$ and $G(A^o)$ are commutative groups. Using (4.1) we see for $u \in U$, $a \in A$, $\chi \in \mathrm{Alg}(A, k)$, and $u' \in G(U)$ that

$$\begin{aligned}(u\otimes a){\cdot_\chi}u' &= u\tau(u', a_{(1)})u'\chi(a_{(2)})\tau^{-1}(u', a_{(3)})\\ &= u\left((\tau_\ell(u')\chi\tau_\ell(u')^{-1})(a)\right)u'\\ &= uu'\chi(a)\end{aligned}$$

and therefore $(u\otimes a){\cdot_\chi}u' = uu'\chi(a)$ for all $u \in U, a \in A$ and $u' \in U'$. Likewise using (4.2) it follows that $a'{\cdot_\rho}(u\otimes a) = a'a\rho(u)$ for all $a' \in A'$, $\rho \in \mathrm{Alg}(U, k)$, $u \in U$, and $a \in A$.

Theorem 4.1. *Let U and A be bialgebras over the field k, suppose $\tau : U\otimes A \longrightarrow k$ is convolution invertible and satisfies (A.1)–(A.4), and let $H = (U\otimes A)^\sigma$. Suppose that all $\rho \in \mathrm{Alg}(U, k)$, $\chi \in \mathrm{Alg}(A, k)$ are determined by their respective restrictions $\rho|G(U)$, $\chi|G(A)$. Then $\mathrm{Alg}(U, k)$, $\mathrm{Alg}(A, k)$ are abelian groups and:*

a) $\mathrm{Alg}(U, k)\times \mathrm{Alg}(A, k) \longrightarrow [{}_H\underline{\mathcal{M}}]$ *given by* $(\rho, \chi) \mapsto [L(\rho, \chi)]$ *is bijective.*
b) *Every object of ${}_H\underline{\mathcal{M}}$ has a unique one-dimensional A-submodule and a unique codimension one U-submodule.*
c) *Suppose that the finite-dimensional irreducible left U-modules and the finite-dimensional irreducible left A-modules are one-dimensional. Then the finite-dimensional irreducible left H-modules are the finite-dimensional objects of ${}_H\underline{\mathcal{M}}$.*

PROOF: The hypothesis of Theorem 3.3 holds for H with $U' = kG(U)$ and $A' = kG(A)$. Thus part a) follows. Part b) is part a) of Corollary 3.5 and part c) is part a) of Corollary 3.6. □

We will show that the preceding theorem applies to a wide class of pointed Hopf algebras. First we recall a basic Hopf module construction.

Let A be a Hopf algebra with sub-Hopf algebra B. Suppose that D, C are subcoalgebras of A which satisfy $D \subseteq C$, $BD \subseteq D$, $BC \subseteq C$, and $\Delta(C) \subseteq B \otimes C + C \otimes D$. Set $M = C/D$ and write $\overline{c} = c + C$ for all $c \in C$. Then M is a left B-Hopf module, where $b \cdot \overline{c} = \overline{bc}$ and $\rho(\overline{c}) = c_{(1)} \otimes \overline{c_{(2)}}$ for all $c \in C$. By the Fundamental Theorem for Hopf modules [17, Theorem 4.1.1] it follows that M is (0) or a free left B-module with basis any linear basis of $M^{\mathrm{co}B} = \{\overline{c} \,|\, \rho(\overline{c}) = 1 \otimes \overline{c}\}$.

Now suppose that $B = A_0$ is a sub-Hopf algebra of A and let $n > 0$. Then $C = A_n$ and $D = A_{n-1}$ satisfy the conditions of the preceding paragraph. Therefore A_n/A_{n-1} is (0) or a free left B-module. As a consequence A_n is a free left B-module for all $n \geq 0$; thus A is a free left B-module.

Any two bases for a free module over a Hopf algebra B have the same cardinality, and thus rank of the free module is well-defined, since Hopf algebras are augmented algebras. For the same reason, if M is a free left B-submodule of a free left B-module N then $\operatorname{rank} M \leq \operatorname{rank} N$.

Suppose that a is a skew primitive element of A. We say that a is of *finite type* if the sub-Hopf algebra of A generated by $A_0 \cup \{a\}$ is a free left A_0-module of finite rank.

Corollary 4.2. *Let U and A be pointed Hopf algebras over and algebraically closed field k of characteristic zero and suppose $\tau : U \otimes A \longrightarrow k$ is convolution invertible and satisfies (A.1)–(A.4). Suppose further that U and A are generated by skew primitives of finite rank and have commutative coradicals. Then the conclusions of the preceding theorem hold for $H = (U \otimes A)^{\sigma}$.*

PROOF: We need only show that all $\rho \in G(U^o)$, $\chi \in G(A^o)$ are determined by their respective restrictions $\rho|G(U)$, $\chi|G(A)$. We give an argument for A which is automatically an argument for U also. To this end we need only show that there is an indexed set of skew primitive elements $\{a_i\}_{i \in I}$, which together with Γ generate A as an algebra, and there is an indexed set of non-trivial characters $\{\chi\}_{i \in I}$ such that $h a_i h^{-1} = \chi_i(h) a_i$ for all $h \in \Gamma$ and $i \in I$. See the opening commentary for Section 3.

Let $B = A_0$. Let $\Gamma = G(A)$. Since A_0 is cocommutative and k is algebraically closed it follows that $B = k[\Gamma]$. Since B is commutative Γ is a commutative group.

Suppose that $a \in A$ is a skew primitive element of finite rank. We may assume that $\Delta(a) = a \otimes g + 1 \otimes a$ for some $g \in \Gamma$ and that $a \notin B$. Let E be the sub-Hopf algebra of A generated by $B \cup \{a\}$. Then E is a free left B-module of finite rank by assumption.

Let $V = \{v \in E \,|\, \Delta(v) = v \otimes g + 1 \otimes v\}$. Then $C = BVB$ is a left B-module, a subcoalgebra of E, and $\Delta(C) \subseteq C \otimes B + B \otimes C$. Thus $M = C/B$ is a left B-Hopf module. Since $1 + \operatorname{rank} M = \operatorname{rank} C \leq \operatorname{rank} E$, and the latter is finite, it follows that $\operatorname{rank} M$ is finite. Now $\overline{V} \subseteq M^{\mathrm{co}B}$. Thus $\operatorname{Dim} \overline{V} \leq \operatorname{rank} M$ is finite. Since $V \cap B = k(g-1)$ we conclude that V is a finite-dimensional vector space.

Let $h \in \Gamma$. Since Γ is commutative and E is a subalgebra of A which contains h it follows that $hVh^{-1} \subseteq V$. By assumption $V \neq (0)$. Since V is finite-dimensional and k is an algebraically closed field of characteristic zero, there is a basis $v_1, \ldots, v_n$ for V consisting of common eigenvectors for the conjugation action by Γ. Therefore there are characters $\chi_1, \ldots, \chi_n \in \widehat{\Gamma}$ such that $hv_ih^{-1} = \chi_i(h)v_i$ for all $h \in \Gamma$ and $1 \leq i \leq n$.

Fix $1 \leq i \leq n$. Observe that v_i and Γ generate sub-Hopf algebra of A of finite rank. By calculations found in [16, Section 3] it follows that $\chi_i(g) = 1$ implies that $v_i \in B$. Since a is in the span of the v_i's, it is clear how to form the families $\{a_i\}_{i \in I}$ and $\{\chi_i\}_{i \in I}$ which satisfy the conditions outlined at the beginning of the proof. □

For later use we note

Lemma 4.3. *Let U and A be bialgebras over the field k and assume that $G(U^0)$ is abelian. Suppose $\tau : U \otimes A \longrightarrow k$ is convolution invertible and satisfies (A.1)–(A.4), and let $H = (U \otimes A)^\sigma$. Let $u \in U$ and $g \in G(A)$ and assume that $u \otimes g$ is central in H. Then for all $\rho \in \mathrm{Alg}(U, k)$ and $\chi \in \mathrm{Alg}(A, k)$ the following are equivalent:*

a) *$u \otimes g - 1 \otimes 1$ acts as zero on $L(\rho, \chi)$.*
b) *$\rho(u)\chi(g) = 1$.*

Proof: Part a) implies

$$(4.4) \quad (u \otimes g - 1 \otimes 1) \cdot_\chi v = uv_{(2)}\tau(v_{(1)}, g)\chi(g)\tau^{-1}(v_{(3)}, g) \in I(\rho, \chi)$$

for all $v \in U$. If $v = 1$ then (4.4) implies $u\chi(g) - 1 \in I(\rho, \chi)$, hence $\rho(u\chi(g) - 1) = 0$ or equivalently $\rho(u)\chi(g) = 1$. Thus part a) implies part b).

Conversely, assume part b). Since $u \otimes g$ is central in H, the k-span of $(u \otimes g - 1 \otimes 1) \cdot_\chi v, v \in U$, is an H-submodule of U_χ. Moreover, for any $v \in U$,

$$\tau(v_{(1)}, g)\rho(v_{(2)})\tau^{-1}(v_{(3)}, g) = \rho(v),$$

since $\tau_r(g) \in \mathrm{Alg}(U, k)$ and $G(U^0) = \mathrm{Alg}(U, k)$ is abelian. Hence

$$\begin{aligned}\rho((u \otimes g) \cdot_\chi v) &= \rho(uv_{(2)}\tau(v_{(1)}, g)\chi(g)\tau^{-1}(v_{(3)}, g)) \\ &= \rho(u)\chi(g)\rho(v) \\ &= \rho(v)\end{aligned}$$

by b). This proves (4.4) by definition of $I(\rho, \chi)$. □

In connection with Propositions 2.7 and 2.8 we will be interested in bialgebra maps of bialgebras of the type $(U \otimes A)^\sigma$.

Proposition 4.4. *Let U, $\overline{U}$, A, and $\overline{A}$ be bialgebras over k, suppose $\tau : U \otimes A \longrightarrow k$ and $\overline{\tau} : \overline{U} \otimes \overline{A} \longrightarrow k$ satisfy (A.1)–(A.4), and suppose that $f : U \longrightarrow \overline{U}$ and $g : A \longrightarrow \overline{A}$ are bialgebra maps which satisfy $\overline{\tau} \circ (f \otimes g) = \tau$. Then $f \otimes g : (U \otimes A)^\sigma \longrightarrow (\overline{U} \otimes \overline{A})^{\overline{\sigma}}$ is a bialgebra map.*

Proof: Since f and g are coalgebra maps $f \otimes g : U \otimes A \longrightarrow \overline{U} \otimes \overline{A}$ is a coalgebra map of the tensor product of coalgebras. As the underlying coalgebra structures of $(U \otimes A)^\sigma$ and $(\overline{U} \otimes \overline{A})^{\overline{\sigma}}$ are $U \otimes A$ and $\overline{U} \otimes \overline{A}$ respectively, it follows that $f \otimes g : (U \otimes A)^\sigma \longrightarrow (\overline{U} \otimes \overline{A})^{\overline{\sigma}}$ is a coalgebra map. Since $(f \otimes g)^* : (\overline{U} \otimes \overline{A})^* \longrightarrow (U \otimes A)^*$ is an algebra map $\overline{\tau}^{-1} = ((f \otimes g)^*(\tau))^{-1} = (f \otimes g)^*(\tau^{-1}) = \tau^{-1} \circ (f \otimes g)$. At this point is it easy to see that $f \otimes g : (U \otimes A)^\sigma \longrightarrow (\overline{U} \otimes \overline{A})^{\overline{\sigma}}$ is an algebra map. □

Suppose that U and A are bialgebras over k and $\tau : U \otimes A \longrightarrow k$ is a linear form which satisfies (A.1)–(A.4). Then τ is convolution invertible by Lemma 1.2 if U or A^{op} is a Hopf algebra, in particular if A is a Hopf algebra with bijective antipode.

Suppose in addition that A has bijective antipode, set $H = (U \otimes A)^\sigma$, and let $f : H \longrightarrow D(A)$ be the bialgebra map defined at the end of Section 1 by $f(u \otimes a) = \tau_\ell(u) \otimes a$ for all $u \in U$ and $a \in A$. Let $\rho \in G(U^o)$ and $\chi \in G(A^o)$. We will examine $L(\rho, \chi)$ in the context of part c) of Proposition 2.6 and find a condition for there to be a left $D(A)$-module such that pullback along f explains $L(\rho, \chi)$. To understand modules for $D(A)$ we need to review a variant of the Yetter–Drinfeld category discussed in many places; in particular in [2].

Let B be any bialgebra over k and let ${}_B\mathcal{Y}D^B$ be the category whose objects are triples $(M, \cdot, \delta)$, where $(M, \cdot)$ is a left B-module and (M, δ) is a right B-comodule which are compatible in the sense

$$b_{(1)} \cdot m_{(0)} \otimes b_{(2)} m_{(1)} = (b_{(2)} \cdot m)_{(0)} \otimes (b_{(2)} \cdot m)_{(1)} b_{(1)} \tag{4.5}$$

for all $b \in B$ and $m \in M$, and whose morphisms are maps of left B-modules and right B-comodules. We observe that when B^{op} has antipode T then (4.5) is equivalent to

$$\delta(b{\cdot}m) = b_{(2)}{\cdot}m_{(0)}\otimes b_{(3)}m_{(1)}T(b_{(1)}) \tag{4.6}$$

for all $b \in B$ and $m \in M$. An example, which is the centerpiece of [13] in the study of simple modules for the double, is the following [13, Lemma 2]:

Example 4.5. *Let B be a bialgebra over k, suppose that B^{op} is a Hopf algebra with antipode T, and let $\beta \in G(B^o)$. Then $(B, \succ_\beta, \Delta) \in {}_B\mathcal{Y}D^B$, where*

$$b{\succ_\beta}m = (b_{(2)}{\leftharpoonup}\beta)mT(b_{(1)})$$

for all $b, m \in B$.

The map $b \mapsto b{\leftharpoonup}\beta$ of the example is an algebra automorphism of B. Thus the module $(B, \succ_\beta)$ can be regarded as a generalized adjoint action.

