MATH 121 Sample Exam Solution

Radford 12/06/02

1. Identities or not? We provide answers and suggest a related identity in the case of non-identities.

(a) **F**. With $x = 0$ the left hand side of the equation is -1 and the right hand side is 1.

Comment: $\cos^2(x) - \sin^2(x) = \cos(2x)$ is an identity.

(b) **F**. With $x = y = 0$ the left hand side of the equation is 1 and the right hand side is $1 + 1 = 2$.

Comment: $e^{x+y} = e^x e^y$ is an identity.

(c) **F**. With $x = y = 1$ the left hand side of the equation is $\log(2)$, which is not 0, and the right hand side is $0 + 0 = 0$.

Comment: $\log(xy) = \log(x) + \log(y)$ is an identity.

(d) **T**. No explanation necessary. However, recall that the period of $\tan(x)$ is π.

(e) **T**. No explanation necessary. However, recall that $0 \leq \cos^{-1}(x) \leq \pi$ and thus $0 \leq \sin(\cos^{-1}(x)) = \sqrt{1 - (\cos(\cos^{-1}(x)))^2} = \sqrt{1 - x^2}$.

(f) **F**. With $x = 0, y = 1$ the left hand side of the equation is $(-\imath)(\imath) = -\imath^2 = -(-1) = 1$ and the right hand side is -1.

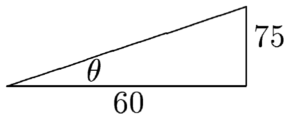
Comment: $(x + \imath y)(x - \imath y) = x^2 - (\imath y)^2 = x^2 - \imath^2 y^2 = x^2 - (-1)y^2 = x^2 + y^2$; thus $(x + \imath y)(x - \imath y) = x^2 + y^2$ is an identity. The latter is simply $(x + \imath y)\overline{(x + \imath y)} = x^2 + y^2 = |x + \imath y|^2$.

2. **E)**. Solve $y = f(x) = \dfrac{2x + 1}{3x - 4}$ in terms of x:

$$\begin{aligned} y &= \frac{2x+1}{3x-4}, \\ y(3x-4) &= 2x + 1, \\ (3y - 2)x &= 4y + 1, \\ x &= \frac{4y+1}{3x-2}. \end{aligned}$$

Therefore the inverse of $f(x)$ is given by $f^{-1}(x) = \dfrac{4x + 1}{3x - 4}$.

3. **E)**. We first draw a picture (not necessarily to scale).



Thus $\tan\theta = \dfrac{75}{60} = \dfrac{5}{4}$ which means, as $-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$, that $\theta = \tan^{-1}(\dfrac{5}{4}) \approx 51.34^o$.

4. D). The inequality $\dfrac{x+1}{x-3} \le 0$ holds if and only if

$$
\begin{array}{c}
(x+1 \ge 0 \;\text{ and }\; x-3>0) \quad \textit{or} \quad (x+1 \le 0 \;\text{ and }\; x-3<0) \\
\text{if and only if} \\
(x \ge -1 \;\text{ and }\; x>3) \quad \textit{or} \quad (x \le -1 \;\text{ and }\; x<3) \\
\text{if and only if} \\
(x>3) \quad \textit{or} \quad (x \le -1).
\end{array}
$$

5. B). By definition of addition and scalar multiplication of vectors

$$
\begin{aligned}
2\mathbf{u} - \mathbf{v} &= 2<1,-2> - <2,-3> \\
&= <2,-4> - <2,-3> \\
&= <2-2,-4-(-3)> \\
&= <0,-1>.
\end{aligned}
$$

6. D). $3P = P(16) = Pe^{r16}$, so $3 = e^{16r}$. Therefore $\ln 3 = 16r$ from whence we deduce that $r = \dfrac{\ln 3}{16} \approx .06866$. Therefore $100r\% \approx 6.866\%$.

Comments: The other type of investment scheme we studied is invest at a fixed annual rate compounded n-times per year; see section 5.2.A.

