Fibers, Surjective Functions, and Quotient Groups

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Let $f : X \longrightarrow Y$ be a function. For a subset Z of X the subset

$$f(Z) = \{f(z) \,|\, z \in Z\}$$

of Y is the *image of* Z *under* f. For a subset W of Y the subset

$$f^{-1}(W) = \{x \in X \,|\, f(x) \in W\}$$

of X is the *pre-image of* W *under* f.

1 Fibers

For $y \in Y$ the subset

$$f^{-1}(y) = \{x \in X \,|\, f(x) = y\}$$

of X is *the fiber of* f *over* y. By definition $f^{-1}(y) = f^{-1}(\{y\})$. Observe that $f^{-1}(y) = \emptyset$ if $y \notin \mathrm{Im}\, f$.

Let $y, y' \in Y$. If $x \in f^{-1}(y) \cap f^{-1}(y')$ then $y = f(x) = y'$ which means $f^{-1}(y) = f^{-1}(y')$. Therefore

$$f^{-1}(y) \cap f^{-1}(y') = \emptyset \quad \text{or} \quad f^{-1}(y) = f^{-1}(y') \tag{1}$$

for all $y, y' \in Y$ which implies the set X/f of non-empty fibers of f partition X. Let $\pi : X \longrightarrow X/f$ be the function defined by $\pi(x) = f^{-1}(f(x))$ for all $x \in X$. The functions π and f, and the sets X/f and Y, are closely related.

Proposition 1 *Let $f : X \longrightarrow Y$ be a surjective function. Then there is a unique bijection $F : X/f \longrightarrow Y$ which satisfies $F \circ \pi = f$.*

PROOF: Uniqueness. Suppose $F, F' : X/f \longrightarrow Y$ satisfy $F \circ \pi = F' \circ \pi$. Since π is surjective and $F(\pi(x)) = F'(\pi(x))$ for all $x \in X$ necessarily $F = F'$.

Existence. The non-empty fibers of f are the sets $f^{-1}(f(x))$, where $x \in X$. If $x, x' \in X$ and $f^{-1}(f(x)) = f^{-1}(f(x'))$ then $f(x) = f(x')$. Thus $F : X/f \longrightarrow Y$ given by $F(f^{-1}(f(x))) = f(x)$ is a well-defined function and it satisfies $F \circ \pi = f$. Since f is surjective F is surjective. It is is easy to see that F is injective. $\square$

By the proposition the set X/f of fibers of f is in bijective correspondence with Y when f is surjective. In this case the inverse of F is given by

$$y \mapsto f^{-1}(y)$$

for all $y \in Y$.

There are some interesting philosophical points which are suggested by the proposition. Suppose that f is surjective. If one does not distinguish between sets which are in bijective correspondence, then one does not distinguish between X/f and Y. From this point of view Y can be thought of as resulting from a construction in X, the formation of a partition of X whose cells are the fibers of f, and f can be thought of as π.

Equivalence relations can be formulated in terms of fibers. First of all, suppose that $f : X \longrightarrow Y$ is any function. Then $x \sim x'$ if and only if $f(x) = f(x')$ defines an equivalence relation on X since "=" is an equivalence relation. Thus $x \sim x'$ if and only if $x' \in f^{-1}(f(x))$. In particular $[x] = f^{-1}(f(x))$ for all $x \in X$.

Conversely, suppose that X is a non-empty set and $\sim$ is an equivalence relation on X. Let $\widehat{X}$ be the set of equivalence classes of the relation and define $\pi : X \longrightarrow \widehat{X}$ by $\pi(x) = [x]$ for all $x \in X$. Then $x \sim x'$ if and only if $\pi(x) = \pi(x')$ which means $x \sim x'$ if and only if $x' \in \pi^{-1}([x])$. Thus $[x] = \pi^{-1}([x])$ for all $x \in X$ and $\widehat{X} = X/\pi$.