Now suppose that B is a Hopf algebra with bijective antipode S and let $(M, \cdot, \delta) \in {}_B\mathcal{Y}D^B$. Then $(M, \bullet) \in {}_{D(B)}\mathcal{M}$ where

$$(p\otimes b){\bullet}m = p{\rightharpoonup}(b{\cdot}m) = (b{\cdot}m)_{(0)}{<}p, (b{\cdot}m)_{(1)}{>}$$

for all $p \in B^o$, $b \in B$, and $m \in M$. When B is finite-dimensional the preceding equation describes the essence of a categorical isomorphism of ${}_{D(B)}\mathcal{M}$ and ${}_B\mathcal{Y}D^B$. See the primary reference [11] as well as [8, Proposition 3.5.1].

It is also interesting to note that ${}_{B^o}\mathcal{Y}D^{B^o}$ also accounts for left modules for $D(B)$. For suppose that $(M, \cdot, \delta) \in {}_{B^o}\mathcal{Y}D^{B^o}$ and let $i_B : B \longrightarrow (B^o)^*$ be the algebra map defined by $i_B(b)(p) = p(b)$ for all $b \in B$ and $p \in B^o$. Set $i = i_B$. Then $(M, \bullet) \in {}_{D(B)}\mathcal{M}$, where

$$(p\otimes b){\bullet}m = p{\cdot}(i(b){\rightharpoonup}m) = p{\cdot}m_{(0)}{<}m_{(1)}, b{>}$$

for all $p \in B^o$, $b \in B$, and $m \in M$. Observe that the action on $(M, \bullet)$ restricted to B is locally finite. The preceding equation describes the essence of a categorical isomorphism between the full subcategory of ${}_{D(B)}\mathcal{M}$ whose objects are locally finite as left B-modules and ${}_{B^o}\mathcal{Y}D^{B^o}$. Thus when B is finite-dimensional there is a categorical isomorphism of ${}_{D(B)}\mathcal{M}$ and ${}_{B^o}\mathcal{Y}D^{B^o}$. It is the Yetter–Drinfeld category ${}_{B^o}\mathcal{Y}D^{B^o}$ which is most appropriate here.

Using Lemma 1.2 and (4.3) we have

$$\Psi_r(u) = \tau_\ell(\rho{\rightharpoonup}u_{(1)})\chi S^{-1}(\tau_\ell(u_{(2)})) = \tau_\ell(u_{(1)})\chi S^{-1}(\tau_\ell(u_{(2)}){\leftharpoonup}\rho) \tag{4.7}$$

for all $u \in U$.

Regard A^* as a left H-module with the transpose action arising from the right H-module structure $(A, \cdot_\rho)$. By part c) of Proposition 2.6 there is an isomorphism of left H-modules $F : L(\rho,\chi) \longrightarrow U\cdot_\rho\chi \subseteq A^*$ given by $F(\overline{u}) = \Psi_r(u) = u\cdot_\rho\chi$ for all $u \in U$, where $\overline{u} = u + I(\rho,\chi)$. Thus

$$F(\overline{u}) = \tau_\ell(\rho \rightharpoonup u_{(1)})\chi S^{-1}(\tau_\ell(u_{(2)})) = u\cdot_\rho\chi \tag{4.8}$$

for all $u \in U$ by (4.3). In particular $U\cdot_\rho\chi = \operatorname{Im} F \subseteq A^o$.

The H-module $L(\rho,\chi)$ can be explained in terms of $D(A)$ when a very natural condition is satisfied. Let $i = i_A$. Then $i(g) \in \operatorname{Alg}(A^o, k) = G((A^o)^o)$ and the calculation

$$\begin{aligned} a\cdot_\rho(u\otimes a') &= \tau(u_{(1)}, a_{(1)})\rho(u_{(2)})a_{(2)}\tau^{-1}(u_{(3)}, a_{(3)})a' \\ &= \tau(u_{(1)}, a_{(1)})\tau(u_{(2)}, g)a_{(2)}\tau^{-1}(u_{(3)}, a_{(3)})a' \\ &= \tau(u_{(1)(1)}, a_{(1)})\tau(u_{(1)(2)}, g)a_{(2)}\tau(u_{(2)}, s^{-1}(a_{(3)}))a' \\ &= \left(\tau_\ell(u_{(1)})_{(2)}(a_{(1)})\right)\left(\tau_\ell(u_{(1)})_{(1)}(g)\right)a_{(2)}\left(S^{-1}(\tau_\ell(u_{(2)}))(a_{(3)})\right)a' \\ &= \left((\tau_\ell(u_{(1)}) \leftharpoonup i(g))(a_{(1)})\right)a_{(2)}\left(S^{-1}(\tau_\ell(u_{(2)}))(a_{(3)})\right)a' \end{aligned}$$

shows that

$$a\cdot_\rho(u\otimes a') = \left(S^{-1}(\tau_\ell(u)_{(1)}) \rightharpoonup a \leftharpoonup (\tau_\ell(u)_{(2)} \leftharpoonup i(g))\right)a' \tag{4.9}$$

for all $a, a \in A$ and $u \in U$. As a consequence

$$(u\otimes a')\cdot_\rho p = \left(\tau_\ell(u)_{(2)} \leftharpoonup i(g)\right)\left(i(a') \rightharpoonup p\right)\left(S^{-1}(\tau_\ell(u)_{(1)})\right) \tag{4.10}$$

for all $u \in U$, $a' \in A$, and $p \in A^o$.

Let $(A^o, \succ_{i(g)}, \Delta)$ be the object of ${}_{A^o}\mathcal{Y}D^{A^o}$ defined in Example 4.5 and let $(M, \bullet)$ be associated left $D(A)$-module structure. Using (4.10) it is not hard to see that

$$(u\otimes a')\cdot_\rho p = f(u\otimes a')\bullet p$$

for all $u \in U$, $a' \in A$, and $p \in A^o$. Since $\chi \in G(A^o)$ we conclude that $H\cdot_\rho\chi = U\cdot_\rho\chi \subseteq A^o\bullet\chi$, the latter is a $D(A)$-submodule of A^o, and that the map $F : L(\rho,\chi) \longrightarrow A^o\bullet\chi$ given by $\overline{u} \mapsto \Psi_r(u) = u\cdot_\rho\chi$ is a one-one map of left H-modules, where $A^o\bullet\chi$ has the left H-module structure action obtained by pullback along f.

Note that

$$\Psi_\ell(a) = \tau_r(a_{(1)})\rho S(\tau_r(a_{(2)} \leftharpoonup \chi)) \tag{4.11}$$

follows for all $a \in A$ by (4.3) also, where here S is the antipode of U. A similar treatment of $R(\chi,\rho)$ can be given based on this equation.

5. Certain Algebras Whose Finite-Dimensional Simple Modules Are One-Dimensional

In light of Corollary 3.6 we wish to consider conditions under which the irreducible representations of an algebra are one-dimensional with an eye towards applications of the results of Sections 1–3 to certain classes of pointed Hopf algebras. Throughout this section the field k is algebraically closed. We are interested in algebras A satisfying the following condition:

> *A is generated by an abelian group Γ of units of A together with finitely many elements $a_1, \ldots, a_\theta$ and there are non-trivial characters $\chi_1, \ldots, \chi_\theta$ such that*

$$ga_ig^{-1} = \chi_i(g)a_i \text{ for all } g \in \Gamma \text{ and } 1 \leq i \leq \theta. \tag{5.1}$$

We denote the preceding condition by (C). Many pointed Hopf algebras satisfy condition (C). See the beginning of Section 6.1.

Suppose A is an algebra which satisfies condition (C). We will find sufficient conditions for all finite-dimensional simple left A-modules M to be one-dimensional. Finding a non-zero $m \in M$ which satisfies $a_1{\cdot}m = \cdots = a_\theta{\cdot}m = 0$ is the key. The theorem of this section gives such a condition which relates the values $\chi_j(g_i)$ to a Cartan matrix of finite type, where $g_1, \ldots, g_\theta \in \Gamma$.

Lemma 5.1. *Let A be an algebra satisfying (C).*

a) *Suppose $M = km$ is a one-dimensional left A-module. Then $a_i{\cdot}m = 0$ for all $1 \leq i \leq \theta$.*

Suppose that M is a non-zero finite-dimensional left A-module.

b) *Assume that M has a non-zero element m such that $a_i{\cdot}m = 0$ for all $1 \leq i \leq \theta$. Then M contains a one-dimensional left A-module.*

c) *Suppose that $I_1, \ldots, I_r$ partition $\{1, \ldots, \theta\}$ and a_i, a_j skew commute whenever i, j belong to different I_ℓ's. Let A_i be the subalgebra of A generated by the a_j's belonging to I_i and Γ. For all $1 \leq i \leq \theta$ assume that non-zero finite-dimensional left A_i-modules contain a one-dimensional submodule. Then M contains a one-dimensional left A-submodule.*

Proof: Assume the hypothesis of part a). Then there is a $\rho \in \mathrm{Alg}(A, k)$ such that $a{\cdot}m = \rho(a)m$ for all $a \in A$. Let $g \in \Gamma$ and $1 \leq i \leq \theta$. From the calculation

$$\rho(ga_i)m = ga_i{\cdot}m = \chi_i(g)a_ig{\cdot}m = \chi_i(g)\rho(a_ig)m$$

we see that $\rho(a_i) = \chi_i(g)\rho(a_i)$. Since $\chi_i \neq 1$ it follows that $\rho(a_i) = 0$. We have shown that $a_i{\cdot}m = 0$ and thus part a) is established.

As for part b), let $M' = \{m \in M \mid a_1{\cdot}m = \cdots = a_\theta{\cdot}m = 0\}$. Since $a_i g = \chi_i(g)^{-1} g a_i$ for all $1 \leq i \leq \theta$ and $g \in \Gamma$, we conclude that M' is a left Γ-module. Now the A-submodules of M' are the Γ-submodules of the same. Since M' is finite-dimensional, Γ is abelian, and k is algebraically closed, M' contains a one-dimensional left Γ-submodule. This concludes our proof of part b).

To show part c) we may assume $r = 2$ by induction on r. Thus $\theta > 1$ and without loss of generality we may assume $S_1 = \{a_1, \ldots, a_s\}$ and $S_2 = \{a_{s+1}, \ldots, a_\theta\}$ for some $1 \leq s < \theta$. Since a_i and a_j skew commute whenever $1 \leq i \leq s < j \leq \theta$, and the elements of the commutative group Γ skew commute with $a_1, \ldots, a_\theta$, we conclude that $A_2 a_i = a_i A_2$ for all $1 \leq i \leq s$.

Let M' be the set of all $m \in M$ such that $a_1{\cdot}m = \cdots = a_s{\cdot}m = 0$. By assumption M contains a one-dimensional left A_1-submodule. Thus $M' \neq (0)$ by part a). Since $A_2 a_i = a_i A_2$ for all $1 \leq i \leq s$ it follows that M' is a (non-zero) left A_2-submodule of M. By assumption M' contains a one-dimensional left A_2-submodule km. Now $a_{s+1}{\cdot}m = \cdots = a_\theta{\cdot}m = 0$ by part a) again. Since $m \in M'$ by definition $a_1{\cdot}m = \cdots = a_s{\cdot}m = 0$. Therefore M contains a one-dimensional left A-module by part b). □

Corollary 5.2. *Let A be an algebra satisfying condition (C). Assume that A' is a subalgebra of A generated by $a_1, \ldots, a_\theta$ and a subgroup Γ' of Γ such that the restrictions $\chi_1|\Gamma', \ldots, \chi_\theta|\Gamma' \neq 1$. Then finite-dimensional simple left A-modules are one-dimensional if the same is true for A'.*

Proof: Suppose that finite-dimensional simple left A'-modules are one-dimensional and let M be a finite-dimensional simple left A-module. Then M contains a finite-dimensional simple left A'-module which must have the form km by assumption. By part a) of Lemma 5.1 we have that $a_1{\cdot}m = \cdots = a_\theta{\cdot}m = 0$. By part b) of the same M contains a one-dimensional A-submodule M'. Since M is simple $M = M'$. □

Apropos of Lemma 5.1, finite-dimensional simple left A-modules are one-dimensional when $\chi_1, \ldots, \chi_\theta$ are free monoid generators. For more generally:

Proposition 5.3. *Let A be an algebra satisfying condition (C). Suppose further that $\chi_1^{k_1} \cdots \chi_\theta^{k_\theta} = 1$, where $k_1, \ldots, k_\theta \geq 0$, implies $k_1 = \cdots = k_\theta = 0$. Then finite-dimensional simple left A-modules are one-dimensional.*

PROOF: Let M be a finite-dimensional non-zero left A-module. Regarding M as a left Γ-module we may write $M = \bigoplus_{\lambda \in \widehat{\Gamma}} M_\lambda$ as the direct sum of weight spaces, where $M_\lambda = \{m \in M \mid g{\cdot}m = \lambda(g)m \ \forall\ g \in \Gamma\}$. Since M is finite-dimensional all but finitely many of the M_λ's are zero. Let $1 \leq i \leq \theta$. Our assumption $ga_ig^{-1} = \chi(g)a_i$ for all $g \in \Gamma$ means that $a_i{\cdot}M_\lambda \subseteq M_{\chi_i\lambda}$ for all $\lambda \in \widehat{\Gamma}$.

