

# Math 215, Fall 05    Homework #11

## Solution

11/29/05

1. **(20 points total)** Suppose that  $p = r^2$ , where  $r$  is rational. Since  $p \neq 0$  it follows that  $r \neq 0$ . Therefore we may write  $r = a/b$ , where  $a, b \in \mathbf{Z}$  and the only common positive divisor of  $a$  and  $b$  is 1.

Since  $p = r^2 = (a/b)^2 = a^2/b^2$  we have  $pb^2 = a^2$ . Therefore  $p|a^2$  which means  $p|a$ . Write  $a = pc$ , where  $c \in \mathbf{Z}$ . Then  $pb^2 = (pc)^2 = p^2c^2$  implies  $b^2 = pc^2$ . But then  $p|b^2$  which means  $p|b$ . Since  $p|a$  and  $p|b$  we have a contradiction. Therefore  $p \neq r^2$  after all.

2. **(20 points total)** Both parts are done in the same manner.

a) Let  $x = 0.\overline{731}$ . Then  $10x = 7.\overline{31}$  and  $1000x = 731.\overline{31}$ . Therefore

$$1000x - 10x = 731.\overline{31} - 7.\overline{31} = 731 - 7 = 724.$$

Therefore  $990x = 724$  which means  $x = \frac{724}{990}$ . **(10 points)**

b) Let  $x = 12.\overline{8697}$ . Then  $10000x = 128697.\overline{8697}$ . Therefore

$$10000x - x = 128697.\overline{8697} - 12.\overline{8697} = 128697 - 12 = 128685.$$

Therefore  $9999x = 128685$  which means  $x = \frac{128685}{9999}$ . **(10 points)**

3. **(20 points total)** We wish to show that if  $A$  is denumerable then  $A^n$  is denumerable for all  $n \geq 1$ . The proof is to be based on the fact that the Cartesian product of two denumerable sets is denumerable.

Suppose that  $A$  is a denumerable set. Then  $A^1 = A$  is denumerable. Assume that  $n \geq 1$  and  $A^n = A \times \cdots \times A$  ( $n$ -factors) is denumerable. Then

$$\begin{aligned} A^{n+1} &= A \times \cdots \times A && n+1\text{-factors} \\ &= (A \times \cdots \times A) \times A && n\text{-factors in the parentheses} \\ &= A^n \times A \end{aligned}$$

is the Cartesian product of two denumerable sets, and is therefore denumerable. We shown that if  $A^n$  denumerable then  $A^{n+1}$  is denumerable. Therefore  $A^n$  is denumerable for all  $n \geq 1$ .

4. **(20 points total)** We will show that  $f$  is a surjection and an injection. (An alternative way to show that  $f$  is a bijection is to construct and inverse for  $f$ .)

We show first of all that  $f$  is a surjection. Suppose that  $m > 0$ . Then  $2m \in \mathbf{Z}^+$  and  $f(2m) = (2m)/2 = m$ . Now suppose that  $m \geq 0$ . Then  $2m + 1 \in \mathbf{Z}^+$  and  $f(2m + 1) = -[(2m + 1) - 1]/2 = -2m/2 = -m$ . As every element of  $\mathbf{Z}$  is positive or the additive inverse of a non-negative integer,  $f$  is a surjection. **(10 points)**

We next show that  $f$  is an injection. Let  $n \in \mathbf{Z}^+$ . Note that  $f(n) > 0$  if  $n$  is even and  $f(n) \leq 0$  if  $n$  is odd.

Let  $n, n' \in \mathbf{Z}^+$  and suppose that  $f(n) = f(n')$ . We will show that  $n = n'$ . There are two cases to consider.

*Case 1:*  $f(n) > 0$ . Then  $f(n') > 0$  also. Thus  $n, n'$  are even positive integers which means  $n = 2m$  and  $n' = 2m'$  for some positive integers  $m, m'$ . Since

$$m = f(2m) = f(2m') = m'$$

we conclude that  $m = m'$  and thus  $n = 2m = 2m' = n'$ .

*Case 2:*  $f(n) \leq 0$ . Then  $f(n') \leq 0$  also. Thus  $n, n'$  are odd positive integers which means that  $n = 2m + 1$  and  $n' = 2m' + 1$ , where  $m, m' \geq 0$ . Therefore

$$-m = f(2m + 1) = f(2m' + 1) = -m'$$

means  $m = m'$  and therefore  $n = 2m + 1 = 2m' + 1 = n'$ .

In both cases  $n = n'$ . Therefore  $f$  is an injection. **(10 points)**