Math 215, Fall 05 Homework #7 Solution

10/31/05

1. (**20 points total**) a) "P implies Q" is false if and only if P is true and Q is false. Thus "P implies (Q implies R)" is false if and only if P is true and Q is true and R is false. On the other hand "(P and Q) implies R" is false if and only if "P and Q" is true and R is false which his the case if and only if P and Q are true and R is false. Therefore the two statements are false under the exact same conditions which means they are true under the exact same conditions. Consequently their truth tables are the same so the statements are logically equivalent. (**15 points**)

b) Since "P implies (Q implies R)" and "(P and Q) implies R" are logically equivalent by part a) and an implication "S implies T" is true if whenever S is true T is true. Since "P and Q" is true when P and Q are true, part b) follows. (**5 points**)

2. Here is another example of an induction argument which comes down to the case $n = 2$.

a) We are to show by induction the assertion: Suppose that $A_1, \ldots, A_n$ are sets, where $n \geq 2$. If $a_\imath, a'_\imath \in A_\imath$ for all $1 \leq \imath \leq n$ then $(a_1, \ldots, a_n) = (a'_1, \ldots, a'_n)$ only if (implies) $a_\imath = a'_\imath$ for all $1 \leq \imath \leq n$.

That the assertion is true for $n = 2$ is given. Suppose that $n \geq 2$ and the assertion is true for all sets $A_1, \ldots, A_n$. Let $A_1, \ldots, A_{n+1}$ be sets and suppose that $a_\imath, a'_\imath \in A_\imath$ for all $1 \leq \imath \leq n + 1$ satisfy

$$(a_1, \ldots, a_{n+1}) = (a'_1, \ldots, a'_{n+1}).$$

Since $n + 1 \geq 2$ this equation can be rewritten

$$((a_1, \ldots, a_n), a_{n+1}) = ((a'_1, \ldots, a_n), a'_{n+1})$$

which is true if and only if $(a_1, \ldots, a_n) = (a'_1, \ldots, a'_n)$ and $a_{n+1} = a'_{n+1}$ by the $n = 2$ case. By the induction hypothesis $(a_1, \ldots, a_n) = (a'_1, \ldots, a'_n)$ implies $a_\imath = a'_\imath$ for all $1 \leq \imath \leq n$. We have shown that if the assertion is true for

$n \geq 2$ then it is true for $n+1$. Thus the assertion is true for all $n \geq 2$. (**10 points**)

b) We are to show by induction the assertion: Suppose that $A_1, \ldots, A_n$ are finite sets, where $n \geq 2$. Then $|A \times \cdots \times A_n| = |A_1| \cdots |A_n|$.

That the assertion is true for $n = 2$ is given. Suppose that $n \geq 2$ and the assertion is true for all finite sets $A_1, \ldots, A_n$. Let $A_1, \ldots, A_{n+1}$ be finite sets. Since $n + 1 \geq 2$ we have

$$|A_1 \times \cdots \times A_{n+1}| = |(A_1 \times \cdots \times A_n) \times A_{n+1}| = |A_1 \times \cdots \times A_n||A_{n+1}|$$

from the $n = 2$ case. Since $|A_1 \times \cdots \times A_n| = |A_1| \cdots |A_n|$ by the induction hypothesis, we have $|A_1 \times \cdots \times A_{n+1}| = (|A_1| \cdots |A_n|)|A_{n+1}|$ by the preceding calculation. We have shown that if the assertion is true for $n \geq 2$ then it is true for $n + 1$. Therefore the assertion is true for all $n \geq 2$. (**10 points**)

3. (**20 points total**) Two functions $f, g : X \longrightarrow Y$ are equal if and only if $\forall x \in X,\ f(x) = g(x)$; thus $f \neq g$ if and only if $\exists x \in X, f(x) \neq g(x)$. The latter applies to Problem 3c).

a) (**4 points**) Let $a \in A$. Then by definition of composition

$$((h \circ g) \circ f)(a) = (h \circ g)(f(a)) = h(g(f(a)))$$

and

$$(h \circ (g \circ f))(a) = h((g \circ f)(a)) = h(g(f(a))).$$

Therefore $((h \circ g) \circ f)(a) = (h \circ (g \circ f))(a)$ for all $a \in A$ which means $(h \circ g) \circ f = h \circ (g \circ f)$.

b) Let $a \in A$. Then

$$(I_B \circ f)(a) = I_B(f(a)) = f(a)$$

which means $I_B \circ f = f$. Likewise, for $a \in A$ we compute

$$(f \circ I_A)(a) = f(I_A(a)) = f(a)$$

which means $f \circ I_A = f$. (**4 points**)

c) (**4 points**) Constant functions will work. Define f by $f(x) = a$ for all $x \in A$ and g by $g(x) = b$ for all $x \in A$. Then

$$(g \circ f)(x) = g(f(x)) = b$$

and

$$(f \circ g)(x) = f(g(x)) = a$$

for all $x \in A$. Since

$$(f \circ g)(a) = a \neq b = (g \circ f)(a)$$

we conclude that $(f \circ g)(a) \neq (g \circ f)(a)$ and thus $f \circ g \neq g \circ f$.

d) Since A is non-empty there is an element $a \in A$. Let $b \in A$. Now define $f, g : A \longrightarrow A$ by $f(x) = a$ and $g(x) = b$ for all $x \in A$. By assumption $f \circ g = g \circ f$. Thus

$$a = f(g(a)) = (f \circ g)(a) = (g \circ f)(a) = g(f(a)) = b$$

which implies $b = a$. We have shown that $A = \{a\}$. (**4 points**)

e) We use parts a) and b). Since $g \circ f = I_A$ we have $(g \circ f) \circ h = I_A \circ h$ from which $(g \circ (f \circ h) = h$ follows. Since $f \circ h = I_B$ we have $g \circ I_B = h$ and thus $g = h$. (**4 points**)

4. (**20 points total**) Our assumption is $g \circ f = I_A$. Therefore $g(f(a)) = (g \circ f)(a) = I_A(a) = a$ for all $a \in A$, or

$$g(f(a)) = a \tag{1}$$

for all $a \in A$.

a) Let $a \in A$ and set $b = f(a)$. Then $g(b) = g(f(a)) = a$ by (1). (**8 points**)

b) Suppose that $a, a' \in A$ and $f(a) = f(a')$. Then

$$a = g(f(a)) = g(f(a')) = a'$$

by (1). (**6 points**)

c) The contrapositive of the predicate "$f(a) = f(a')$ implies $a = a'$" is "not $(a = a')$ implies not $(f(a) = f(a'))$". An answer is: For all $a, a' \in A$, if a and a' are not the same then $f(a)$ and $f(a')$ are not the same. (**6 points**)