By Lemma 5.1 it suffices to show that there is a non-zero $m \in M$ such that $a_i{\cdot}m = 0$ for all $1 \leq i \leq \theta$. Suppose this is not the case. Since $M \neq (0)$ there is a $\lambda \in \widehat{\Gamma}$ such that $M_\lambda \neq (0)$. Choose a non-zero $m \in M_\lambda$. By induction there is an infinite sequence of integers $i_1, i_2, \ldots$ such that $1 \leq i_j \leq \theta$ for all $j \geq 1$ and $a_{i_r} \cdots a_{i_1}{\cdot}m \neq 0$ for all $r \geq 1$. Now $a_{i_r} \cdots a_{i_1}{\cdot}m \in M_{\chi_r \cdots \chi_1 \lambda}$. Our assumption on products of the characters $\chi_1^{k_1} \cdots \chi_\theta^{k_\theta}$ means that $\lambda, \chi_{i_1}\lambda, \chi_{i_2}\chi_{i_1}\lambda, \ldots$ are all distinct. But this is impossible since all but finitely many weight spaces are zero. Therefore there is a non-zero $m \in M$ such that $a_i{\cdot}m = 0$ for all $1 \leq i \leq \theta$ after all. $\square$

We recall some notions from [1]. A *datum of Cartan type*

$$\mathcal{D} = \mathcal{D}(\Gamma, (g_i)_{1 \leq i \leq \theta}, (\chi_i)_{1 \leq i \leq \theta}, (a_{ij})_{1 \leq i,j \leq \theta})$$

consists of an abelian group Γ, elements $g_i \in \Gamma, \chi_i \in \widehat{\Gamma}, 1 \leq i \leq \theta$, and a $\theta \times \theta$ Cartan matrix (a_{ij}) satisfying

$$q_{ij}q_{ji} = q_{ii}^{a_{ij}}, \ q_{ii} \neq 1, \text{ with } q_{ij} = \chi_j(g_i) \text{ for all } 1 \leq i,j \leq \theta. \tag{5.2}$$

We define $q_i = q_{ii}$ for all $1 \leq i \leq \theta$. Note that by (5.2)

$$q_i^{a_{ij}} = q_j^{a_{ji}} \text{ for all } 1 \leq i,j \leq \theta. \tag{5.3}$$

Recall that a (generalized) Cartan matrix $(a_{ij})_{1 \leq i,j \leq \theta}$ is a matrix whose entries are integers such that $a_{ii} = 2$ for all i, $a_{ij} \leq 0$ for all $i \neq j$, and if $a_{ij} = 0$ then $a_{ji} = 0$ for all i, j. A datum $\mathcal{D}$ of Cartan type will be called of finite Cartan type if (a_{ij}) is of finite type.

Let $\mathcal{D} = \mathcal{D}(\Gamma, (g_i)_{1 \leq i \leq \theta}, (\chi_i)_{1 \leq i \leq \theta}, (a_{ij})_{1 \leq i,j \leq \theta})$ be a datum of Cartan type. Suppose $1 \leq i, j \leq \theta$. We say that i is connected to j, denoted by $i \sim j$, if there are indices $1 \leq i_1, \cdots, i_t \leq \theta$, where $t \geq 2$, with $i = i_1, j = i_t$ and $a_{i_l i_{l+1}} \neq 0$ for all $1 \leq l < t$. In this case we define $a(i,j) = a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{t-1} j}$, and $b(i,j) = a_{i_2 i_1} a_{i_3 i_2} \cdots a_{j i_{t-1}}$. Then it follows from (5.2) that

$$q_i^{a(i,j)} = q_j^{b(i,j)}. \tag{5.4}$$

Connectivity is an equivalence relation and the equivalence classes are called the *connected components* of $\{1, \ldots, \theta\}$.

More generally, let R be a ring and $(a_{ij}) \in \mathrm{M}_\theta(R)$ be a $\theta \times \theta$ matrix with coefficients in R. Let $1 \leq i, j \leq \theta$. We say that i is connected to

j if there are indices $1 \leq i_1, \cdots, i_t \leq \theta$, where $t \geq 2$, with $i = i_1, j = i_t$ and $a_{i_l i_{l+1}} \neq 0$ for all $1 \leq l < t$. In this generality connectivity may not be an equivalence relation.

In the proof of the main theorem in this section we use the following lemma in the special case of data of Cartan type.

Lemma 5.4. *Suppose $(a_{ij}) \in \mathrm{M}_\theta(\mathbb{Z})$ is a non-zero matrix and all indices $1 \leq i, j \leq \theta$ are connected. Assume further that $q_1, \dots, q_\theta \in k$ are non-zero, that (5.3) holds, and that one of the q_i's is not a root of unity. Then:*

a) *None of $q_1, \dots, q_\theta$ is a root of unity.*
b) *There are roots of unity $\omega_1, \dots, \omega_\theta$ in k, an element $q \in k$, and non-zero integers $d_1, \dots, d_\theta$ with $q_i = \omega_i q^{d_i}$ for all $1 \leq i \leq \theta$.*
c) *$d_i a_{ij} = d_j a_{ji}$ for all $1 \leq i, j \leq \theta$.*
d) *Suppose Γ is a group, $g_1, \dots, g_\theta \in \Gamma$, and $\chi_1, \dots, \chi_\theta \in \widehat{\Gamma}$ satisfy $\chi_i(g_i) = q_i$ and $\chi_j(g_i)\chi_i(g_j) = q_i^{a_{ij}}$ for all $1 \leq i, j \leq \theta$. Consider the quadratic form*

$$Q(x_1, \dots, x_\theta) = \sum_{i=1}^{\theta} 2x_i^2 d_i + \sum_{1 \leq i < j \leq \theta} 2x_i x_j d_i a_{ij}.$$

Let $k_1, \dots, k_\theta \in \mathbb{Z}$ and suppose $\chi_1^{k_1} \cdots \chi_\theta^{k_\theta} = 1$ or $g_1^{k_1} \cdots g_\theta^{k_\theta} = 1$. Then $Q(k_1, \dots, k_\theta) = 0$.

Proof: Suppose that q_i is not a root of unity and $a_{ij} \neq 0$ where $1 \leq j \leq \theta$. Since $q_i^{a_{ij}} = q_j^{a_{ji}}$, necessarily $a_{ji} \neq 0$ and therefore q_j is not a root of unity. We now conclude that if $1 \leq \ell \leq \theta$ and i is connected to ℓ then q_ℓ is not a root of unity. We have shown part a) and the hypotheses of Lemma 5.5 are met. Thus part b) follows from Lemma 5.5.

By parts a) and b), q is not a root of unity. For $1 \leq i, j \leq \theta$ we calculate $\omega_i q^{d_i a_{ij}} = q_i^{a_{ij}} = q_j^{a_{ji}} = \omega_j q^{d_j a_{ji}}$. Choose a positive integer N such that $\omega_i^N = 1$ for all $1 \leq i \leq \theta$. Then $q^{N d_i a_{ij}} = q^{N d_j a_{ji}}$. Since q is not a root of unity and N is a positive integer the preceding equation implies part c).

It remains to show part d). Let $1 \leq i \leq \theta$. By assumption $q_i^2 = \chi_i(g_i)^2 = q_i^{a_{ii}}$ and thus $a_{ii} = 2$ since q_i is not a root of unity.

Now let $k_1, \dots, k_\theta \in \mathbb{Z}$ and suppose that $\chi_1^{k_1} \cdots \chi_\theta^{k_\theta} = 1$. Then $\prod_{j=1}^{\theta} \chi_j(g_i)^{k_j} = 1$, and hence $\prod_{j=1}^{\theta} \chi_j(g_i)^{k_i k_j} = 1$, for all $1 \leq i \leq \theta$. Suppose that $g_1^{k_1}\cdots g_\theta^{k_\theta}=1$. Then $\prod_{i=1}^{\theta}\chi_j(g_i)^{k_i}=1$, and hence $\prod_{i=1}^{\theta}\chi_j(g_i)^{k_ik_j}=1$, for all $1\le j\le\theta$. Thus in both cases

$$\begin{aligned} 1 &= \prod_{1\le i,j\le\theta} q_{ij}^{k_ik_j} \\ &= \prod_{i=1}^{\theta} q_i^{k_i^2} \prod_{1\le i<j\le\theta} q_{ij}^{k_ik_j} \prod_{\theta\ge i>j\ge1} q_{ji}^{k_ik_j} \\ &= \prod_{i=1}^{\theta} q_i^{k_i^2} \prod_{1\le i<j\le\theta} (q_{ij}q_{ji})^{k_ik_j} \\ &= \prod_{i=1}^{\theta} q_i^{k_i^2} \prod_{1\le i<j\le\theta} q_i^{a_{ij}k_ik_j}. \end{aligned}$$

Raising the last expression to the $2N$ power we have $1=(q^N)^{Q(k_1,\dots,k_\theta)}$. Since q^N is not a root of unity necessarily $Q(k_1,\dots,k_\theta)=0$. □

We remark that part c) of the previous lemma was shown in [3, Lemma 2.4] for matrices of Cartan type in a different way.

Lemma 5.5. *Suppose that S is a finite non-empty subset of non-zero elements of k which satisfies the following property: For all $x,y\in S$ there is an $r\ge1$, a sequence $x=x_0,x_1,\dots,x_r=y$ in S, and there are sequences $n_1,\dots,n_r$ and $m_0,\dots,m_{r-1}$ of non-zero integers such that $x_{i-1}^{m_{i-1}}=x_i^{n_i}$ for all $1\le i\le r$. Then there is a $q\in k$ such that each $x\in S$ can be written as $x=\omega q^L$ for some root of unity $\omega\in k$ and non-zero integer L.*

Proof: We may as well assume that there is a non-zero $x_0\in S$. Consider tuples $\mathcal{C}=(x_0,\dots,x_r,n_1,\dots,n_r,m_0,\dots,m_{r-1})$, where the condition of the lemma is satisfied and let $|\mathcal{C}|=n_1\cdots n_r$. By assumption there are tuples $\mathcal{C}_1,\dots,\mathcal{C}_s$ such that each $x\in S$ appears as an x_i in one of them.

Let $N=|\mathcal{C}_1|\cdots|\mathcal{C}_s|$. Then N is a non-zero integer. Since k is algebraically closed there is a $q\in k$ which satisfies $q^N=x_0$. Let $\mathcal{C}=(x_0,\dots,x_r,n_1,\dots,n_r,m_0,\dots,m_{r-1})$ be one of the $\mathcal{C}_i$'s. To complete the proof it suffices to show for all $0\le i\le r$ that

$$x_i=\omega'\left(q^{\frac{N}{n_1\cdots n_i}}\right)^{\ell_i}$$

for some root of unity $\omega'\in k$ and non-zero integer ℓ_i. The case $i=0$ is trivial. (By convention $n_1\cdots n_i=1$ when $i=0$.)

Suppose that $1 < i \le r$ and x_{i-1} has this form. Then we can write $x_{i-1} = \omega\left(q^{\frac{N}{n_1\cdots n_{i-1}}}\right)^{\ell_{i-1}}$ for some root of unity $\omega \in k$ and non-zero integer ℓ_{i-1}. From the calculation

$$x_i^{n_i} = x_{i-1}^{m_{i-1}} = \omega^{m_{i-1}}\left(q^{\frac{N}{n_1\cdots n_i}}\right)^{n_i\ell_{i-1}m_{i-1}} = \omega^{m_{i-1}}\left[\left(q^{\frac{N}{n_1\cdots n_i}}\right)^{m_{i-1}\ell_{i-1}}\right]^{n_i}$$

we deduce that $x_i \neq 0$ and that $x_i = \omega'\left(q^{\frac{N}{n_1\cdots n_i}}\right)^{m_{i-1}\ell_{i-1}}$, where $\omega' \in k$ is a root of unity. Take $l_i = m_{i-1}l_{i-1}$. $\square$

The main result of this section is:

Theorem 5.6. *Let A be an algebra satisfying (C) such that Γ and $(\chi_i)_{1\le i\le\theta}$ are part of a datum $\mathcal{D} = \mathcal{D}(\Gamma, (g_i)_{1\le i\le\theta}, (\chi_i)_{1\le i\le\theta}, (a_{ij})_{1\le i,j\le\theta})$ of finite Cartan type. Suppose that all q_i, where $1 \le i \le \theta$, are not roots of unity, and that if $1 \le i, j \le \theta$ are in different connected components of $\{1, \dots, \theta\}$, then a_i and a_j skew commute. Then finite-dimensional simple left A-modules are one-dimensional.*

Proof: Let $I_1, \dots, I_r$ be the components of $\{1, \dots, \theta\}$. Since the matrix $(a_{ij})_{1\le i,j\le\theta}$ is of finite type so are the matrices $(a_{ij})_{(i,j)\in I_l\times I_l}$ of I_l for all $1 \le l \le r$. Thus by virtue of part c) of Lemma 5.1 we may assume $r = 1$; that is $\{1, \dots, \theta\}$ is connected.

Suppose that $\{1, \dots, \theta\}$ is connected. We will use Proposition 5.3 to complete the proof. Let $k_1, \dots, k_\theta \ge 0$ and suppose that $\chi_1^{k_1}\cdots\chi_\theta^{k_\theta} = 1$. Then $Q(k_1, \dots, k_\theta) = 0$ by part d) of Lemma 5.4. Let $b_{ij} = d_i a_{ij}$ for all $1 \le i, j \le \theta$ and set $B = (b_{ij})$. Then B is a symmetric matrix by part c) of the same lemma. Since $a_{ii} = 2$ for all $1 \le i \le \theta$ we have

$$0 = Q(k_1, \dots, k_\theta) = \sum_{1\le i,j\le\theta} k_i b_{ij} k_j = (k_1 \cdots k_\theta)B\begin{pmatrix} k_1 \\ \vdots \\ k_\theta \end{pmatrix}.$$

It will follow by Proposition 5.3 that all finite-dimensional simple left A-modules are one-dimensional once we show that $k_1 = \cdots = k_\theta = 0$.

To show the latter we follow [7, Chapter III]. Let $\alpha_1, \dots, \alpha_\theta$ be a basis for the root system Φ corresponding to the Cartan matrix (a_{ij}). Let $(,)$ be the inner product of the euclidean vector space spanned by Φ. By definition $a_{ij} = \dfrac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$. Set $x = \sum_{i=1}^{\theta} k_i\alpha_i$. Then

$$(x, x) = \sum_{i,j=1}^{\theta} k_i k_j (\alpha_i, \alpha_j) = \sum_{i,j=1}^{\theta} k_i k_j a_{ij}\frac{(\alpha_i, \alpha_i)}{2}.$$

Since (a_{ij}) is connected there is a non-zero $c \in \mathbb{Q}$ such that $d_i = c\frac{(\alpha_i, \alpha_i)}{2}$ for all $1 \leq i \leq \theta$. Thus $0 = Q(k_1, \ldots, k_\theta) = c(x, x)$ which means $x = 0$ and consequently $k_1 = \cdots = k_\theta = 0$. □

Let $\mathcal{D}$ be a datum of Cartan type and assume that no q_i is a root of unity. We have seen that the characters $\chi_1, \ldots, \chi_\theta$ are $\mathbb{Z}$-linearly independent if $\mathcal{D}$ is connected and of finite type. If $\mathcal{D}$ is not connected, then linear independency fails for any non-trivial linking. The next example shows that for connected data $\mathcal{D}$ the characters are in general not linearly independent if $\mathcal{D}$ is not of finite type.

