Math 215, Fall 05 Homework #6 Solution

10/18/05

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1. (**20 points total**) The statements of parts b) and c) are logically equivalent. (**12 points**) The statement of part a) implies the statement of part b), and thus the statement of part c) also (**8 points**).

2. (**20 points total**)

a) Suppose that $a, b > 0$ and $a^2 < b^2$. Then $a + b > 0$ and $0 < b^2 - a^2 = (b-a)(b+a)$. If $b-a = 0$ then $(b-a)(b+a) = 0$, a contradiction. If $b-a < 0$ then $(b-a)(b+a) < 0$, a contradiction. Therefore $b - a > 0$ or equivalently $b > a$. (**6 points**)

b) In any event $|a| \geq 0$. If $a \geq 0$ then $|a| = a$ by definition. If $a < 0$ then $a < 0 \leq |a|$. In both cases $a \leq |a|$. (**2 points**)

If $a > 0$ then $|a| = a$ which means $|a|^2 = a^2$. If $a < 0$ then $|a| = -a$ which means $|a|^2 = (-a)^2 = (-a)(-a) = -(-aa) = a^2$. (**2 points**)

c) In order to use the hint, one needs to show (or at least point out the use of) Equations (1)and (2) below.

$$a^2 = b^2 \text{ implies } a = b \text{ for all } a, b \geq 0. \tag{1}$$

To prove this, assume that $a, b \in \mathbf{R}$ are non-negative and $a^2 = b^2$. Then $0 = a^2 - b^2 = (a-b)(a+b)$ which means $a - b = 0$ or $a + b = 0$. In the first case $a = b$. In the second $a + b = 0$. Here $0 \leq a = -b \leq 0$ from which we conclude that $0 = a = -b$. Thus $a = 0 = b$ in the second case. In any event $a = b$.

As a consequence of (1) we have:

$$|ab| = |a||b| \text{ for all } a, b \in \mathbf{R}. \tag{2}$$

To see this we first note that $|ab|, |a|, |b| \geq 0$, and thus $|a||b| \geq 0$. By part b) we have $|ab|^2 = (ab)^2 = a^2b^2 = |a|^2|b|^2 = (|a||b|)^2$ which means $|ab| = |a||b|$ by (1). (**2 points for all of the above**)

We can now calculate, using part b) and (2),

$$\begin{aligned}
|a+b|^2 &= (a+b)^2 \\
&= a^2 + 2ab + b^2 \\
&\leq |a|^2 + 2|ab| + |b|^2 \\
&= |a|^2 + 2|a||b| + |b|^2 \\
&= (|a| + |b|)^2.
\end{aligned}$$

Therefore $|a+b|^2 \leq (|a|+|b|)^2$. Thus $|a+b|^2 \leq (|a|+|b|)^2$. As $|a+b|, |a|, |b| \geq 0$ and hence $|a| + |b| \geq 0$, using part a) and (1) we deduce $|a+b| \leq |a| + |b|$. (**8 points**)

3. (**20 points total**) Let $\epsilon > 0$. We wish to find $\delta > 0$ so that $0 < |x-a| < \delta$ implies $|f(x) - L| < \epsilon$, where L is given below.

a) $f(x) = c$ for all $x \in \mathbf{R}$ and $L = c$. Therefore

$$|f(x) - L| = |c - c| = 0 < \epsilon$$

for all $x \in \mathbf{R}$. Thus any $\delta > 0$ will do. (**Calculation 2 points, δ 3 points**)

b) $f(x) = cx + d$ for all $x \in \mathbf{R}$ and $L = ca + d$. If $c = 0$ then $f(x) = d$ is a constant function and

$$\lim_{x \longrightarrow a} f(x) = d = 0a + d = ca + d$$

by part a). Suppose that $c \neq 0$. Then

$$|f(x) - (ca + d)| = |(cx + d) - (ca + d)| = |c(x - a)| = |c||x - a| < \epsilon;$$

the inequality holds if $|x - a| < \dfrac{\epsilon}{|c|}$. Take $\delta = \dfrac{\epsilon}{|c|}$. (**Calculation 10 points, δ 5 points**)

4. (**20 points total**) The containment $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$ is equality in some cases and not in general.

a) We are to show that $(A\times B)\cup(A\times D) = (A\cup A)\times(B\cup D)$ which is better written as $(A\times B)\cup(A\times D) = A\times(B\cup D)$. To do this we show that each side of the equation is a subset of the other.

Suppose that $(x, y) \in (A\times B)\cup(A\times D)$. Then $(x, y) \in A\times B$ or $(x, y) \in A\times D$. In the first case $x \in A$ and $y \in B$. In the second $x \in A$ and $y \in D$. Thus $x \in A$ and $y \in B\cup D$ which is to say $(x, y) \in A\times(B\cup D)$. We have shown $(A\times B)\cup(A\times D) \subseteq A\times(B\cup D)$.

Conversely, suppose that $(x, y) \in A\times(B\cup D)$. Then $x \in A$ and $y \in B\cup D$. Therefore $(x, y) \in A\times B$ or $(x, y) \in A\times D$ which means $(x, y) \in (A\times B)\cup(A\times D)$. We have shown that $A\times(B\cup D) \subseteq A\times B)\cup(A\times D)$. Therefore $(A\times B)\cup(A\times D) = A\times(B\cup D)$. (**7 points**)

b) We are to show that $(A\times B)\cup(C\times B) = (A\cup C)\times(B\cup B)$ which is better written as $(A\times B)\cup(C\times B) = (A\cup C)\times B$. To do this we show that each side of the equation is a subset of the other.

Suppose that $(x, y) \in (A\times B)\cup(C\times B)$. Then $(x, y) \in A\times B$ or $(x, y) \in C\times B$. In the first case $x \in A$ and $y \in B$. In the second $x \in C$ and $y \in B$. Thus $x \in A\cup C$ and $y \in B$ which is to say $(x, y) \in (A\cup C)\times B$. We have shown $(A\times B)\cup(C\times B) \subseteq (A\cup C)\times B$.

Conversely, suppose that $(x, y) \in (A\cup C)\times B$. Then $x \in A\cup C$ and $y \in B$. Therefore $(x, y) \in A\times B$ or $(x, y) \in C\times B$ which means $(x, y) \in (A\cup C)\times B$. We have shown that $(A\cup C)\times B \subseteq (A\times B)\cup(C\times B)$. Therefore $(A\times B)\cup(C\times B) = (A\cup C)\times B$. (**7 points**)

c) Let $A = \{1\}$ and $B = \{2\}$ for example. Then $A\times B = \{(1, 2)\}$ and $B\times A = \{(2, 1)\}$. Therefore

$$A\cup B = \{(1, 2), (2, 1)\}.$$

On the other hand, since $A\cup B = B\cup A = \{1, 2\}$, we have

$$(A\cup B)\times(B\cup A) = \{(1, 1), (1, 2), (2, 1), (2, 2)\}.$$

Therefore $(A\times B)\cup(B\times A) \subset (A\cup B)\times(B\cup A)$. (**6 points**)