MATH 516 # Hour Exam **Radford** **10/17/08**

Name (print) ____________________

(1) *Return* this exam copy with your exam booklet. (2) *Write* your solutions in your exam booklet. (3) *Show* your work. (4) There are *four questions* on this exam. (5) Each question counts 25 points. (6) *You are expected to abide by the University's rules concerning academic honesty.*

---

1. Let $G = \langle a \rangle$ be a cyclic group of order 35.

(a) Find the number of subgroups of G.

(b) Find $|a^{-77}|$.

(c) List the generators of G in the form a^ℓ, where $0 \le \ell < 35$.

(d) List the elements of $\langle a^{205} \rangle$ in the form a^ℓ, where $0 \le \ell < 35$.

(e) Is G the only abelian group of order 35? Justify your answer.

2. Let G be a group, let $H \le G$, let A be the set of left cosets of H in G, and finally let $\pi : G \longrightarrow S_A$ be the permutation representation induced by the left action of G on A given by $g{\cdot}(aH) = gaH$ for all $g \in G$ and $aH \in A$.

(a) Show that $\mathrm{Ker}\,\pi$ is the largest normal subgroup of G which is contained in H.

(b) Now suppose that G is finite, $|G : H| = n$, and $|G| > n!$. Show that H contains a normal subgroup $(e) \neq N$ of G.

3. Let $f, g : G \longrightarrow G'$ be group homomorphisms.

(a) Suppose that $S \subseteq G$ is a non-empty set. Show that $f(\langle S \rangle) = \langle f(S) \rangle$.

(b) Suppose that f is surjective. Use part (a) to show that if G is finitely generated (respectively cyclic) implies G' is finitely generated (respectively cyclic).

(c) Show that $H = \{a \in G \,|\, f(a) = g(a)\} \le G$.

(d) Suppose $S \subseteq G$ generates G and $f(s) = g(s)$ for all $s \in S$. Show that $f = g$.

4. Let G be a finite group of order $5{\cdot}7{\cdot}17$.

(a) Show that G has a normal subgroup of order 7 or 17.

(b) Show that G has a subgroup of index 5. [Hint: Consider the product of two appropriate Sylow p-subgroups.]