Example 5.7. *Let Γ be a free abelian group of rank two with basis g_1, g_2, and $q \in k$ not a root of unity. Let $(a_{ij}) = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$. Define $\chi_1, \chi_2 \in \widehat{\Gamma}$ by $\chi_j(g_i) = q^{a_{ij}}$ for all $1 \leq i, j \leq 2$. Then $q_1 = q_2 = q^2$ is not a root of unity, $\mathcal{D}(\Gamma, (g_i)_{1\leq i\leq 2}, (\chi_i)_{1\leq i\leq 2}, (a_{ij})_{1\leq i,j\leq 2})$ is a connected datum of Cartan type, but not of finite type, and $\chi_1\chi_2 = 1$.*

Finally we note that Theorem 5.6 is false if the q_i's are roots of unity.

Example 5.8. *Let $N \geq 2$ be an integer, let q a primitive N-th root of 1, and let $A = k\langle g, a \mid g^N = 1, gag^{-1} = qa\rangle$. The algebra A satisfies (C) with $\Gamma = \langle g\rangle$, and $\chi(g) = q$, and Γ and χ are part of a Cartan datum of type A_1. Let $\mathbb{Z}_N$ be the additive cyclic group of order N and M be an N-dimensional vector space with basis $m_i, i \in \mathbb{Z}_N$. Then the rules $gm_i = q^i m_i$, and $xm_i = m_{i+1}$ for all $i \in \mathbb{Z}_N$ determine an irreducible left A-module structure on M.*

6. Application to Some Classes of Pointed Hopf Algebras

We apply the results of the preceding sections to certain classes of pointed Hopf algebras which arise in recent classification work [1],[3]. The Hopf algebras of interest to us are quotients of two-cocycle twists $H = (U \otimes A)^\sigma$, where U and A have the form $B\#k[\Gamma]$, where Γ is an abelian group and B is a left $k[\Gamma]$-module algebra and a left $k[\Gamma]$-comodule coalgebra.

In the smash product, for $b \in B$ we identify b with $b\#1$ and for $h \in k[\Gamma]$ we identify h with $1\#h$. Thus we write $bh = (b\#1)(1\#h) = b\#h$ and $hb = (1\#h)(1\#b) = (h_{(1)} \cdot b)h_{(2)}$.

For $\chi \in \widehat{\Gamma}$ we set $\widetilde{\chi} = \varepsilon \otimes \gamma$. Then $\widetilde{\chi} : B \to k$ is an algebra homomorphism which extends χ.

In most cases $B = \mathfrak{B}(X)$ is a Nichols algebra of a finite-dimensional Yetter-Drinfeld module over the group algebra $k[\Gamma]$. See [2, Section

2] for a discussion of Nichols algebras and Yetter-Drinfeld modules in general.

6.1. **2-cocycle Twists of the Tensor Product of Biproducts over Abelian Groups.** Let Γ be an abelian group. A Yetter-Drinfeld module over $k[\Gamma]$ can be described as a Γ-graded vector space which is a Γ-module such that all g-homogeneous components, where $g \in \Gamma$, are stable under the Γ-action. We denote the category of Yetter-Drinfeld modules over $k[\Gamma]$ by ${}^{k[\Gamma]}_{k[\Gamma]}\mathcal{Y}D = {}^{\Gamma}_{\Gamma}\mathcal{YD}$.

For $V \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$, $g \in \Gamma$, and $\chi \in \widehat{\Gamma}$ we define

$$V_g = \{v \in V \mid \delta(v) = g \otimes v\}$$

and

$$V_g^{\chi} = \{v \in V_g \mid h \cdot v = \chi(h)v \text{ for all } h \in \Gamma\}.$$

In this section we fix abelian groups Λ and Γ, integers $n, m \geq 1$, elements $z_1, \dots, z_n \in \Lambda$ and $g_1, \dots, g_m \in \Gamma$, and nontrivial characters $\eta_1, \dots, \eta_n \in \widehat{\Lambda}$ and $\chi_1, \dots, \chi_m \in \widehat{\Gamma}$.

Let $W \in {}^{\Lambda}_{\Lambda}\mathcal{YD}$ have basis $u_i \in W_{z_i}^{\eta_i}, 1 \leq i \leq n$, and $V \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$ have basis $a_j \in V_{g_j}^{\chi_j}, 1 \leq j \leq m$. Note that the restriction maps

$$\mathrm{Alg}(\mathfrak{B}(W)\#k[\Lambda], k) \to \widehat{\Lambda},\ \mathrm{Alg}(\mathfrak{B}(V)\#k[\Gamma], k) \to \widehat{\Gamma}, \tag{6.1}$$

are bijective since the characters η_i, χ_j are all non-trivial.

Let $U = \mathfrak{B}(W)\#k[\Lambda]$ and $A = \mathfrak{B}(V)\#k[\Gamma]$. For all $z \in \Lambda, g \in \Gamma$ and $1 \leq i \leq n, 1 \leq j \leq m$

$$zu_i = \eta_i(z)u_i z, \tag{6.2}$$

$$ga_j = \chi_j(g)a_j g, \tag{6.3}$$

$$\Delta(z) = z \otimes z, \qquad \Delta(u_i) = z_i \otimes u_i + u_i \otimes 1, \tag{6.4}$$

$$\Delta(g) = g \otimes g, \qquad \Delta(a_j) = g_j \otimes a_j + a_j \otimes 1. \tag{6.5}$$

Let $\varphi : R \to S$ be any ring homomorphism. Then the restriction functor $\varphi^* : {}_S\mathcal{M} \to {}_R\mathcal{M}$ maps a left S-module M to itself thought of as an R-module via pullback along φ.

Theorem 6.1. *In addition to the above let $\varphi : \Lambda \to \widehat{\Gamma}$ be a group homomorphism, $s : \{1, \dots, n\} \to \{1, \dots, m\}$ be a function, and let $\lambda_1, \dots, \lambda_n \in k$. Let $I' = \{1 \leq i \leq n \mid \lambda_i \neq 0\}$. Assume that for all $i \in I'$ and $z \in \Lambda$*

$$\varphi(z_i) = \chi_{s(i)}^{-1} \text{ and } \eta_i(z) = \varphi(z)(g_{s(i)}). \tag{6.6}$$

Then:

a) *There is a Hopf algebra map* $\Phi : U \to A^{o\,\mathrm{cop}}$ *determined by* $\Phi(z) = \widetilde{\varphi(z)}$ *for all* $z \in \Lambda$ *and for all* $1 \le i \le n$ *the equations* $\Phi(u_i)(g) = 0$ *for all* $g \in \Gamma$ *and* $\Phi(u_i)(a_j) = \delta_{s(i),j}\lambda_i$ *for all* $1 \le j \le m$ *hold.*

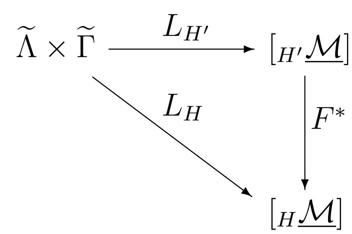
b) *Let* σ *be the 2-cocycle determined by* Φ *and* $H = (U \otimes A)^\sigma$. *Then*

$$L_H : \widehat{\Lambda}\times\widehat{\Gamma} \to [{}_H\underline{\mathcal{M}}],\ \text{given by } (\rho,\chi) \mapsto [L(\rho,\chi)], \tag{6.7}$$

is a bijection, where $L(\rho,\chi) = L(\widetilde{\rho},\widetilde{\chi})$.

c) *Assume that the restriction of* s *to* I' *is injective and let* $V' \subset V$ *and* $W' \subset W$ *be the Yetter–Drinfeld submodules with bases* $a_{s(i)}, i \in I'$, *and* $u_i, i \in I'$. *Let* $U' = \mathfrak{B}(W')\#k\Lambda$, $A' = \mathfrak{B}(V')\#k\Gamma$ *and let* σ' *be the restriction of* σ *to* $U' \otimes A'$. *Then the projections* $\pi_W : W \to W'$, $\pi_V : V \to V'$ *define a surjective bialgebra map* $F : H \to H'$ *determined by* $F|W = \pi_W$, $F|V = \pi_V$, $F|\Gamma = \mathrm{id}_\Gamma$, *and* $F|\Lambda = \mathrm{id}_\Lambda$.

d) *The diagram*



commutes. Thus the restriction functor F^* *defines a bijection* $F^* : [{}_{H'}\underline{\mathcal{M}}] \to [{}_H\underline{\mathcal{M}}]$.

Proof: Part a) is shown in [15, Corollary 9.1]. By (6.1) part b) follows from part a) and from Theorem 4.1 a). Part c) follows by [15, Corollary 9.2]. Commutativity of the diagram in part d) follows by Proposition 2.7 with $U' = \overline{U}$, $A' = \overline{A}$, $f = F|U$, and $g = F|A$. Now F^* is bijective by part b) for H and H'. □

Remark 6.2. *The restriction* $s|I'$ *of the previous theorem is injective when the braiding matrix* $(q_{ij} = \eta_j(z_i))$ *of* W *satisfies the following condition*

$$q_{ij}q_{ji} = q_{ii}^{a_{ij}} \text{ for all } i, j,$$

where (a_{ij}) *is a Cartan matrix of finite type and for all* i *the order of* q_{ii} *is greater than 3.*

The remark is easily justified. By (6.6) we have $q_{ij} = \eta_j(z_i) = \varphi(z_i)(g_{s(j)}) = \chi^{-1}_{s(i)}(g_{s(j)})$ for all $i \in I'$ and $1 \le j \le \theta$.

Assume $s(j) = s(l)$, where $j, l \in I'$. Then $q_{ij} = q_{il}$ and $q_{ji} = q_{li}$ for all $i \in I'$. Thus $q_{ii}^{a_{ij}} = q_{ij}q_{ji} = q_{il}q_{li} = q_{ii}^{q_{il}}$. Since $|a_{ij} - a_{il}| \leq 3$ and the order of q_{ii} is larger than 3 we have $a_{ij} = a_{il}$ for all $i \in I'$. Since $(a_{rs})_{r,s \in I'}$ is invertible necessarily $j = l$.

The results of Section 9 of [15] are given in terms of the bilinear form $\beta : W \otimes V \to k$ defined by $\beta(u_i \otimes a_j) = \lambda_i \delta_{s(i)j}$ for all $1 \leq i \leq n$ and $1 \leq j \leq m$. In part b) of the preceding theorem, this form for $W' \otimes V'$ is $\beta' = \beta|(W' \otimes V')$ and is non-degenerate. Most applications for us will be in the context of Theorem 6.1 and the map s will be injective by virtue of Remark 6.2. In light of part c) of the theorem to study the $L(\rho, \chi)'s$ we may assume that β is non-degenerate.

6.2. **The Modules $L(\rho, \chi)$ in the Non-degenerate Case.** We continue in the context of Theorem 6.1. In particular

$$U = \mathfrak{B}(W)\#k[\Lambda],\ A = \mathfrak{B}(V)\#k[\Gamma].$$

Assume that β is non-degenerate. Then $n = m$, $\lambda_1, \ldots, \lambda_n \neq 0$ and without loss of generality we assume $s = \mathrm{id}$. By rescaling the generators a_i we could further assume that $\lambda_i = 1$ for all $1 \leq i \leq n$; however this might not be convenient for computations.

In this case

$$\varphi(z_i) = \chi_i^{-1} \text{ and } \eta_i(z) = \varphi(z)(g_i) \text{ for all } 1 \leq i \leq n, z \in \Lambda, \tag{6.8}$$

$$\Phi(z) = \widetilde{\varphi(z)},\ \Phi(u_i) = \delta_i \text{ for all } z \in \Lambda, 1 \leq i \leq n, \tag{6.9}$$

where $\delta_i : A \to k$ is the $(\varepsilon, \chi_i^{-1})$-derivation with

$$\delta_i(g) = 0,\ \delta_i(a_j) = \delta_{ij}\lambda_i \text{ for all } g \in \Gamma, 1 \leq i, j \leq n. \tag{6.10}$$

Remark 6.3. *The Hopf algebra map $\Phi : U \to A^{o\,cop}$ is injective since it is injective on the primitive elements $P(\mathfrak{B}(W)) = W$ by our assumption that $\lambda_i \neq 0$ for all i (see* [17, Lemma 11.01]*).*

Let $H = (U \otimes A)^\sigma$, where σ is the 2-cocycle defined by τ with $\tau(u, a) = \Phi(u)(a)$ for all $u \in U, a \in A$. For simplicity we will write ua instead of $u \otimes a \in H$ for all $u \in U, a \in A$.

For the rest of this subsection we fix characters $\rho \in \widehat{\Lambda}$ and $\chi \in \widehat{\Gamma}$. As in section 2.1 we consider $U = U_\chi$ and the dual $A_\rho)^*$ of A as left H-modules by

$$(u'a) \cdot_\chi u = u'\tau(u_{(1)}, a_{(1)})u_{(2)}\chi(a_{(2)})\tau^{-1}(u_{(3)}, a_{(3)}), \text{ and} \tag{6.11}$$

$$((ua') \cdot_\rho p)(a) = \tau(u_{(1)}, a_{(1)})\rho(u_{(2)})p(a_{(2)}a')\tau^{-1}(u_{(3)}, a_{(3)}) \tag{6.12}$$

for all $u, u' \in U, a, a' \in A$, and $p \in A^*$.

Note that by (6.12)

$$u \cdot_\rho p = \Phi(u_{(1)})\rho(u_{(2)})p\Phi(S(u_{(3)})) \text{ for all } u \in U, p \in A^*. \tag{6.13}$$

We recall that $L(\rho,\chi) = U_\chi/I(\rho,\chi)$ is a cyclic left H-module and left U-module with generator $m =$ residue class of 1. By part c) of Proposition 2.6 there is a left H-isomorphism

$$U_\chi/I(\rho,\chi) \cong U \cdot_\rho \chi, \text{ with } m \mapsto \chi. \tag{6.14}$$

We denote the H-action on the quotient $L(\rho,\chi)$ by $\cdot$ and we write $um = u \cdot m$ for all $u \in U$. Recall that $u \cdot_\chi u' = uu'$ for all $u, u' \in U$ by 2.3. Therefore for all $u \in U$, um is the residue class of u in $L(\rho,\chi)$ and the H-action on um is given for all $u' \in U$ and $a \in A$ by

$$u' \cdot um = u'um, \; a \cdot um = (a \cdot_\chi u)m. \tag{6.15}$$

The latter holds since $a \cdot_\chi (u \cdot_\chi 1) = a \cdot_\chi u = (a \cdot_\chi u) \cdot_\chi 1$.