7. Since $f(x) = x^3 - 3x^2 + 5x - 3$ has integer coefficients, the rational roots of $f(x)$ are among $\pm 1, \pm 3$. Since $f(1) = 0$ it follows that $x = 1$ is a root of $f(x)$. Therefore $x - 1$ divides $f(x)$. Dividing $x - 1$ into $f(x)$ gives the quotient of $x^2 - 2x + 3$ and remainder 0. Thus

$$x^3 - 3x^2 + 5x - 3 = (x-1)(x^2 - 2x + 3).$$

By the quadratic formula the roots of $x^2 - 2x + 3$ are $\dfrac{-(-2) \pm \sqrt{(-2)^2 - (4)(1)(3)}}{2} = \dfrac{2 \pm 2\sqrt{2}\imath}{2}$. Thus the roots of $f(x)$ are

$$1, \quad 1 + \sqrt{2}\imath, \quad 1 - \sqrt{2}\imath.$$

Comment: As a product of linear factors (degree 1 factors), note that

$$x^3 - 3x^2 + 5x - 3 = (x-1)(x - (1 + \sqrt{2}\imath))(x - (1 - \sqrt{2}\imath)).$$

Comment: As a product of linear and quadratic polynomials with real coefficients, where the quadratic factors have no reals roots (see page 292), note that

$$x^3 - 3x^2 + 5x - 3 = (x-1)(x^2 - 2x + 3).$$

8. Let $z = 3x$. Then $2\sin(3x) = 1$ is the same as $\sin(z) = \frac{1}{2}$. All solutions $0 \le z < 2\pi$ to the latter are $z = \frac{\pi}{6}, \frac{5\pi}{6}$. Therefore all solutions to the latter

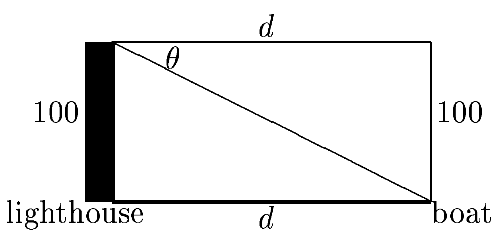
$$z = \frac{\pi}{6} + 2\pi n, \qquad \frac{5\pi}{6} + 2\pi n,$$

where n is any integer. since $z = 3x$ all solutions to the original equation are

$$x = \frac{\pi}{18} + \frac{2\pi}{3}n, \qquad \frac{5\pi}{18} + \frac{2\pi}{3}n,$$

where n is any integer.

9. Let θ be the angle of depression and d be the distance from the base of the lighthouse to the boat.



Then $\tan\theta = \frac{100}{d}$ and thus $d = \frac{100}{\tan\theta}$. Assume that the angle of depression changes from 30^o to 40^o (a second value is missing in the problem description). The change in distance is therefore

$$\frac{100}{\tan 30^o} - \frac{100}{\tan 40^o} \approx 54.030 \ \ feet.$$

Comments: There are other very important types of word problems involving triangles which you should master; see the assigned word problems at the end of sections 7.1–7.3. For the meaning of "angle of depression" and "angle of inclination" see pages 473–474.

10. (a) The modulus of $z = 1 - \imath = 1 + (-1)\imath$ is $|z| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$. Therefore $1 - \imath = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \left(-\frac{1}{\sqrt{2}}\right)\imath\right) = \sqrt{2}\left(\cos(-\frac{\pi}{4}) + \imath\sin(-\frac{\pi}{4})\right)$, so

$$z = \sqrt{2}\left(\cos(-\frac{\pi}{4}) + \imath\sin(-\frac{\pi}{4})\right)$$

is a polar form of $z = 1 - \imath$.

Comment: If $0 \le \theta < 2\pi$ were stipulated, then we would have to take $\theta = \frac{7\pi}{4}$.