2 Subgroups Revisited

Let G be a group. Here we describe useful necessary and sufficient conditions for a non-empty subset of G to be a subgroup in terms of set multiplication and sets of inverses. For non-empty subsets S, T of G let

$$ST = \{st \,|\, s \in S, t \in T\} \qquad \text{and} \qquad S^{-1} = \{s^{-1} \,|\, s \in S\}.$$

Lemma 1 *Let G be a group and $H \subseteq G$. Then the following are equivalent:*

(a) $H \leq G$.

(b) $H \neq \emptyset$, $HH = H$, *and* $H^{-1} = H$.

(c) $H \neq \emptyset$, $HH \subseteq H$, *and* $H^{-1} \subseteq H$.

PROOF: Suppose that $H \leq G$. Then $HH \subseteq H$ and $H^{-1} \subseteq H$. Since $e \in H$, $H \neq \emptyset$, and $h = he$ for all $h \in H$ from which $H \subseteq HH$ follows. In particular $HH = H$. Since $h = (h^{-1})^{-1}$ for all $h \in H$, from $H^{-1} \subseteq H$ we deduce $H = (H^{-1})^{-1} \subseteq H^{-1}$ as well. Therefore $H^{-1} = H$. We have shown part (a) implies part (b).

Part (b) implies part (c) since sets are equal if and only if they contain each other. Suppose that the hypothesis of (c) holds and let $a, b \in H$. Then $ab^{-1} \in HH^{-1} \subseteq HH \subseteq H$ which implies $ab^{-1} \in H$. Since $H \neq \emptyset$ it follows that $H \leq G$.

$\square$

3 Homomorphisms

We want to apply our discussion of fibers to homomorphisms. First some basic properties which homomorphisms satisfy.

Proposition 2 *Let $f : G \longrightarrow G'$ be a homomorphism. Then:*

(a) $f(e) = e'$

(b) $f(a^{-1}) = f(a)^{-1}$ *for all* $a \in G$.

(c) $f(a^n) = f(a)^n$ *for all* $a \in G$ *and* $n \in \mathbf{Z}$.

(d) *If* $H \leq G$ *then* $f(H) \leq G'$.

(e) *If* $H' \leq G'$ *then* $f^{-1}(H') \leq G$.

PROOF: $f(e) = f(e^2) = f(e)^2$ which means that $f(e)$ is a solution to $x^2 = x$ in G'. In any group this equation has a unique solution, namely the neural element. We have established part (a). Let $a \in G$. Since f is a homomorphism $e' = f(e) = f(aa^{-1}) = f(a)f(a^{-1})$ by part (a). This is enough to show that $f(a^{-1}) = f(a)^{-1}$ and part (b) follows. For $n \geq 0$, part (c) follows by part (a) and induction on n. Suppose $n < 0$. Then $-n > 0$. Using part (b) and part (c) for non-negative integers we have

$$f(a^n) = f((a^{-1})^{-n}) = f(a^{-1})^{-n} = (f(a)^{-1})^{-n} = f(a)^n$$

which completes the argument for part (c). Parts (d) and (e) are left as exercises. □

For a homomorphism $f : G \longrightarrow G'$ the fiber $f^{-1}(e') = f^{-1}(\{e'\})$ is called the *kernel of* f and is denoted by $\operatorname{Ker} f$. Thus

$$\operatorname{Ker} f = \{a \in G \,|\, f(a) = e'\}.$$

This particular fiber of f relates to the others in special ways. For $a \in G$ and a non-empty subset S of G we let

$$aS = \{as \,|\, s \in S\} \qquad \text{and} \qquad Sa = \{sa \,|\, s \in S\}.$$

Proposition 3 *Let $f : G \longrightarrow G'$ be a homomorphism. Then:*

(a) $\operatorname{Ker} f \leq G$.

(b) *Let $N = \operatorname{Ker} f$, let $a \in G$, and $b = f(a)$. Then $f^{-1}(b) = aN = Na$ for all $a \in G$.*

PROOF: To show $\operatorname{Ker} f \leq G$ let $a, b \in \operatorname{Ker} f$. Then $f(ab^{-1}) = f(a)f(b^{-1}) = f(a)f(b)^{-1} = e'e'^{-1} = e'$ by part (b) of Proposition 2. Thus $ab^{-1} \in \operatorname{Ker} f$. Since $f(e) = e'$ by part (a) of the same, it follows that $e \in \operatorname{Ker} f$ which means $\operatorname{Ker} f \neq \emptyset$. Therefore $\operatorname{Ker} f \leq G$.

