Math 517 **Spring 2007** **Radford**

Written Homework # 3

Due at the beginning of class 03/16/07

Throughout R is a ring with unity and all R-modules are unital.

1. Suppose that R is commutative and $a, b, c \in R$ satisfy $c|a$ and $c|b$.

(1) Show that there is a homomorphism of R-modules

$$f : R/Ra \otimes_R R/Rb \longrightarrow R/Rc$$

such that $f((r + Ra) \otimes (s + Rb)) = rs + Rc$ for all $r, s \in R$.

(2) Suppose further that $c = xa + yb$ for some $x, y \in R$. Show that the homomorphism of (1) is an isomorphism. [Hint: Show that there is a homomorphism of R-modules $g : R/Rc \longrightarrow R/Ra \otimes_R R/Rb$ given by $g(r + Rc) = (r + Ra) \otimes (1 + Rb)$ for all $r \in R$.]

(3) Suppose that R is a Principal Ideal Domain (for example $\mathbf{Z}$) and c is a greatest common divisor of a and b. Show that $R/Ra \otimes_R R/Rb \simeq R/Rc$.

2. Let D be an integral domain, regard it as a subring of its field of quotients F, and regard F as a left D-module under multiplication.

(1) Suppose that F is submodule of a free D-module. Show that $F = D$.

(2) Show that F is a projective D-module if and only if $F = D$.

[Hint: For part (1) let M be a free left D-module with basis $\{m_i\}_{i \in I}$. For a non-zero $m \in M$ there is a finite list of distinct elements $m_{i_1}, \ldots, m_{i_s}$ of M such that $m = r_1 \cdot m_{i_1} + \cdots + r_s \cdot m_{i_s}$ where $r_1, \ldots, r_s \in D \backslash 0$. This list is unique up to order. Define

$$\text{supp}(m) = \{m_{i_1}, \ldots, m_{i_s}\}.$$

You may assume the preceding (which holds for any ring with unity R).

Now suppose that F is a submodule of M. Show that $\text{supp}(x) = \text{supp}(1)$ for all $x \in F \backslash 0$. Let $\text{supp}(1) = \{m_1, \dots m_s\}$, where the m_i's are distinct. Use the fact that $F \subseteq D{\cdot}m_1 \oplus \cdots \oplus D{\cdot}m_s$ to construct a surjective homomorphism of D-modules $f : F \longrightarrow D$. You may use Exercise 5 of Written Homework #2.]

In the next two exercises we will consider the details of the proof of Baer's Criterion. The notion of injective module can be formulated in a slightly more convenient way:

Lemma 1 *Let R be a ring with unity. Then a left R-module Q is injective if and only if whenever M' is a submodule of left R-module M, every R-module homomorphism $f' : M' \longrightarrow Q$ extends to an R-module homomorphism $f : M \longrightarrow Q$.*

From this point on M and Q are fixed left R-modules. The notation (f', M') will mean that M' is a submodule of M and $f : M' \longrightarrow Q$ is a homomorphism of left R-modules. For such pairs (f', M') and (f'', M'') the notation $(f', M') \leq (f'', M'')$ means that $M' \subseteq M''$ and f'' extends f'; that is $f''(m) = f'(m)$ for all $m \in M'$.

3. Consider a fixed (f_0, M_0) and let

$$\mathcal{S} = \{(f', M') \,|\, (f_0, M_0) \leq (f', M')\}.$$

(1) Suppose that $(f', M'), (f'', M'') \in \mathcal{S}$ and satisfy $f'(m) = f''(m)$ for all $m \in M' \cap M''$. Show that $(f' + f'', M' + M'') \in \mathcal{S}$, where $(f' + f'')(m' + m'') = f'(m') + f''(m'')$ for all $m' \in M'$ and $m'' \in M''$. [Hint: Be sure to show that $f' + f''$ is well-defined.]

(2) Show that $\mathcal{S}$ is a partially ordered set under $\leq$ and that $\mathcal{S}$ has a maximal element (f_e, M_e).

(3) Show that $(f', M') \leq (f_e, M_e)$ for all $(f', M') \in \mathcal{S}$. [Hint: (f_e, M_e) maximal means whenever $(f', M') \in \mathcal{S}$ and $(f_e, M_e) \leq (f', M')$ then $(f_e, M_e) = (f', M')$. Thus part (3) is saying something much more than maximal.]

4. Suppose that $(f', M') \in \mathcal{S}$ and let $m \in M$.

(1) Show that $L = \{r \in R \,|\, r{\cdot}m \in M'\}$ is a left ideal of R.

(2) Show that $F : L \longrightarrow Q$ defined by $F(r) = f'(r{\cdot}m)$ for all $r \in L$ is a homomorphism of left R-modules.

(3) Suppose that F has an extension to an R-module homomorphism $G : R \longrightarrow Q$. Show that $g : R{\cdot}m \longrightarrow Q$ given by $g(r{\cdot}m) = G(r)$ for all $r \in R$ is a well-defined homomorphism of left R-modules and $g(x) = f'(x)$ for all $x \in M' \cap R{\cdot}m$.

(4) Suppose for all left ideals L of R that every homomorphism of left R-modules $F : L \longrightarrow Q$ has an R-module homomorphism extension $G : R \longrightarrow Q$. Show that $M_e = M$.