(b) Using the polar form of $1 - \imath$ of part (a) we calculate

$$\begin{aligned}
(1+\imath)^{10} &= \left(\sqrt{2}\left(\cos(-\frac{\pi}{4}) + \imath\sin(-\frac{\pi}{4})\right)\right)^{10} \\
&= (2^{1/2})^{10}\left(\cos(-\frac{\pi}{4}) + \imath\sin(-\frac{\pi}{4})\right)^{10} \\
&= 2^5\left(\cos(-\frac{10\pi}{4}) + \imath\sin(-\frac{10\pi}{4})\right) \\
&= 32\left(\cos(-\frac{2\pi}{4}) + \imath\sin(-\frac{2\pi}{4})\right) \\
&= 32\left(\cos(-\frac{\pi}{2}) + \imath\sin(-\frac{\pi}{2})\right) \\
&= 32(0 + \imath(-1)) \\
&= -32\imath.
\end{aligned}$$

Thus $(1+\imath)^{10} = -32\imath = 0 + (-32)\imath$ is the required answer.

Comments: Note that $32\left(\cos(-\frac{\pi}{2}) + \imath\sin(-\frac{\pi}{2})\right)$ is the answer in polar form. The choice of $\theta = -\frac{\pi}{4}$, instead of $\theta = \frac{7\pi}{4}$, makes the calculation of part (b) easier.

11. $v_0 = -56$ *feet/second* and $h_0 = 780$ *feet*.

Comment: Setting $v_0 = 56$ would imply that the object is traveling *upward* initially.

$h(t) = -16t^2 - 56t + 780$. When the object is on the ground, $h(t) = 0$ and $t \ge 0$. Thus, by the quadratic formula,

$$t = \frac{-(-56) \pm \sqrt{(-56)^2 - 4(-16)(780)}}{2(-16)} \approx \frac{56 \pm \sqrt{53056}}{-32}.$$

Since $t \ge 0$ necessarily $t \approx 5.448$ *seconds*.

Comment: In this problem the object is thrown downward. For the "thrown upward" case, see page 213, exercises 41 and 42.

12. Perhaps the best way of relating the graphs to the listed functions is to extend the graphs in fairly obvious ways. We provide answers to **1**–**5** with detailed explanations.

1 (O). The graph is not that of a periodic function or a polynomial. Therefore all rows are ruled out except for the last. Since $y = e^{bx} > 0$ always, (M) is ruled out. This leaves (N) or (O). Since the graph of (N) is the graph of $y = \ln x$ shifted to the right, (N) is ruled out.

2 (G). Since the graph is not that of a periodic function, an exponential or logarithm function, all rows but the third and fourth are ruled out. Since the graph crosses the x-axis in four different places, the fourth is ruled out and we are down to (G) or (I). For large x the value of y in (I) is negative.

3 (C). The only functions whose graphs have vertical asymptotes lie in the first row. The graph has a vertical asymptote at $x = c$; moreover the function is not defined at $x = c$. The function (B) is defined at $x = c$. Thus we are down to (A) and (C). Note that the function of (C) is not defined at $x = c$ for all $c > 0$ whereas this is not the case for (A). (That $c > 0$ seems to be implicit in the diagram.) Actually the function of (A) works when $c = \frac{1}{\sqrt{2}}$.

4 (L). Since the graph is not that of a periodic function, an exponential or logarithm function, all rows but the third and fourth are ruled out. The function of (G) is *even*; thus (G) is ruled out. For large values of x the function of (I) is negative; thus (I) is ruled out. Thus we are left with (H) from the third row. With the stipulation that $a, b > 0$ note that (H) is ruled out. (This case can be ruled out, but I am not sure what the exam writers had in mind for justification.)

Consider the fourth row. The graph of the function of (J) is the graph of $y = x^2$ shifted to the right by $a > 0$ units and then shifted up by a units. Therefore (J) is ruled out. The graph of the function of (K) is the graph of $y = x^2$ shifted to the left by $a > 0$ units and then shifted down by a units. Thus (K) is ruled out. This leaves (L) from the fourth row.

5 (D) The ranges of the functions in the rows other that second, are *unbounded*. Therefore the answer lies in the second row. Note that the period of (K) is 4π; thus (K) is ruled out. With the stipulation that $a > 0$, then answer must be (D); if $a < 0$ then the answer must be (F). We will assume that $a > 0$ was intended.