Assume the hypothesis of part (b). We first show that $aN = f^{-1}(b)$. Let $n \in N$. Then $f(an) = f(a)f(n) = be' = b$ which shows that $an \in f^{-1}(b)$. Therefore $aN \subseteq f^{-1}(b)$. To show the other inclusion, let $x \in f^{-1}(b)$. Then $f(x) = b = f(a)$ which means that $f(a^{-1}x) = f(a)^{-1}f(x) = b^{-1}b = e'$ by part (a) of Proposition 2. We have shown that $a^{-1}x \in N$ which implies

$x = a(a^{-1}x) \in aN$. Therefore $f^{-1}(b) \subseteq aN$ from which $f^{-1}(b) = aN$ follows. To show that $f^{-1}(b) = Na$ also, we need only observe that $f : G^{op} \longrightarrow G'^{op}$ is a homomorphism and use our preceding calculation to deduce $Na = a \cdot^{op} N = f^{-1}(b)$. $\Box$

A subgroup H of a group G is a *normal subgroup of* G if $aH = Ha$ for all $a \in G$. In this case we write $H \trianglelefteq G$. Kernels of homomorphisms are normal by part (b) of Proposition 3.

Corollary 1 *Let $f : G \longrightarrow G'$ be a homomorphism. Then the following are equivalent:*

(a) *f is injective.*

(b) *$\mathrm{Ker}\, f = (e)$.*

PROOF: Any set function $f : X \longrightarrow Y$ is injective if and only if each of its fibers have at most one element. Thus the corollary follows by part (b) of the preceding proposition. $\Box$

Now let $f : G \longrightarrow G'$ be a surjective homomorphism and $F : G/f \longrightarrow G'$ be the set bijection defined at the end of Section 1. Then $F(f^{-1}(a')) = a'$ for all $a' \in G'$. There is a unique group structure $(G/f, \bullet)$ defined on G/f such that F is an isomorphism. Let $a', b' \in G'$. Since F is injective, the calculation

$$F(f^{-1}(a')\bullet f^{-1}(b')) = F(f^{-1}(a'))F(f^{-1}(b')) = a'b' = F(f^{-1}(a'b'))$$

shows that

$$f^{-1}(a')\bullet f^{-1}(b') = f^{-1}(a'b'). \tag{2}$$

Now let $xN \in G/N$. Then $xN = f^{-1}(f(x))$ by part (b) of Proposition 3. Let $a, b \in G$. Using (2) we have

$$aN\bullet bN = f^{-1}(f(a)f(b)) = f^{-1}(f(ab)) = abN.$$

On the other hand the product of sets $(aN)(bN) = a(Nb)N = a(bN)N = abNN = abN$ by part (b) of Proposition 1 and Lemma 1. Thus

$$aN\bullet bN = abN = (aN)(bN) \tag{3}$$

for all $a, b \in G$; in particular *the multiplication of* $(G/f, \bullet)$ *is set multiplication.* Combining these observations with Propositions 1 and 3 we have a version of the First Isomorphism Theorem for groups:

Theorem 1 *Suppose that* $f : G \longrightarrow G'$ *is a surjective homomorphism and let* $N = \mathrm{Ker}\, f$*. Then:*

(a) $N \trianglelefteq G$ *and* G/f *is the set of left cosets of* N *in* G*.*

(b) G/f *is a group under set multiplication and* $\pi : G \longrightarrow G/f$ *defined by* $\pi(a) = aN$ *for all* $a \in G$ *is a surjective homomorphism.*

(c) *There a unique isomorphism* $F : G/f \longrightarrow G'$ *which satisfies* $F \circ \pi = f$*.*

□

4 Quotient Groups

In the preceding section we showed that kernels are normal subgroups. Let G be a group, $N \trianglelefteq G$, and let G/N denote the set of left (or equivalently right) cosets of N in G. We will show that G/N is group under set multiplication and that there is a homomorphism $\pi : G \longrightarrow G/N$ with $N = \mathrm{Ker}\, \pi$. *Thus kernels and normal subgroups are one in the same.* When $f : G \longrightarrow G'$ is a homomorphism and $N = \mathrm{Ker}\, f$ then $G/f = G/N$ as sets.