The next propositions contain some information about the modules $L(\rho,\chi)$. We denote

$$q_{ij} = \eta_j(z_i) \text{ for all } 1 \leq i,j \leq n. \tag{6.16}$$

Proposition 6.4. *$L(\rho,\chi)$ is the k-span of all*

$$u_{i_1} \cdots u_{i_t} m, \text{ where } 1 \leq i_1, \ldots, i_t \leq n \text{ and } t \geq 0.$$

The H-action on these elements is given for all $z \in \Lambda, g \in \Gamma$, and $1 \leq j \leq n$ by

a) $z \cdot u_{i_1} \cdots u_{i_t} m = (\eta_{i_1} \cdots \eta_{i_t} \rho)(z) u_{i_1} \cdots u_{i_t} m$,
b) $g \cdot u_{i_1} \cdots u_{i_t} m = (\chi_{i_1}^{-1} \cdots \chi_{i_t}^{-1} \chi)(g) u_{i_1} \cdots u_{i_t} m$,
c) $u_j \cdot u_{i_1} \cdots u_{i_t} m = u_j u_{i_1} \cdots u_{i_t} m$,
d) $a_j \cdot u_{i_1} \cdots u_{i_t} m = \sum_{l=1}^t \alpha_l(j, i_1, \ldots, i_t) u_{i_1} \cdots u_{i_{l-1}} u_{i_{l+1}} \cdots u_{i_t} m$,

with $\alpha_l(j, i_1, \ldots, i_t) = \delta_{i_l j} \lambda_j \prod_{r=1}^{l-1} q_{i_r j} \left(1 - \prod_{s=l+1}^t q_{i_s j} q_{j i_s} \rho(z_j)\chi(g_j)\right)$.

Proof: By (6.13) we have $z \cdot_\rho \chi = \Phi(z)\rho(z)\chi\Phi(z^{-1})\rho(z) = \rho(z)\chi$ since $\mathrm{Alg}(A,k)$ is commutative by (6.1). Hence by (6.14)

$$z \cdot m = \rho(z)m.$$

Since $zu_i = \eta_i(z)u_i z$ for all $1 \leq i \leq n$, part a) follows by the first equation of (6.15). Note that part a) implies that the elements $u_{i_1} \cdots u_{i_t} m$ span $L(\rho,\chi)$.

Let $u = u_{i_1} \cdots u_{i_t}$. Note that by (6.4)

$$\begin{aligned} \Delta^2(u) &= u_{(1)} \otimes u_{(2)} \otimes u_{(3)} \\ &= (z_{i_1} \otimes z_{i_1} \otimes u_{i_1} + z_{i_1} \otimes u_{i_1} \otimes 1 + u_{i_1} \otimes 1 \otimes 1) \cdots \\ &\cdots (z_{i_t} \otimes z_{i_t} \otimes u_{i_t} + z_{i_t} \otimes u_{i_t} \otimes 1 + u_{i_t} \otimes 1 \otimes 1). \end{aligned} \tag{6.17}$$

By (6.15) $g \cdot um = (g \cdot_\chi u)m$, and by (6.11) and (6.17)

$$\begin{aligned} g \cdot_\chi u &= \Phi(u_{(1)})(g)u_{(2)}\chi(g)\Phi(u_{(3)})(g^{-1}) \\ &= \varphi(z_{i_1} \cdots z_{i_t})(g)u_{i_1} \cdots u_{i_t}\chi(g), \end{aligned}$$

since $\Phi(u_{(1)})(g) = 0$ resp. $\Phi(u_{(3)})(g^{-1}) = 0$ if the term $u_{(1)}$ resp. $u_{(3)}$ contains a factor u_{i_l}. This proves part b) since

$$\varphi(z_{i_1} \cdots z_{i_t})(g) = (\chi_{i_1}^{-1} \cdots \chi_{i_t}^{-1})(g)$$

by (6.8).

Part c) is trivial, and to prove part d) we compute $a_j \cdot_\chi u$. Since $\Delta^2(a_j) = g_j \otimes g_j \otimes a_j + g_j \otimes a_j \otimes 1 + a_j \otimes 1 \otimes 1$, and $\chi(a_j) = 0$ it follows from (6.11) that

$$a_j \cdot_\chi u = \Phi(u_{(1)})(g_j)u_{(2)}\chi(g_j)\Phi(u_{(3)})(-a_j g_j^{-1}) + \Phi(u_{(1)})(a_j)u_{(2)}. \tag{6.18}$$

Here we used that $S^{-1}(a_j) = -a_j g_j^{-1}$.

To compute the first term in (6.18) we first note that

$$\Phi(u_{(1)})(g_j)u_{(2)} = q_{i_1 j} \cdots q_{i_t j} u, \tag{6.19}$$

since by part a) of Theorem 6.1 we have $\Phi(u_{(1)})(g_j) = 0$ if $u_{(1)}$ contains at least one factor u_i, hence $\Phi(u_{(1)})(g_j)u_{(2)} = \Phi(z_{i_1} \cdots z_{i_t})(g_j)u$. Next we see that

$$u_{(1)}\Phi(u_{(2)})(a_j g_j^{-1}) = \sum_{l=1}^{t} u_{i_1} \cdots u_{i_{l-1}} z_{i_l} u_{i_{l+1}} \cdots u_{i_t} \Phi(u_{i_l})(a_j g_j^{-1}) \tag{6.20}$$

since $\Phi(u_{(2)})(a_j g_j^{-1}) = 0$, if $u_{(2)}$ contains no or at least two factors u_{i_l}. Using (6.2), (6.10) and (6.16) we obtain from (6.20)

$$\begin{aligned} &u_{(1)}\Phi(u_{(2)})(a_j g_j^{-1}) = \\ &\sum_{l=1}^{t} \delta_{i_l j}\lambda_j q_{jj}^{-1} q_{i_l i_{l+1}} \cdots q_{i_l i_t} u_{i_1} \cdots u_{i_{l-1}} u_{i_{l+1}} \cdots u_{i_t} z_{i_l}. \end{aligned} \tag{6.21}$$

We now use (6.21) in Δ applied to (6.19) and obtain for the first term in (6.18)

$$\begin{aligned} &\Phi(u_{(1)})(g_j)u_{(2)}\chi(g_j)\Phi(u_{(3)})(-a_j g_j^{-1}) = \\ &-q_{i_1 j} \cdots q_{i_t j}\chi(g_j) \sum_{l=1}^{t} \delta_{i_l j}\lambda_j q_{jj}^{-1} q_{i_l i_{l+1}} \cdots q_{i_l i_t} u_{i_1} \cdots u_{i_{l-1}} u_{i_{l+1}} \cdots u_{i_t} z_{i_l}. \end{aligned} \tag{6.22}$$

Similarly we compute the second term in (6.18)

$$\Phi(u_{(1)})(a_j)u_{(2)} = \sum_{l=1}^{t} \delta_{i_l j}\lambda_j q_{i_1 i_l} \ldots q_{i_{l-1} i_l} u_{i_1} \cdots u_{i_{l-1}} u_{i_{l+1}} \cdots u_{i_t}. \tag{6.23}$$

Finally (6.18), (6.22) and (6.23) prove part d) in view of (6.15) and part a). □

The presentation of $L(\rho,\chi)$ in (6.14) as the subspace $U \cdot_\rho \chi$ of A^* is helpful to actually calculate the elements of $L(\rho,\chi)$ as linear functions on A. The next proposition in principle describes an inductive procedure to calculate $L(\rho,\chi)$.

Proposition 6.5. *Let* $1 \le i, i_1, \dots, i_t \le n$, *where* $t \ge 1$, *and define* $u = u_{i_1} \cdots u_{i_t}$. *Then*

a) $u_i \cdot_\rho \chi = (1 - \rho(z_i)\chi(g_i))\Phi(u_i)\chi$.
b) $u_i \cdot_\rho (\Phi(u)\chi) = \Phi(\big[u_i u - \prod_{r=1}^{t} q_{ii_r}\rho(z_i)\chi(g_i)uu_i\big])\chi$.
c) $u_i^t \cdot_\rho \chi = \prod_{l=0}^{t-1}\big(1 - q_{ii}^l \rho(z_i)\chi(g_i)\big)\,\Phi(u_i^t)\chi$.

Proof: Since $\Delta^2(u_i) = z_i \otimes z_i \otimes u_i + z_i \otimes u_i \otimes 1 + u_i \otimes 1 \otimes 1$, and $\rho(u_i) = 0, S(u_i) = -z_i^{-1}u_i$ it follows from (6.13) that for any $p \in A^*$

$$u_i \cdot_\rho p = \Phi(z_i)\rho(z_i)p\Phi(-z_i^{-1}u_i) + \Phi(u_i)p. \tag{6.24}$$

In particular

$$\begin{aligned} u_i \cdot_\rho (\Phi(u)\chi) &= \Phi(u_i)\Phi(u)\chi - \rho(z_i)\Phi(z_i)\Phi(u)\chi\Phi(z_i^{-1})\Phi(u_i) \\ &= \Phi(u_i u)\chi - q_{ii_1}\cdots q_{ii_t}\rho(z_i)\Phi(u)\chi\Phi(u_i) \end{aligned} \tag{6.25}$$

since, as $\mathrm{Alg}(A,k)$ is abelian,

$$\begin{aligned} \Phi(z_i)\Phi(u)\chi\Phi(z_i^{-1}) &= \Phi(z_i)\Phi(u)\Phi(z_i^{-1})\chi \\ &= \Phi(z_i u z_i^{-1})\chi \\ &= q_{ii_1}\cdots q_{ii_t}\Phi(u)\chi. \end{aligned}$$

Since $\chi\Phi(u_i)\chi^{-1}$ and $\chi(g_i)\Phi(u_i)$ are both $(\varepsilon,\chi_i^{-1})$-derivations taking the same values on the generators $a_j, 1 \le j \le n$, and $g \in \Gamma$ of A we see that

$$\chi\Phi(u_i)\chi^{-1} = \chi(g_i)\Phi(u_i). \tag{6.26}$$

Part b) now follows from (6.25) and (6.26), and part a) is the special case of part b) with $t = 0$.

Part c) then follows by induction on t. The case $t = 1$ is part a), and the induction step is

$$\begin{aligned} u_i^{t+1} \cdot_\rho \chi &= u_i \cdot_\rho (u_i^t \cdot_\rho \chi) \\ &= \prod_{l=0}^{t-1} (1 - q_{ii}^l \rho(z_i)\chi(g_i)) u_i \cdot_\rho (\Phi(u_i^t)\chi) \\ &= \prod_{l=0}^{t} (1 - q_{ii}^l \rho(z_i)\chi(g_i)) \Phi(u_i^{t+1})\chi \end{aligned}$$

since by part b) $u_i \cdot_\rho (\Phi(u_i^t)\chi) = (1 - q_{ii}^t \rho(z_i)\chi(g_i))\Phi(u_i^{t+1})\chi$. □

Corollary 6.6. *Assume that*

$$q_{ii} \text{ is not a root of unity for all } 1 \le i \le n \tag{6.27}$$

and that $L(\rho, \chi)$ *is finite-dimensional. Then for all* $1 \le i \le n$ *there is a natural number* $r_i \ge 0$ *with* $q_{ii}^{r_i} \rho(z_i)\chi(g_i) = 1$.

Proof: Let $1 \le i \le n$. The elements $u_i^t m, t \ge 0$, of $L(\rho, \chi)$ are linearly dependent. By part a) of Proposition 6.4, the group Λ acts on $u_i^t m$ via the character $\eta_i^t \rho$. Since by (6.27) $\eta_i^t \rho \neq \eta_i^{t'} \rho$ for all $t \neq t'$, there is an integer $r_i \ge 0$ such that $u_i^t m = 0$ for all $t > r_i$, and $u_i^t m \neq 0$ for all $0 \le t \le r_i$. It is well-known that by (6.27) $u_i^t \neq 0$ for all $t \ge 0$ (see for example [2, Example 2.9]). Since Φ is injective by Remark 6.3, it follows that $\Phi(u_i^t) \neq 0$ for all $t \ge 0$, and thus $q_{ii}^{r_i} \eta(z_i)\chi(g_i) = 1$ by part c) of Proposition 6.5 and (6.14). □

As an example we consider the easiest case where U and A are quantum linear spaces, that is (6.28) holds.

Corollary 6.7. *Assume*

$$q_{ij} q_{ji} = 1 \text{ for all } 1 \le i, j \le n, i \neq j. \tag{6.28}$$

Then for all $t_1, \dots, t_n \ge 0$,

$$(u_1^{t_1} \cdots u_n^{t_n}) \cdot_\rho \chi = \prod_{i=1}^{n} \prod_{l_i=0}^{t_i - 1} (1 - q_{ii}^{l_i} \rho(z_i)\chi(g_i)) \Phi(u_1^{t_1} \cdots u_n^{t_n})\chi.$$

Proof: It follows from (6.28) that

$$u_i u_j = q_{ij} u_j u_i \text{ for all } 1 \le i, j \le n, i \neq j, \tag{6.29}$$

since the skew-commutator $u_i u_j - q_{ij} u_j u_i$ is primitive in $\mathfrak{B}(W)$.

We prove by induction on $1 \leq j \leq n$ that

$$(u_j^{t_j} \cdots u_n^{t_n}) \cdot_\rho \chi = \prod_{i=j}^{n} \prod_{l_i=0}^{t_i-1} (1 - q_{ii}^{l_i} \rho(z_i)\chi(g_i))\Phi(u_j^{t_j} \cdots u_n^{t_n})\chi$$

for all $t_j, \ldots, t_n \geq 0$.

