Math 516 **Fall 2008** **Radford**

Written Homework # 5

Due at the beginning of class 12/05/08

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1. Let R be a non-zero Boolean ring with identity (a ring with identity such that $a^2 = a$ for all $a \in R$).

(a) Show that R is commutative.

(b) Suppose that $e \in R$. Show that Re and $R(1-e)$ are ideals of R and $R = Re \oplus R(1-e)$.

(c) Suppose that R is finite. Show that $R \simeq \mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2$.

2. Suppose that R is any ring.

(a) Show that the direct product of abelian groups $\mathbf{R} = \mathbf{Z} \times R$ is a ring with identity, where the product is defined by $(m, a) \cdot (n, b) = (mn, n \cdot a + m \cdot b + ab)$ for all $(m, a), (n, b) \in \mathbf{R}$, and $\imath : R \longrightarrow \mathbf{R}$ defined by $\imath(a) = (0, a)$ for all $a \in R$ is an injective ring homomorphism.

(b) Show that there is an abelian group A and an injective ring homomorphism $\jmath : R \longrightarrow \mathrm{End}\,(A)$.

Remark: Part (b) may be thought of as Cayley's Theorem for rings.

3. Find all irreducible polynomials in $\mathbf{Z}_2[x]$ of degrees 2, 3, or 4. [Hint: Which ones are reducible?]

4. Let R be a ring, let I be a non-empty set, let $\{M_i\}_{i \in I}$ be an indexed family of left R-modules, and let $\prod\limits_{i \in I} M_i = \{f : I \longrightarrow \cup_{i \in I} M_i \mid \ f(i) \in M_i \ \ \forall i \in I\}$.

(a) Show that $\prod\limits_{i \in I} M_i$ is a left R-module, where

$$(f + g)(i) = f(i) + g(i) \ \text{ and } \ (r \cdot f)(i) = r \cdot (f(i))$$

for all $f, g \in \prod\limits_{i \in I} M_i$, $r \in R$, and $i \in I$.

(b) For $j \in I$ define $\pi_j : \prod\limits_{i \in I} M_i \longrightarrow M_j$ by $\pi_j(f) = f(j)$ for all $f \in \prod\limits_{i \in I} M_i$. Show that $(\{\pi_i\}_{i \in I}, \prod\limits_{i \in I} M_i)$ is a product of $\{M_i\}_{i \in I}$.

5. We continue with Exercise 4. For $i_0 \in I$ define $\jmath_{i_0} : M_i \longrightarrow \prod\limits_{i \in I} M_i$ by $\jmath_{i_0}(m)(i) = \begin{cases} m & : \ i = i_0 \\ 0 & : \ i \neq i_0 \end{cases}$. Show that $(\{\jmath_i\}_{i \in I}, M)$ is a direct sum of $\{M_i\}_{i \in I}$ for some submodule M of $\prod_{i \in I} M_i$.