Math 517 **Spring 2007** **Radford**

Written Homework # 2

Due at the beginning of class 02/23/07[1]

Throughout R and S are rings with unity; $\mathbf{Z}$ denotes the ring of integers and $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{C}$ denote the rings of rational, real, and complex numbers respectively.

1. Regard $\mathbf{C}$ as a (left) $\mathbf{Z}$-module, $\mathbf{Q}$-module, and $\mathbf{R}$-module by multiplication in $\mathbf{C}$. Let $f : \mathbf{C} \longrightarrow \mathbf{C}$ be the function defined by $f(r + s\imath) = s + r\imath$ for all $r, s \in \mathbf{R}$. Prove or disprove:

(1) f is a homomorphism of additive groups (that is a homomorphism of $\mathbf{Z}$-modules);

(2) f is a homomorphism of $\mathbf{Q}$-modules;

(3) f is a homomorphism of $\mathbf{R}$-modules;

(4) f is a homomorphism of $\mathbf{C}$-modules.

To disprove a statement find one specific example for which the statement fails.

2. Suppose that R is commutative, ${}_RL$, ${}_RM$, ${}_RN$, ${}_RL'$, and ${}_RM'$. Using results from ClassNotes:

(1) Show that there are R-module isomorphisms $R\otimes_R M \simeq M$, $M\otimes_R R \simeq M$, and $L\otimes_R(M\otimes_R N) \simeq (L\otimes_R M)\otimes_R N$.

(2) Suppose that $f : L \longrightarrow L'$ and $g : M \longrightarrow M'$ are left R-module homomorphisms. Show that $f\otimes g : L\otimes_R M \longrightarrow L'\otimes_R M'$ is a left R-module homomorphism.

[1]Slightly revised 02/07/07.

3. Suppose L_R, ${}_RM_S$, and ${}_SN$. Use results from ClassNotes, except Propostion 2.1.9, to establish that there is a homomorphism of abelian groups $F : L \otimes_R (M \otimes_S N) \longrightarrow (L \otimes_R M) \otimes_S N$ given by $F(\ell \otimes (m \otimes n)) = (\ell \otimes m) \otimes n$ as follows:

(1) Let $\ell \in L$ be fixed. Show that there is a homomorphism of abelian groups $f_\ell : M \otimes_S N \longrightarrow (L \otimes_R M) \otimes_S N$ given by $f_\ell(m \otimes n) = (\ell \otimes m) \otimes n$ for all $m \in M$ and $n \in N$.

(2) Show that $f_\ell + f_{\ell'} = f_{\ell+\ell'}$ for all $\ell, \ell' \in L$.

(3) Show that $f : L \times (M \otimes_S N) \longrightarrow (L \otimes_R M) \otimes_S N$ defined by $f(\ell, x) = f_\ell(x)$ for all $\ell \in L$ and $x \in M \otimes N$ is R-balanced.

(4) Using (3) deduce the existence of F.

4. Prove Proposition 3.2.1, parts (2) and (3), and Theorem 3.2.2 from ClassNotes.

5. Let D be an integral domain and let F be its field of quotients. We may assume that D is a subring of F and we regard F as a left D-module by multiplication in F.

(1) Suppose that M and N are submodules of F. Show that $M \cap N = (0)$ implies $M = (0)$ or $N = (0)$.

(2) Suppose that F is a free D-module and let $\{m_i\}_{i \in I}$ be a basis for F. Show that $|I| = 1$.

(3) Show that F is a free D-module if and only if $D = F$.

(4) Suppose that P is a non-zero projective R-module and $f : F \longrightarrow P$ is a surjective R-module homomorphism. Show that f is an isomorphism. [Hint: Consider part (1) of Proposition 3.2.1 which you can use without proof.]

(5) Show that there is no surjective homomorphism of D-modules $f : F \longrightarrow D$ unless $D = F$. [Hint: D is a free, hence projective, D-module.]