Math 215 Written Homework 5 Solution 07/21/08

1. (**20 points total**) Let $n \geq 1$ and $A_1, \cdots A_n$ be finite sets. We are to show that the assertion $|A_1 \times \cdots \times A_n| = |A_1| \cdots |A_n|$ is true by induction.

When $n = 1$, $A_1 \times \cdots \times A_n = A_1$. Thus $|A_1 \times \cdots \times A_n| = |A_1|$ in this case. Therefore the assertion is true for $n = 1$. (**5 points**)

Suppose that $n \geq 1$ and the assertion is true. Let $A_1, \cdots, A_{n+1}$ be finite sets. Then $A_1 \times \cdots \times A_n$ is a finite set and $|A_1 \times \cdots \times A_n| = |A_1| \cdots |A_n|$. We may assume that the assertion is true for two sets. Thus

$$\begin{aligned} |A_1 \times \cdots \times A_{n+1}| &= |(A_1 \times \cdots \times A_n) \times A_{n+1}| \\ &= |A_1 \times \cdots \times A_n||A_{n+1}| \\ &= (|A_1| \cdots |A_n|)|A_{n+1}| \\ &= |A_1| \cdots |A_{n+1}|. \end{aligned}$$

(**3 points per line**) Therefore the assertion holds for all $n \geq 1$ by induction. (**3 points**)

2. (**20 points total**) Statements (a) and (b) are negations of each other (**2 points**) as are (c) and (d) (**2 points**). We need to examine the relationship between statements (a) and (c), (a) and (d), (b) and (c), and (b) and (d).

(a) and (c). (a) does not imply (c). Let $A = B = \mathbf{R}$ and P(a, b): $1 \neq 1$. (**2 points**) (c) does not imply (a). Let $A = B = \mathbf{R}$ and P(a, b): $1 = 1$. (**2 points**)

(a) and (d). (d) implies (a). (**2 points**) (a) does not imply (d). Let $A = B = \mathbf{R}$ and P(a, b): $b \neq 1$. (**2 points**)

(b) and (c). (b) implies (c). (**2 points**) (c) does not imply (b). Let $A = B = \mathbf{R}$ and P(a, b): $b \neq 1$. (**2 points**)

(b) and (d). (b) does not imply (d) since a statement can not be true and false at the same time. (**2 points**) (d) does not imply (b). Let $A = B = \mathbf{R}$ and P(a, b): $a = 1$. (**2 points**)

3. (**20 points total**) Let $\epsilon > 0$. When $x \neq 4$ note

$$|f(x) - 17| = |(3x + 5) - 17| = |3x - 12| = 3|x - 4|.$$

(**8 points**) Thus when $0 < |x-4| < \epsilon/3$ we have

$$|f(x) - 17| = 3|x-4| < 3(\epsilon/3) = \epsilon.$$

(**8 points**) Take $\delta = \epsilon/3$. (**4 points**)

4. (**20 points total**) Note the domain of f is $X = \mathbf{R}$ by convention.

(a) $\exists \epsilon > 0$, (**1 point**) $\forall \delta > 0$, (**1 point**) $\exists x \in \mathbf{R}$, (**1 point**) $0 < |x-a| < \delta$ and $|f(x) - b| \geq \epsilon$ (**6 points**).

(b)

5. (**20 points total**) $f : X \longrightarrow Y$ is a function.

(a) $\exists y \in Y$, (**3 points**) $\forall x \in X$, (**3 points**) $f(x) \neq y$ (**3 points**).

(b) Let $y = 0$ for example. (**3 points**) We will show that $f(x) \neq y$ for all $x \in X = \mathbf{R}$.

Let $x \in \mathbf{R}$ and suppose that $f(x) = 0$, or equivalently $x^2 - 2x + 20 = 0$. Then by the quadratic formula $x = \dfrac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(20)}}{2} = \dfrac{2 \pm \sqrt{-76}}{2}$ which is not a real number, contradiction. Therefore $f(x) \neq 0$. (**8 points for justification**)

Comment: There is a more basic argument from which $\operatorname{Im} f$ can be seen. Note that $f(x) = (x-1)^2 + 19 \geq 19$; thus if $y < 19$ then $f(x) \neq y$ for all $x \in \mathbf{R}$.