Math 516 **Fall 2006** **Radford**

Written Homework # 2

Due at the beginning of class 10/06/06

Let G be a non-empty set with binary operation. For non-empty subsets $S, T \subseteq G$ we define the product of the sets S and T by

$$ST = \{st \,|\, s \in S, t \in T\}.$$

If $S = \{s\}$ is a singleton then we set

$$sT = \{s\}T = \{st \,|\, t \in T\}$$

and if $T = \{t\}$ is a singleton we set

$$St = S\{t\} = \{st \,|\, s \in S\}.$$

We denote the set of inverses of elements of S by S^{-1}.

You may assume multiplication of sets is associative and $(ST)^{-1} = T^{-1}S^{-1}$. *From this point on G is a group*, not necessarily finite.

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1. Suppose that $H \leq G$.

   (a) Suppose that G is abelian. Show that $H \trianglelefteq G$.

   (b) Suppose that $a^2 = e$ for all $a \in G$. Show that G is abelian.

   (c) Suppose that G is *finite* and $a^2 = e$ for all $a \in G$. Show, by induction, that $|G| = 2^n$ for some $n \geq 0$. [Hint: Suppose $e \neq a \in G$ and consider the quotient G/H, where $H = <a>$.]

2. Suppose that $H, K \leq G$ and let $f : H{\times}K \longrightarrow HK$ be the set map defined by $f((h,k)) = hk$ for all $(h,k) \in H{\times}K$.

(a) For fixed $h \in H$ and $k \in K$ show that

$$f^{-1}(hk) = \{(hx, x^{-1}k) \mid x \in H{\cap}K\}.$$

(b) For fixed $h \in h$ and $k \in K$ show that the function

$$b : H{\cap}K \longrightarrow f^{-1}(hk)$$

defined by $b(x) = (hx, x^{-1}k)$ for all $x \in H{\cap}K$ is a bijection.

(c) Now suppose that H, K are *finite*. Use parts (a)–(b) to show that

$$|H||K| = |HK||H{\cap}K|.$$

3. Suppose that H is a non-empty subset of G.

(a) Show that $H \leq G$ if and only if $HH = H$ and $H^{-1} = H$.

(b) Suppose that $H, K \leq G$. Using part (a), show that $HK \leq G$ if and only if $HK = KH$.

(c) Suppose that H is *finite*. Show that $H \leq G$ if and only if $HH \subseteq H$. [Hint: Suppose that $HH \subseteq H$ and $a \in H$. Show that the list $a, a^2, a^3, \ldots$ must have a repetition.]

4. Suppose that $|G| = 6$.

(a) Use Exercise 1 to show that $a^2 \neq e$ for some $a \in G$.

(b) Use Exercise 2 to show that G has *at most* one subgroup of order 3. (Thus if G has a subgroup N of order 3 then $N \trianglelefteq G$.)

(c) Use Lagrange's Theorem and parts (a) and (b) to show that G has an element a of order 2 and an element b of order 3.

(d) Let $N = \langle b \rangle$. Show that $|G : N| = 2$. (Thus $N \trianglelefteq G$.) Show that $ab = ba$ or $ab = b^2a = b^{-1}a$.

(e) Suppose that $ab = ba$. Use Lagrange's Theorem to show that G is cyclic. [Hint: Consider $<ab>$.]

5. We continue Exercise 4.

(a) Show that $G = \{e, b, b^2, a, ab, ab^2\}$.

(b) Suppose that $ab = b^2a$. Complete the multiplication table

	e	b	b^2	a	ab	ab^2
e						
b						
b^2						
a						
ab						
ab^2						

for G.

[Hint: Let $N = <b> = \{e, b, b^2\}$. Then $N \trianglelefteq G$ and $|G/N| = 2$. Note that $G/N = \{N, aN\}$ by part (a). Since aN has order 2 the multiplication table for G/N is given by

	N	aN
N	N	aN
aN	aN	N

.]

You can ignore this hint and simply use the relations

$$a^2 = e, \qquad b^3 = e, \qquad ab = b^2a$$

to compute *all* of the products. However, it would be very illuminating to use the hint and see how many calculations you then need to make using the relations.

Comment: The relations $a^2 = e = b^3$ and $ab = b^2a$ completely determine the group table in Exercise 5. In light of Exercise 4 there is *at most one* non-abelian group G (up to isomorphism) of order 6. Since S_3 has order 6 and is non-abelian, $G \simeq S_3$.