The case $j = n$ is part c) of Proposition 6.5. Assume the formula for $j+1 \leq n$. Then we prove the claim for j by induction on t_j. If $t_j = 0$, then the formula is true by assumption, and for all $t_j \geq 0$ we see by induction on t_j that

$$\begin{aligned}(u_j^{t_j+1} \cdots u_n^{t_n}) \cdot_\rho \chi &= u_j \cdot_\rho \left((u_j^{t_j} \cdots u_n^{t_n}) \cdot_\rho \chi \right) \\ &= \prod_{i=j}^{n} \prod_{l_i=0}^{t_i-1} (1 - q_{ii}^{l_i} \rho(z_i)\chi(g_i)) u_j \cdot_\rho (\Phi(u_j^{t_j} \cdots u_n^{t_n})\chi). \qquad (6.30)\end{aligned}$$

By part b) of Proposition 6.5,

$$\begin{aligned} & u_j \cdot_\rho \left(\Phi(u_j^{t_j} \cdots u_n^{t_n})\chi \right) \\ = & \left(\Phi(u_j^{t_j+1} \cdots u_n^{t_n}) - q_{jj}^{t_j} q_{j,j+1}^{t_{j+1}} \cdots q_{jn}^{t_n} \rho(z_j)\chi(g_j)\Phi(u_j^{t_j} \cdots u_n^{t_n} u_j) \right) \chi \\ = & \left(1 - q_{jj}^{t_j} \rho(z_j)\chi(g_j) \right) \Phi(u_j^{t_j+1} \cdots u_n^{t_n})\chi, \end{aligned}$$

since $u_j^{t_j} \cdots u_n^{t_n} u_j = q_{j+1,j}^{t_{j+1}} \cdots q_{nj}^{t_n} u_j^{t_j+1} \cdots u_n^{t_n}$ by (6.29), and all the q_{jl} and q_{lj} except for $l = j$ cancel by (6.28). Hence the claim follows from (6.30). □

Corollary 6.8. *Assume* (6.28) *and* (6.27)*. Then the following are equivalent:*

a) *$L(\rho, \chi)$ is finite-dimensional.*
b) *For all $1 \leq i \leq n$ there is a natural number $r_i \geq 0$ with $q_{ii}^{r_i} \rho(z_i)\chi(g_i) = 1$.*

If condition b) is satisfied, then the elements $u_1^{t_1} \cdots u_n^{t_n} m, 0 \leq t_i \leq r_i$ for all $1 \leq i \leq n$, form a basis of $L(\rho, \chi)$.

Proof: It is well-known that by (6.28) and (6.27) $\mathfrak{B}(W)$ is generated by $u_1, \ldots, u_n$ with defining relations $u_i u_j - q_{ij} u_j u_i = 0$ for all $1 \leq i, j \leq n, i \neq j$, and has PBW-basis $u_1^{t_1} \cdots u_n^{t_n}, t_1, \ldots, t_n \geq 0$ (see for example [3, Theorem 4.2]).

By Corollary 6.6, a) implies b). Conversely, suppose b). Since Φ is injective by Remark 6.3, and the elements $u_1^{t_1} \cdots u_n^{t_n}, 0 \leq t_i \leq r_i$ for all $1 \leq i \leq n$, of the PBW-basis are linearly independent it follows from Corollary 6.7 that the elements $u_1^{t_1} \cdots u_n^{t_n} m, 0 \leq t_i \leq r_i$ for all

$1 \leq i \leq n$, are linearly independent, hence a basis of $L(\rho, \chi)$ since $u_1^{t_1} \cdots u_n^{t_n} = 0$ if $t_i > r_i$ for one i. □

7. Application to pointed Hopf algebras given by data of finite Cartan type

Let $\mathcal{D} = \mathcal{D}(\Gamma, (g_i)_{1\leq i\leq\theta}, (\chi_i)_{1\leq i\leq\theta}, (a_{ij})_{1\leq i,j\leq\theta})$ be a datum of finite Cartan type and suppose that λ is a family of linking parameters for $\mathcal{D}$ in the sense of [1]. We apply the theory developed in this paper to the infinite-dimensional Hopf algebras $U(\mathcal{D}, \lambda)$ in general, and to the finite-dimensional versions $u(\mathcal{D}, \lambda)$ under some restrictions.

Each is a quotient of an $H = (U \otimes A)^\sigma$ where the 2-cocycle σ is determined by the linking parameters. The finite-dimensional irreducible left modules for $U(\mathcal{D}, \lambda)$ are parameterized by a subset of $\widehat{\Gamma}$ and the irreducible modules for $u(\mathcal{D}, \lambda)$ by $\widehat{\Gamma}$; these irreducible modules arise from $L(\rho, \chi)$'s defined for H where χ determines ρ.

The Hopf algebras $U(\mathcal{D}, \lambda)$ and $u(\mathcal{D}, \lambda)$ are described in [2] in a slightly different notation. There are many parallels between the Hopf algebras $U(\mathcal{D}, \lambda)$ and the quantized enveloping algebras $U_q(\mathfrak{g})$ where q is not a root of unity, and there are many parallels between the Hopf algebras $u(\mathcal{D}, \lambda)$ and the Frobenius-Lusztig kernels. We emphasize them here and set the stage for further work on their representation theories.

7.1. The Linking Graph.

We expand the discussion following the proof of Proposition 5.3. Let $\mathcal{D} = \mathcal{D}(\Gamma, (g_i)_{1\leq i\leq\theta}, (\chi_i)_{1\leq i\leq\theta}, (a_{ij})_{1\leq i,j\leq\theta})$ be a datum of finite Cartan type. A family $\lambda = (\lambda_{ij})_{1\leq i,j\leq\theta,\, i\not\sim j}$ of elements in k is called a *family of linking parameters for* $\mathcal{D}$ if the following conditions are satisfied for all $1 \leq i, j \leq \theta$, $i \not\sim j$:

$$\text{If } g_i g_j = 1 \text{ or } \chi_i\chi_j \neq \varepsilon, \text{ then } \lambda_{ij} = 0. \tag{7.1}$$

$$\lambda_{ji} = -q_{ji}\lambda_{ij}. \tag{7.2}$$

This definition is a formal extension of [1, Section 5.1] where λ_{ij} was only defined for $i < j$ and $i \not\sim j$. Note that by (7.2) $\lambda_{ij} = -q_{ji}^{-1}\lambda_{ji} = -q_{ij}\lambda_{ji}$ since $q_{ij}q_{ji} = 1$ for all $i \not\sim j$. Not that (7.1) and (7.2) are met when $\lambda_{ij} = 0$ for all i, j. We let 0 denote this family of linking parameters.

Vertices $1 \leq i, j \leq \theta$ are called *linkable* if $i \not\sim j$, $g_i g_j \neq 1$ and $\chi_i\chi_j = \varepsilon$. Then (see [1, Section 5.1])

$$q_i = q_j^{-1} \text{ if } i, j \text{ are linkable.} \tag{7.3}$$

The next lemma is [1, Lemma 5.6].

Lemma 7.1. *Let* $\mathcal{D} = \mathcal{D}(\Gamma, (g_i)_{1\le i\le\theta}, (\chi_i)_{1\le i\le\theta}, (a_{ij})_{1\le i,j\le\theta})$ *be a datum of finite Cartan type, and assume that* $\mathrm{ord}(q_t) > 3$ *for all* $1 \le t \le \theta$.

a) *If vertices* i, k *and* j, l *are linkable, then* $a_{ij} = a_{kl}$.
b) *A vertex* i *cannot be linkable to two different vertices* j, k.

We say that linkable vertices $1 \le i, j \le \theta$ are *linked* if $\lambda_{ij} \neq 0$.

Let λ be a family of linking parameters for a datum of Cartan type $\mathcal{D} = \mathcal{D}(\Gamma, (g_i)_{1\le i\le\theta}, (\chi_i)_{1\le i\le\theta}, (a_{ij})_{1\le i,j\le\theta})$. Let $\mathcal{X}$ be the set of connected components of $\{1, \dots, \theta\}$. See the discussion preceding Lemma 5.4. We define the *linking graph* of $(\mathcal{D}, \lambda)$ as follows: The set of its vertices is $\mathcal{X}$. There is an edge between $J_1, J_2 \in \mathcal{X}$ if and only there are elements $i \in J_1, j \in J_2$ such that i, j are linked. Recall that a graph is called *bipartite* if the set of its vertices is the disjoint union of subsets X_1, X_2 such that there is no edge between vertices in X_1 or in X_2.

Lemma 7.2. *Let* $\mathcal{D} = \mathcal{D}(\Gamma, (g_i)_{1\le i\le\theta}, (\chi_i)_{1\le i\le\theta}, (a_{ij})_{1\le i,j\le\theta})$ *be a datum of Cartan type, and* λ *a family of linking parameters for* $\mathcal{D}$. *Assume one of the following conditions:*

a) *The Cartan matrix* (a_{ij}) *is simply laced, that is* $a_{ij} \in \{0, -1\}$ *for all* $1 \le i, j \le \theta, i \neq j$, *and* $\mathrm{ord}(q_i)$ *is odd for all* $1 \le i \le \theta$.
b) *For all* $1 \le i \le \theta$, q_i *is not a root of one.*

Then the linking graph of $(\mathcal{D}, \lambda)$ *is bipartite.*

Proof: A graph is bipartite if and only if it contains no cycle of odd length [5, Proposition 0.6.1]. Assume the linking graph has a cycle of odd length n. Then there are connected components $I_1, \dots, I_n$, and $i_l \in I_l, 1 \le l \le n$, and $j_k \in I_k, 2 \le k \le n+1, I_{n+1} = I_1$, with $\lambda_{i_l j_{l+1}} \neq 0$ for all $1 \le l \le n$. By (5.4) we have $q_{i_l}^{a_l} = q_{j_l}^{b_l}$ for all $2 \le l \le n+1$, with $i_{n+1} = i_1$, where $a_{i_l} = a(i_l, j_l)$ and $b_l = b(i_l, j_l)$. And by (7.3), $q_{i_l} = q_{j_{l+1}}^{-1}$ for all $1 \le l \le n$. Hence

$$q_{i_{n+1}}^{a_{n+1}\cdots a_2} = q_{i_1}^{(-1)^n b_{n+1}\cdots b_2}.$$

Since $i_{n+1} = i_1$, and n is odd, the order of q_{i_1} must divide $a_{n+1} \cdots a_2 + b_{n+1} \cdots b_2$. This is impossible in both cases a) and b). $\square$

However, in general the linking graph of $(\mathcal{D}, \lambda)$ is not necessarily bipartite.

Example 7.3. *One can link an odd number of copies of* B_2 *in a circle, where the group* Γ *is* $\mathbb{Z}^{2n}$ *or* $(\mathbb{Z}/(N))^{2n}$ *such that* N *is an integer dividing* $1 + 2^n$, *and where* $g_1, \dots, g_{2n}$ *are the canonical basis elements of* Γ.

7.2. **The Infinite-dimensional Case.** In this section we fix a Cartan datum of finite type $\mathcal{D} = \mathcal{D}(\Gamma, (g_i)_{1\le i\le\theta}, (\chi_i)_{1\le i\le\theta}, (a_{ij})_{1\le i,j\le\theta})$, and a family $\lambda = (\lambda_{ij})_{1\le i,j\le\theta, i\not\sim j}$ of linking parameters for $\mathcal{D}$.

We assume that for all $1 \le i \le \theta$ the character values $q_i = \chi_i(g_i)$ are not roots of unity.

Let $X \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$ be the vector space with basis $x_i \in X^{\chi_i}_{g_i}, 1 \le i \le \theta$. The tensor algebra $T(X)$ is an algebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$. Thus the smash product $T(X)\#k[\Gamma]$ is a biproduct and thus has a Hopf algebra structure. We identify $T(X)$ with the free algebra $k\langle x_1, \dots, x_\theta\rangle$. We use the identification conventions for smash products described in the beginning of Section 6.

Definition 7.4. *Let $U(\mathcal{D}, \lambda)$ be the quotient Hopf algebra of the smash product $k\langle x_1, \dots, x_\theta\rangle\#k[\Gamma]$ modulo the ideal generated by*

(7.4) $\quad \mathrm{ad}_c(x_i)^{1-a_{ij}}(x_j),$ *for all* $1 \le i, j \le \theta, i \sim j, i \ne j,$

(7.5) $\quad x_ix_j - q_{ij}x_jx_i - \lambda_{ij}(1 - g_ig_j),$ *for all* $1 \le i < j \le \theta,\ i \not\sim j.$

Here, the k-linear endomorphism $\mathrm{ad}_c(x_i)$ of the free algebra is given for all y by the braided commutator $\mathrm{ad}_c(x_i)(y) = x_iy - (g_i \cdot y)x_i$.

We denote the images of x_i and $g \in \Gamma$ in $U(\mathcal{D}, \lambda)$ again by x_i and g. Note that in $U(\mathcal{D}, \lambda)$

$$x_ix_j - q_{ij}x_jx_i = \lambda_{ij}(1 - g_ig_j),\ \text{for all } 1 \le i, j \le \theta,\ i \not\sim j$$

by (7.2).

The elements in (7.4) and (7.5) are skew-primitive. Hence $U(\mathcal{D}, \lambda)$ is a Hopf algebra with

$$\Delta(x_i) = g_i \otimes x_i + x_i \otimes 1,\ 1 \le i \le \theta.$$

Then $U(\mathcal{D}, 0)$ is the biproduct $R(X)\#k[\Gamma]$, where

$$R(X) = k\langle x_1, \dots, x_\theta\rangle/(\mathrm{ad}_c(x_i)^{1-a_{ij}} \mid 1 \le i, j \le \theta, i \ne j).$$

Suppose that for any $J \in \mathcal{X}$ there is an element $q_J \in k$ such that $q_i = q_J$ for all $i \in J$. Then it is known that $R(X)$ is the Nichols algebra of X ([3, Section 4]).

Suppose that $\lambda \ne 0$. In [3] a glueing process was used to build $U(\mathcal{D}, \lambda)$ inductively by adding one connected component at a time and thus to obtain a basis of the algebra. The methods of the present paper to parameterize the finite-dimensional irreducible modules do not apply to this description of $U(\mathcal{D}, \lambda)$.

Since by Lemma 7.2 the linking graph of $(\mathcal{D}, \lambda)$ is bipartite, we can give another description of $U(\mathcal{D}, \lambda)$ by one glueing only and show that the algebra is a quotient of $(U \otimes A)^\sigma$ where some central group-likes are identified with 1 and U and A are biproducts. As a consequence Theorem 4.1 applies.

In the inductive construction of [3] each stage yields a quotient of the form just described; however, U is not a biproduct and the finite-dimensional irreducible U-modules are not necessarily one-dimensional.

