Math 516 **Fall 2008** **Radford**

Written Homework # 2

Due at the beginning of class 10/03/08

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1. Let $\emptyset \neq I$ and $\{G_i\}_{i\in I}$ be an indexed family of groups. Set

$$\mathcal{G} = \{f : I \longrightarrow \cup_{i\in I} G_i \,|\, f(i) \in G_i \;\; \forall i \in I\}.$$

(a) Show that $\mathcal{G}$ is a group, where $(fg)(i) = f(i)g(i)$ for all $f, g \in \mathcal{G}$ and $i \in I$.

Suppose that $n \geq 1$ and $I = \{1, 2, \ldots, n\}$. Then the set bijection $F : \mathcal{G} \longrightarrow G_1 \times \cdots \times G_n$ given by $F(f) = (f(1), \ldots, f(n))$ induces a group structure on the Cartesian product, where

$$(g_1, \ldots, g_n)\cdot(h_1, \ldots, h_n) = (g_1h_1, \ldots, g_nh_n),$$

and F is an isomorphism.

(b) Show that there are two groups (up to isomorphism) of order 4 which are $\mathbf{Z}_4$ and $\mathbf{Z}_2 \times \mathbf{Z}_2$. [Hint: Suppose that G has order 4. Then $x^2 = e$ for all $x \in G$ or there is an $a \in G$ such that $a^2 \neq e$; in the latter case set $b = a^2$. Write down the group tables in each case.]

2. Suppose that G is a finite group and $a^2 = e$ for all $a \in G$.

(a) Show that G is abelian.

(b) Show that for some $n \geq 0$ there is a sequence of subgroups $(e) = H_0 \subseteq H_1 \subseteq \cdots \subset H_n = G$ of subgroups which satisfies $|H_i : H_{i-1}| = 2$ for all $0 \leq i < n$. (In particular, by induction $|G| = 2^n$ for some $n \geq 0$.)

3. Let G be a group and suppose that $\mathcal{G}$ is a partition of G into non-empty sets. Suppose that $\mathcal{G}$ is a group under set product and N is the neutral element of $\mathcal{G}$. Show for $S \in \mathcal{G}$ that $S = aN = Na$ for all $a \in S$.

4. Let G, G' be groups.

(a) Suppose that $f, g : G \longrightarrow G'$ are group homomorphisms and set $H = \{a \in G \mid f(a) = g(a)\}$. Show that $H \leq G$.

(b) Continuing with part (a), suppose that $G = <S>$ for some $S \subseteq G$. Show that $f = g$ if and only if $f(s) = g(s)$ for all $a \in S$.

(c) Suppose that $H, K \leq G$ and $H = \lt S \gt$, $K = \lt T \gt$ where $S, T \subseteq G$. Suppose that $st = ts$ for all $s \in S$ and $t \in T$. Show that $hk = kh$ for all $h \in H$ and $k \in K$. [Hint: First consider the relationship between the subgroups H and $C_G(\{t\})$ for all $t \in T$.]

(d) Continuing with part (c), suppose that H, K are abelian. Show that $HK \leq G$ and is abelian.

5. Here we consider subgroups of order 6. Let G be a group. The exercise establishes that there are two such groups up to isomorphism, namely $\mathbf{Z}_6$ and S_3.

(a) Suppose that G is a group and $a, b \in G$ have orders m an n respectively, where $(m, n) = 1$, and $ab = ba$. Show that ab has order mn.

(b) Suppose G has order 6 and is abelian. Show that $G \simeq \mathbf{Z}_6$. (You may assume the exponent law $(ab)^\ell = a^\ell b^\ell$ for all $\ell \in \mathbf{Z}$, which incidently holds if and only if $ab = ba$.)

Now Suppose that G is non-abelian and has order 6.

(c) Use the class equation to show that G has an element a of order 2.

(d) Show that $H = \lt a \gt$ is not a normal subgroup of G. [Hint: Show that $H \trianglelefteq G$ implies $H \subseteq \mathrm{Z}(G)$.]

(e) Let A be the set of left cosets of H and let G act on A by $g\cdot(aH) = gaH$ for all $g \in G$ and $aH \in A$. Show that the induced representation $\pi : G \longrightarrow S_A$ is an isomorphism. (Thus $G \simeq S_A \simeq S_3$ since $|A| = 3$.)