Theorem 2 *Let* G *be a group and suppose that* $N \trianglelefteq G$*. Then:*

(a) G/N *is a group under set multiplication.*

(b) *Let* $\pi : G \longrightarrow G/N$ *be defined by* $\pi(a) = aN$ *for all* $a \in G$*. Then* π *is a homomorphism,* $\mathrm{Ker}\, \pi = N$*, and* $G/N = G/\pi$*.*

Proof: The elements of G/N have the form aN where $a \in G$. Let $aN, bN \in G/N$. In what follows we use Lemma 1 without particular reference.

The set multiplication calculation

$$(aN)(bN) = a(Nb)N = a(bN)N = abNN = abN$$

shows that the set product $(aN)(bN) \in G/N$. Associativity in G/N follows directly from associativity in G. The neutral element of G/N is $N = eN$ since

$$(aN)N = aNN = aN \qquad \text{and} \qquad N(aN) = NaN = aNN = aN.$$

Using various descriptions of the set of inverses $(aN)^{-1} = N^{-1}a^{-1} = Na^{-1} = a^{-1}N$ we have $(aN)^{-1} = a^{-1}N$,

$$(aN)(aN)^{-1} = (Na)(a^{-1}N) = Naa^{-1}N = NN = N$$

and

$$(aN)^{-1}(aN) = (Na^{-1})(aN) = Na^{-1}aN = NN = N.$$

Therefore $(aN)^{-1} \in G/N$ and is a two-sided inverse for aN. We have shown part (a).

That π is a homomorphism and $\operatorname{Ker}\pi = N$ is easy to see. That $G/N = G/\pi$ now follows by part (b) of Proposition 3. $\Box$

5 Normal Subgroups

Let G be group and $H \leq G$. The condition $aH = Ha$ for all $a \in G$, that is $H \trianglelefteq G$, is significant; see Proposition 3.

Theorem 3 *Let G be group and $H \leq G$. Then the following are equivalent:*

(a) *The set of left cosets of H is the set of right cosets of H.*

(b) *For all $a \in G$ there exists a $b \in G$ such that $aH = Hb$.*

(c) *$aH = Ha$ for all $a \in G$.*

(d) *$aHa^{-1} = H$ for all $a \in G$.*

(e) *$aHa^{-1} \subseteq H$ for all $a \in G$.*

(f) *$aH \subseteq Ha$ for all $a \in G$.*

Proof: Part (a) implies part (b). The set of left cosets of H in G partition G as does the of right cosets. Suppose these sets are the same and let $a \in G$. Then the left coset aH is a right coset which has the form Hb for some $b \in G$.

Part (b) implies part (c). Assume part (b) is true and let $a \in G$. Then $aH = Hb$ for some $b \in G$. Now $a = ae \in aH$ implies $a \in Hb$. Since $a = ea \in Ha$ the right cosets Hb, Ha are not disjoint. Therefore $Hb = Ha$.

Part (c) implies part (d) since $aH = Ha$ implies $aHa^{-1} = Haa^{-1} = H$. Part (d) implies part (e) since sets are equal if and only if they contain each other. Part (e) implies part (f) since $aHa^{-1} \subseteq H$ implies $aH = (aHa^{-1})a \subseteq (H)a = Ha$.

Suppose part (f) is true and let $a \in G$. Then $aH \subseteq Ha$ and $a^{-1}H \subseteq Ha^{-1}$. From the last equation we deduce

$$Ha = a(a^{-1}H)a \subseteq a(Ha^{-1})a = aH.$$

Therefore $aH \subseteq Ha \subseteq aH$ which means $aH = Ha$. Thus part (f) implies part (a). $\square$

By virtue of the preceding theorem there several ways of describing normal subgroups.