We now give our construction of $U(\mathcal{D}, \lambda)$. By Lemma 7.1 there are non-empty disjoint subsets $\mathcal{X}_1, \mathcal{X}_2 \subseteq \mathcal{X}$ with $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2$, an integer $n \geq 1$ and an injective map $t : \{1, \dots, 2n\} \to \{1, \dots, \theta\}$, such that

$$t(i) \in I_1 := \bigcup_{J \in \mathcal{X}_1} J,\ t(n+i) \in I_2 := \bigcup_{J \in \mathcal{X}_2} J,$$

for all $1 \leq i \leq n$, and such that $(t(i), t(n+i))$ and $(t(n+i), t(i))$, where $1 \leq i \leq n$ are all the linked pairs of elements in $\{1, \dots, \theta\}$.

Let Λ be the free abelian group with basis $z_i, i \in I_1$. For all $j \in I_1$ let $\eta_j \in \widehat{\Lambda}$ be defined by $\eta_j(z_i) = \chi_j(g_i)$ for all $i \in I_1$. Let

$$\mathcal{D}_1 = \mathcal{D}(\Lambda, (z_i)_{i \in I_1}, (\eta_i)_{i \in I_1}, (a_{ij})_{(i,j) \in I_1 \times I_1}),$$

and let $\mathcal{D}_2 = \mathcal{D}(\Gamma, (g_i)_{i \in I_2}, (\chi_i)_{i \in I_2}, (a_{ij})_{(i,j) \in I_2 \times I_2})$ be the restriction of $\mathcal{D}$ to I_2.

Let $W \in {}^{\Lambda}_{\Lambda}\mathcal{YD}$ with basis $u_i \in W^{\eta_i}_{z_i}, i \in I_1$, and $V \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$ with basis $a_j \in V^{\chi_j}_{g_j}, j \in I_2$. We define

$$U = U(\mathcal{D}_1, 0) \cong R(W)\#k[\Lambda] \text{ and } A = U(\mathcal{D}_2, 0) \cong R(V)\#k[\Gamma].$$

Theorem 7.5. *Assume the situation above.*

For all $i \in I_1$ there is a unique algebra map

$$\gamma_i : A \to k \text{ with } \gamma_i(a_j) = 0, \gamma_i(g) = \chi_i(g) \tag{7.6}$$

for all $j \in I_2$ and $g \in \Gamma$, and a unique (ε, γ_i)-derivation

$$\delta_i : A \to k \text{ with } \delta_i(a_j) = \lambda_{ji}, \delta_i(g) = 0 \tag{7.7}$$

for all $j \in I_2$ and $g \in \Gamma$. Moreover there is an algebra map $\Phi : U \to A^o$ determined by

$$\Phi(z_i) = \gamma_i,\ \Phi(u_i) = \delta_i \tag{7.8}$$

for all $i \in I_1$, and the bilinear form $\tau : U \otimes A \to k$ with $\tau_\ell = \Phi$ satisfies (A.1)-(A.4).

Let σ be the 2-cocycle corresponding to τ (see Section 1). The group-like elements $z_i \otimes g_i^{-1}, i \in I_1$, are central in $(U \otimes A)^\sigma$, and there is an isomorphism of Hopf algebras

$$U(\mathcal{D}, \lambda) \cong (U \otimes A)^\sigma / (z_i \otimes g_i^{-1} - 1 \otimes 1 \mid i \in I_1)$$

mapping x_i with $i \in I_1, x_j$ with $j \in I_2$, respectively $g \in \Gamma$ onto the residue classes of $u_i \otimes 1, 1 \otimes a_j$, respectively $1 \otimes g$.

Proof: It can be checked directly as in [1, Lemma 5.19] that the maps γ_i, δ_i and Φ are well-defined by working with the defining relations.

In the case when the values of q_i are constant in each connected component, $R(W)$ and $R(V)$ are the Nichols algebras. To see that Φ is well-defined without checking the relations we alternatively can then apply Theorem 6.1. We define $\varphi : \Lambda \to \widehat{\Gamma}, (\lambda_i)_{i \in I_1}$, and $s : I_1 \to I_2$ for all $i \in I_1$ by

$$\varphi(z_i) = \chi_i,$$
$$\lambda_i = \begin{cases} \lambda_{t(n+l),t(l)} & \text{if } i = t(l) \text{ for some } 1 \leq l \leq n, \\ 0 & \text{otherwise,} \end{cases}$$
$$s(i) = \begin{cases} t(n+l) & \text{if } i = t(l) \text{ for some } 1 \leq l \leq n, \\ j_0 & \text{otherwise,} \end{cases}$$

where j_0 is any element in I_2.

Hence $\lambda_i \delta_{s(i),j} = \lambda_{ji}$ for all $i \in I_1, j \in I_2$. We have to check the conditions in (6.6). Let $i \in I_1$ with $\lambda_i \neq 0$, that is $i = t(l)$ for some $1 \leq l \leq n$. Then $\varphi(z_{t(l)}) = \chi_{t(l)} = \chi^{-1}_{t(n+l)}$ since $t(l)$ and $t(n+l)$ are linked. This proves the first part of (6.6) since $i = t(l)$ and $s(i) = t(n+l)$.

The second part of (6.6) says that $\eta_i(z_k) = \chi_k(g_{s(i)})$ for all $k \in I_1$. Since k and $s(i)$ are in different connected components, it follows from the Cartan condition that $\chi_k(g_{s(i)}) = \chi_{s(i)}(g_k)^{-1}$. Since i and $s(i)$ are linked, $\chi_i = \chi^{-1}_{s(i)}$. Hence $\chi_k(g_{s(i)}) = \chi_i(g_k) = \eta_i(g_k)$ by definition of η_i.

The remaining claims of the theorem follow by direct calculations as in [1, Theorem 5.17, end of the proof]. □

For any $\chi \in \widehat{\Gamma}$ define $\rho \in \widehat{\Lambda}$ by $\rho(z_i) = \chi(g_i)$ for all $i \in I_1$. Then the left $(U \otimes A)^\sigma$-module $L(\rho, \chi)$ defined in part b) of Theorem 6.1 is annihilated by all $z_i \otimes g_i^{-1} - 1 \otimes 1, i \in I_1$ by Lemma 4.3. Thus using Theorem 7.5 we can define $L(\chi) = L(\rho, \chi)$ as a left module over

$$U(\mathcal{D}, \lambda) \cong (U \otimes A)^\sigma / (z_i \otimes g_i^{-1} - 1 \otimes 1 \mid i \in I_1)$$

.

Theorem 7.6. *The function*

$$\{\chi \in \widehat{\Gamma} \mid \dim L(\chi) \textit{ is finite } \} \to \mathrm{Irr}(U(\mathcal{D}, \lambda)), \ \chi \mapsto [L(\chi)],$$

is bijective.

Proof: By Theorem 5.6 the finite-dimensional simple U- and A-modules are one-dimensional. Hence

$$\{(\rho, \chi) \in \widehat{\Lambda} \times \widehat{\Gamma} \mid \dim L(\rho, \chi) \text{ is finite } \} \to \mathrm{Irr}((U \otimes A)^\sigma),$$

given by $(\rho, \chi) \mapsto [L(\rho, \chi)]$, is bijective by Theorem 4.1 and Theorem 5.6. The claim now follows from Lemma 4.3. □

By Theorem 6.1 the computation of the finite-dimensional simple $U(\mathcal{D}, \lambda)$-modules can be reduced to the non-degenerate case. To describe this reduction we define the non-degenerate case abstractly.

Definition 7.7. *Let Γ be an abelian group, $n \geq 1, g_i, h_i \in \Gamma, \chi_i \in \widehat{\Gamma}$ for all $1 \leq i \leq n$, and $(a_{ij})_{1\leq i,j\leq n}$ a Cartan matrix of finite type. We say that $\mathcal{D}_{red} = \mathcal{D}_{red}(\Gamma, (g_i)_{1\leq i\leq n}, (h_i)_{1\leq i\leq n}, (\chi_i)_{1\leq i\leq n}, (a_{ij})_{1\leq i,j\leq n})$ is a* reduced datum of finite Cartan type *if for all $1 \leq i, j \leq n$*

$$\chi_i(g_j)\chi_j(g_i) = \chi_i(g_i)^{a_{ij}}, \tag{7.9}$$

$$\chi_i(g_j) = \chi_j(h_i), \tag{7.10}$$

$$g_i h_i \neq 1 \tag{7.11}$$

Definition 7.8. *Let $\mathcal{D}_{red}$ be a reduced datum of finite Cartan type, and $X \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$ with basis $x_1, \dots, x_n, y_1, \dots, y_n$ and $x_i \in X^{\chi_i}_{g_i}, y_i \in X^{\chi_i^{-1}}_{h_i}$ for all $1 \leq i \leq n$.*

Then we define $U(\mathcal{D}_{red})$ as the quotient Hopf algebra of the smash product $k\langle x_1, \dots, x_n, y_1, \dots, y_n\rangle \# k[\Gamma]$ modulo the ideal generated by

$$\mathrm{ad}_c(x_i)^{1-a_{ij}}(x_j) \text{ for all } 1 \leq i, j \leq n, i \neq j, \tag{7.12}$$

$$\mathrm{ad}_c(y_i)^{1-a_{ij}}(y_j) \text{ for all } 1 \leq i, j \leq n, i \neq j, \tag{7.13}$$

$$x_i y_j - \chi_j^{-1}(g_i) y_j x_i - \delta_{ij}(1 - g_i h_j) \text{ for all } 1 \leq i, j \leq n. \tag{7.14}$$

The relations of $U(\mathcal{D}_{red})$ are very similar to the relations of $U_q(\mathfrak{g}), \mathfrak{g}$ a semisimple Lie algebra. Note that if we set $e_i = x_i, f_i = y_i h_i^{-1}$, then in the quotient algebra (7.14) can be rewritten as

$$e_i f_j - f_j e_i = \delta_{ij}(h_i^{-1} - g_i) \text{ for all } 1 \leq i, j \leq n, \tag{7.15}$$

(7.12) and (7.12) are the Serre relations for the e_i and the f_i, and the action of Γ is given for all $g \in \Gamma$ and all $1 \leq i \leq n$ by

$$g e_i g^{-1} = \chi_i(g) e_i, \tag{7.16}$$

$$g f_i g^{-1} = \chi_i^{-1}(g) f_i. \tag{7.17}$$

The definition of $U(\mathcal{D}_{red})$ is a special case of Definition 7.4. Indeed define $g_{n+i} = h_i, \chi_{n+i} = \chi_i^{-1}$ for all $1 \leq i \leq n$, and let $(a_{ij})_{1\leq i,j\leq 2n}$ be the diagonal block matrix consisting of two identical blocks $(a_{ij})_{1\leq i,j\leq n}$ on the diagonal. Then

$$\mathcal{D}' = \mathcal{D}(\Gamma, (g_i)_{1\leq i\leq 2n}, (\chi_i)_{1\leq i\leq 2n}, (a_{ij})_{1\leq i,j\leq 2n})$$

is a datum of finite Cartan type, and $U(\mathcal{D}_{red}) = U(\mathcal{D}', \lambda')$ where for all $1 \leq i < j \leq 2n, i \not\sim j$, $\lambda'_{ij} = \begin{cases} 1 & \text{if } 1 \leq i \leq n, j = n+i, \\ 0 & \text{otherwise} \end{cases}$. The remaining values of λ' are determined by (7.2). Thus the Dynkin diagram of $\mathcal{D}'$ consists of two copies of the Dynkin diagram of $(a_{ij})_{1 \leq i,j \leq n}$, and each vertex is linked with its copy.

In very specific terms Theorem 7.5 for $\mathcal{D}_{red}$ says the following. Let Λ the free abelian group with basis $z_i, 1 \leq i \leq n$, and define characters $\eta_j, 1 \leq j \leq n$, by

$$\eta_j(z_i) = \chi_j(g_i) = q_{ij}.$$

Let

$$\begin{aligned}
W &\in {}^{\Lambda}_{\Lambda}\mathcal{YD} \text{ with basis } u_i \in W^{\eta_i}_{z_i}, 1 \leq i \leq n, \\
V &\in {}^{\Gamma}_{\Gamma}\mathcal{YD} \text{ with basis } a_i \in V^{\chi_i^{-1}}_{h_i}, 1 \leq i \leq n, \\
\mathcal{D}_1 &= \mathcal{D}(\Lambda, (z_i)_{1 \leq i \leq n}, (\eta_i)_{1 \leq i \leq n}, (a_{ij})_{1 \leq i,j \leq n}), \\
\mathcal{D}_2 &= \mathcal{D}(\Gamma, (h_i)_{1 \leq i \leq n}, (\chi_i^{-1})_{1 \leq i \leq n}, (a_{ij})_{1 \leq i,j \leq n}), \\
U &= U(\mathcal{D}_1, 0) \cong R(W)\#k[\Lambda], \\
A &= U(\mathcal{D}_2, 0) \cong R(V)\#k[\Gamma].
\end{aligned}$$

Then by Theorem 6.2

$$U(\mathcal{D}_{red}) \cong (U \otimes A)^{\sigma}/(z_i \otimes g_i^{-1} - 1 \otimes 1 \mid 1 \leq i \leq n)$$

where $x_i, y_i, 1 \leq i \leq n$, resp. $g \in \Gamma$ are mapped onto the residue classes of $u_i \otimes 1, 1 \otimes a_i$ resp. $1 \otimes g$. The Hopf algebra map

$$\Phi : U \to A^{o\,cop} \text{ is defined by } \Phi(z_i) = \gamma_i, \Phi(u_i) = \delta_i, 1 \leq i \leq n,$$

where $\gamma_i : A \to k$ is the algebra map with $\gamma_i(g) = \chi_i(g)$ for all $g \in \Gamma$, and $\delta_i : A \to k$ is the (ε, χ_i)-derivation with $\delta_i(a_j) = -\delta_{ij} q_{ii}$ for all $1 \leq i, j \leq n$ (since $\lambda'_{n+j,i} = -\delta_{ij}\chi_i(g_{n+j})$, and $\chi_i(g_{n+j}) = \chi_i(h_j) = \chi_j(g_i) = q_{ij}$).

In the notation of Section 6.2 Φ is given by the group homomorphism $\varphi : \Lambda \to \widehat{\Gamma}$ with $\varphi(z_i) = \chi_i$, and the non-zero scalars $\lambda_i = -q_{ii}, 1 \leq i \leq n$. Thus the χ_i and g_i in Section 6.2 are the χ_i^{-1} and h_i of $\mathcal{D}_{red}$.

Theorem 7.9. *In the situation of Theorem 7.5 let*

$$\mathcal{D}_{red} = \mathcal{D}_{red}(\Gamma, (g_{t(i)})_{1 \leq i \leq n}, (g_{t(n+i)})_{1 \leq i \leq n}, (\chi_{t(i)})_{1 \leq i \leq n}, (a_{t(i)t(j)})_{1 \leq i,j \leq n})$$

and let $F : U(\mathcal{D}, \lambda) \to U(\mathcal{D}_{red})$ *be the Hopf algebra projection determined by*

$$F \mid \Gamma = \mathrm{id},$$
$$F(x_{t(i)}) = x_i \lambda_{t(i),t(n+i)} \text{ and } F(x_{t(n+i)}) = y_i \text{ for all } 1 \leq i \leq n,$$
$$F(x_l) = 0 \text{ for all } l \in \{1, \dots, \theta\} \setminus t(\{1, \dots, 2n\}).$$

Then pullback along $F : U(\mathcal{D}, \lambda) \to U(\mathcal{D}_{red})$ *defines a bijection*

$$F^* : \mathrm{Irr}(U(\mathcal{D}_{red})) \to \mathrm{Irr}(U(\mathcal{D}, \lambda)).$$

Proof: By Theorem 7.5

$$U(\mathcal{D}, \lambda) \cong (U \otimes A)^{\sigma} / (z_i \otimes g_i^{-1} - 1 \otimes 1 \mid i \in I_1),$$

and similarly

$$U(\mathcal{D}', \lambda') \cong (U' \otimes A')^{\sigma'} / (z_i \otimes g_i^{-1} - 1 \otimes 1 \mid i \in I_1),$$

where we use the notation of Theorem 6.1 for U' and A'. Hence the claim follows from Theorem 7.6, Theorem 6.1 and the preceding discussion. □

Example 7.10. *Let* $H = (U \otimes A)^{\sigma}$ *be the Hopf algebra defined in the beginning of Section 6.2 with Cartan matrix* $a_{ij} = 2\delta_{ij}, 1 \leq i, j \leq n$, *that is, each connected component of the Dynkin diagram is of type* A_1. *In Corollary 6.8 we explicitly described the finite-dimensional simple* H*-modules, and by Theorems 4.1 and 5.6 we constructed a bijection between* $\mathrm{Irr}(H)$ *and the set of all character pairs* $(\rho, \chi) \in \widehat{\Lambda} \times \widehat{\Gamma}$ *such that for all* $1 \leq i \leq n, q_i^{r_i} \rho(z_i) \chi(g_i) = 1$, *where* $r_i \geq 0$, *and* $q_i = \eta_i(z_i)$.

Example 7.11. *We consider a reduced datum of finite Cartan type* $\mathcal{D}_{red} = \mathcal{D}_{red}(\Gamma, (g_i)_{1 \leq i \leq n}, (h_i)_{1 \leq i \leq n}, (\chi_i)_{1 \leq i \leq n}, (a_{ij})_{1 \leq i,j \leq n})$ *and assume that each connected component of the Dynkin diagram of* (a_{ij}) *is of type* A_1. *For all* $1 \leq i \leq n$, *let* $q_i = \chi_i(g_i)$. *By Example 7.10 and Theorem 7.6 there is a bijection*

$$\{\chi \in \widehat{\Gamma} \mid \text{ for all } 1 \leq i \leq n, q_i^{r_i} \chi(g_i)^2 = 1, \text{ with } r_i \geq 0\} \cong \mathrm{Irr}(U(\mathcal{D}_{red}))$$

given by $\chi \mapsto L(\chi)$.

Example 7.12. *Suppose that* $\mathcal{D}$ *is a datum of finite Cartan type such that the Dynkin diagram of the Cartan matrix of* $\mathcal{D}$ *is the disjoint union of an even number of components of type* $A_{n_l}, n_l \geq 2$. *Suppose that* λ *is a family of linking parameters such that the connected components are linked in a circle, the end of one* A_{n_l} *being linked to the beginning of the next* $A_{n_{l+1}}$. *Then the Dynkin diagram of* $\mathcal{D}_{red}$ *corresponding to* $\mathcal{D}, \lambda$ *in Theorem 7.9 is a union of components of type* A_1, *and by Example*

7.11 and Theorem 7.9 we have an explicit description of the finite-dimensional simple $U(\mathcal{D},\lambda)$-modules.

7.3. **The Finite-dimensional Case.** In this section we fix a finite abelian group Γ, a datum $\mathcal{D} = \mathcal{D}(\Gamma, (g_i)_{1\le i\le\theta}, (\chi_i)_{1\le i\le\theta}, (a_{ij})_{1\le i,j\le\theta})$ of finite Cartan type, and a family λ of linking parameters for $\mathcal{D}$.

We assume that for all $1 \le i,j \le \theta$, $\mathrm{ord}(q_{ij})$ *is odd and* $\mathrm{ord}(q_i)$ *is* > 3 *and prime to* 3 *if* i *is in a component* G_2.

For each connected component $J \in \mathcal{X}$, and positive root α of the root system Φ_J of J let x_α be the root vector in the free algebra $k\langle x_j \mid j \in J\rangle$ defined in [1, Section 4.1] generalizing the root vectors in [10].

Definition 7.13. *Let $u(\mathcal{D},\lambda)$ be the quotient Hopf algebra of the smash product $k\langle x_1,\dots,x_\theta\rangle\#k[\Gamma]$ modulo the ideal generated by* (7.4), (7.5) *and*

$$x_\alpha^{N_J} \text{ for all } \alpha \in \Phi_J^+, J \in \mathcal{X}. \tag{7.18}$$

In addition we assume that *the linking graph of $\mathcal{D},\lambda$ is bipartite.* By Lemma 7.2 this holds in particular if (a_{ij}) is simply laced.

We proceed exactly as in the previous section and use the above notations. The only difference is the definition of Λ as in [1, proof of Theorem 5.17].

Let Λ be the abelian group with generators $z_i, i \in I_1$, and relations $z_i^{n_i} = 1$ for all $i \in I_1$, where n_i is the least common multiple of $\mathrm{ord}(g_i)$ and $\mathrm{ord}(\chi_i)$. For all $j \in I_1$ let $\eta_j \in \widehat{\Lambda}$ be defined by $\eta_j(z_i) = \chi_j(g_i)$ for all $i \in I_1$. Let

$$\mathcal{D}_1 = \mathcal{D}(\Lambda, (z_i)_{i\in I_1}, (\eta_i)_{i\in I_1}, (a_{ij})_{(i,j)\in I_1\times I_1}),$$

and let $\mathcal{D}_2 = \mathcal{D}(\Gamma, (g_i)_{i\in I_2}, (\chi_i)_{i\in I_2}, (a_{ij})_{(i,j)\in I_2\times I_2})$ be the restriction of $\mathcal{D}$ to I_2.

Let $W \in {}^{\Lambda}_{\Lambda}\mathcal{YD}$ with basis $u_i \in W_{z_i}^{\eta_i}, i \in I_1$, and $V \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$ with basis $a_j \in V_{g_j}^{\chi_j}, j \in I_2$. We define

$$U = u(\mathcal{D}_1, 0) \text{ and } A = u(\mathcal{D}_2, 0).$$

Then $U = \mathfrak{B}(W)$ and $A = \mathfrak{B}(V)$ by [1, Theorem 4.5].

Using Theorem 6.1 we define for all $i \in I_1$ the algebra map $\gamma_i : A \to k$, the (ε,γ_i)-derivation $\delta_i : A \to k$ and the Hopf algebra map $\Phi : U \to A^{0\,cop}$ as in the infinite case. Let σ be the 2-cocycle corresponding to Φ. As in the proof of [1, Theorem 5.17] we obtain

Theorem 7.14. *The group-like elements $z_i \otimes g_i^{-1}, i \in I_1$, are central in $(U \otimes A)^\sigma$, and there is an isomorphism of Hopf algebras*

$$u(\mathcal{D},\lambda) \cong (U \otimes A)^\sigma/(z_i \otimes g_i^{-1} - 1 \otimes 1 \mid i \in I_1)$$

mapping x_i with $i \in I_1, x_j$ with $j \in I_2$, resp. $g \in \Gamma$ onto the residue classes of $u_i \otimes 1, 1 \otimes a_j$ resp. $1 \otimes g$.

For any $\chi \in \widehat{\Gamma}$ define $\rho \in \widehat{\Lambda}$ by $\rho(z_i) = \chi(g_i)$ for all $i \in I_1$. As above we define $L(\chi) = L(\chi, \rho)$ as a left module over $U(\mathcal{D}, \lambda)$. As in the infinite case we define $\mathcal{D}_{red}$, and $u(\mathcal{D}_{red})$ by adding the root vector relations, and the Hopf algebra projection $F : u(\mathcal{D}, \lambda) \to u(\mathcal{D}_{red})$.

Theorem 7.15. *The function*

$$\widehat{\Gamma} \to \mathrm{Irr}(u(\mathcal{D}, \lambda)),\ \chi \mapsto [L(\chi)],$$

is bijective, and pullback along F defines a bijection

$$F^* : \mathrm{Irr}(u(\mathcal{D}_{red})) \to \mathrm{Irr}(u(\mathcal{D}, \lambda)).$$

PROOF: This is shown as in the proof of Theorems 7.6 and 7.9. Instead of Theorem 5.6 we use the fact $\mathfrak{B}(W)$ and $\mathfrak{B}(V)$ are finite-dimensional $\mathbb{N}$-graded algebras and hence have nilpotent augmentation ideals. Since $U/U\mathfrak{B}(W)^+ \cong k[\Lambda]$, and $A/A\mathfrak{B}(V)^+ \cong k[\Gamma]$ the simple U- and simple A-modules are simple $k[\Lambda]$- and $k[\Gamma]$-modules, hence one-dimensional. □

In particular each x_i is contained in the kernel of F, hence x_i lies in the Jacobson radical of $u(\mathcal{D}, \lambda)$ by Theorem 7.15. We give another proof of this fact without using the bipartiteness assumption.

Theorem 7.16. *Let $\mathcal{D}, \lambda$ be as in the beginning of this section but where the linking graph of $\mathcal{D}, \lambda$ is not necessarily bipartite. Let $1 \le i \le \theta$ be a vertex which is not linked to any other vertex. Then x_i is contained in the Jacobson radical of $u(\mathcal{D}, \lambda)$.*

PROOF: Let J be the connected component of the Dynkin diagram containing i. Let u_J be the subalgebra of $u(\mathcal{D}, \lambda)$ generated by all $x_j, j \in J$, and let u' be the subalgebra of $u(\mathcal{D}, \lambda)$ generated by all $g \in \Gamma$ and $x_l, l \notin J$. Then using the PBW-basis in [4, Theorem 3.3] it follows that

$$u(\mathcal{D}, \lambda) = u' u_J.$$

Since i is not linked, x_i skew-commutes with all the generators of u'. Hence $u(\mathcal{D}, \lambda) x_i u(\mathcal{D}, \lambda) \subseteq u(\mathcal{D}, \lambda) u_J^+$. Since the augmentation ideal u_J^+ of u_J is nilpotent we see that x_i generates a nilpotent ideal in $u(\mathcal{D}, \lambda)$. □

References

[1] Nicolás Andruskiewitsch and Hans-Jürgen Schneider, Finite quantum groups over abelian groups of prime exponent *Ann. Sci. École Norm. Sup. (4)* **35** (2002), 1–26.

[2] Nicolás Andruskiewitsch and Hans-Jürgen Schneider, Pointed Hopf algebras. New directions in Hopf algebras, 1–68, *Math. Sci. Res. Inst. Publ.* **43**, Cambridge Univ. Press, Cambridge, 2002.

[3] Nicolás Andruskiewitsch and Hans-Jürgen Schneider, A characterization of quantum groups, *J. reine angew. Math.* **577** (2004), 81–104.

[4] Nicolás Andruskiewitsch and Hans-Jürgen Schneider, Small quantum groups and the classification of pointed Hopf algebras, preprint 2005.

[5] R. Diestel, Graphentheorie, Springer, 1996.

[6] Yukio Doi and Mitsuhiro Takeuchi, Multiplication alteration by two-cocycles—the quantum version, *Comm. Algebra* **22** (1994), 5715–5732.

[7] James E. Humphreys, Introduction to Lie algebras and representation theory. Second printing, revised. *Graduate Texts in Mathematics*, **9**. Springer-Verlag, New York-Berlin, 1978. xii+171

[8] Larry A. Lambe and David E. Radford, Algebraic aspects of the quantum Yang–Baxter equation, *J. of Alg.* **154** (1993), 228–288.

[9] Larry A. Lambe and David E. Radford, Introduction to the quantum Yang-Baxter equation and quantum groups: an algebraic approach. Mathematics and its Applications **423**, Kluwer Academic Publishers, Dordrecht 1997.

[10] G. Lusztig, Introduction to quantum groups, Birkhäuser, 1993.

[11] S. Majid, Quasitriangular Hopf algebras and Yang–Baxter equations, *Internat. J. of Modern Phys. A.* **5** (1990), 1–91.

[12] S. Montgomery, "Hopf Algebras and their actions on rings," **82**, *Regional Conference Series in Mathematics*, AMS, Providence, RI, 1993.

[13] David E. Radford, On oriented quantum algebras derived form representations of the quantum double of a finite-dimensional Hopf algebra, *J. Algebra* **270** (2003), 670–695.

[14] D. E. Radford, The structure of Hopf algebras with a projection, *J. Algebra* **92** (1985), 322–347.

[15] David E. Radford and Hans-Jürgen Schneider, Biproducts and two-cocycle twists of Hopf algebras, preprint (2005)

[16] David E. Radford, Finite-dimensional simple-pointed Hopf algebras, *J. Algebra* **211** (1999), 686–710.

[17] Moss E. Sweedler, Hopf algebras. Mathematics Lecture Note Series W. A. Benjamin, Inc., New York, 1969.

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