MATH 531 Written Homework 1 Radford DUE FRIDAY 09/14/07
(09/11/07: Revisions in boldface with asterisks.)

In the following exercises F is a field and $\boldsymbol{R}$ is the field of real numbers. We follow the notation of the text and that used in class.

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1. Let A be an algebra over F which satisfies $a^2 = 0$ for all $a \in A$. Let $J : A \times A \times A \longrightarrow A$ be the function defined by

$$J(a, b, c) = a(bc) + b(ca) + c(ab)$$

for all $a, b, c \in A$.

(a) Show that $J(a, a, b) = 0$ for all $a, b \in A$.

(b) Show that $J(a, c, b) = J(b, a, c) = J(c, b, a) = -J(a, b, c)$ for all $a, b, c \in A$. (Thus if $J(a, b, c) = 0$ then $J(x, y, z) = 0$ for any rearrangement x, y, z of a, b, c since the equation holds for rearrangements which are transpositions.)

(c) Suppose that A is finite-dimensional and $\{a_1, \ldots, a_n\}$ is a basis for A. Show that A is a Lie algebra if and only if $J(a_i, a_j, a_k) = 0$ for all $1 \leq i < j < k \leq n$. (You may assume that if $J(a, b, c) = 0$ for all a, b, c in some spanning set then $J = 0$.)

(d) Use part (b) to show that a 2-dimensional algebra ***B*** over F with basis $\{a, b\}$ and multiplication table

	a	b
a	0	c
b	-c	0

,

where $c \in A$, is a Lie algebra.

(e) Show that a 3-dimensional algebra ***B*** over F with basis $\{x, y, z\}$ and multiplication table

	x	y	z
x	0	cz	by
y	-cz	0	ax
z	-by	-ax	0

,

where $a, b, c \in F$, is a Lie algebra.

(f) Show that $\mathbf{R}^3$ with the cross product is a Lie algebra. [Hint: Recall that

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} = \begin{vmatrix} \boldsymbol{\imath} & \boldsymbol{\jmath} & \boldsymbol{k} \\ a & b & c \\ a' & b' & c' \end{vmatrix},$$

where

$$\boldsymbol{\imath} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \boldsymbol{\jmath} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \text{and} \qquad \boldsymbol{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.]$$

2. Let $n \geq 2$ *****and the characteristic of F is not two**.***

(a) Show that $sl(n, F) = L_1 + \cdots + L_r$, where $L_i \simeq sl(2, F)$ for all $1 \leq i \leq r$. [Hint: Consider the Lie subalgebra of $sl(n, F)$ generated by $e_{i\,j}$ and $e_{j\,i}$, where $1 \leq i < j \leq n$.]

(b) Show that $[L\,L] = L$, where $L = sl(n, F)$. [Hint: Show that part (b) reduces to the case $n = 2$.]

(c) Find a derivation D of $sl(n, F)$ such that D^2 is not a derivation. [Hint: Find a necessary and sufficient condition for D^2 to be a derivation, where D is a derivation of an algebra A over F.]

3. Let $n \geq 1$. For $1 \leq r, r', c, c' \leq n$ let $L_{r,r':c,c'}$ be the span of all $e_{i\,j} \in \mathrm{M}(n, F)$ such that $r \leq i \leq r'$ and $c \leq j \leq c'$.

(a) Show that $L_{r,r':c,c'}$ is a Lie subalgebra of $gl(n, F)$.

Let $L = L_{1,1:1,n}$ and $a_i = e_{1\,i}$ for all $1 \leq i \leq n$. Then $\{a_1, \ldots, a_n\}$ is a basis for L.

(b) Find $[a_i\,a_j]$ in terms of this basis for all $1 \leq i, j \leq n$.

(c) Determine $\mathrm{Z}(L)$, $\mathrm{N}_L(Fa_1)$, and $\mathrm{C}_L(Fa_1)$.

(d) Determine the terms of the derived series and the lower central series of L.

4. Suppose that V is a finite-dimensional vector space over F and a is a diagonalizable endomorphism of V. Show that $\mathrm{ad}\, a$ is a diagonalizable endomorphism of $\mathrm{End}(V)$. [Hint: We take for the definition of diagonalizable "V has a basis of eigenvectors for a". Suppose that $\{v_i\}_{1 \leq i \leq n}$ is a basis for V. Show that $\{E_{i\,j}\}_{1 \leq i,j \leq n}$ is a basis for $\mathrm{End}(V)$, where $E_{i\,j}(v_k) = \delta_{j,k} v_i$, and find a formula for $E_{i\,j} \circ E_{k\,\ell}$.]

5. Let $s \in \mathrm{M}(n, F)$ and set $\mathcal{L}_s = \{x \in \mathrm{M}(n, F) \,|\, x^t s = -sx\}$.

(a) Show that $\mathcal{L}_s$ is a Lie subalgebra of $gl(n, F)$.

(b) Suppose that the characteristic of F is not 2 and s is invertible. Show that $\mathcal{L}_s$ is a Lie subalgebra of $sl(n, F)$.

(c) Suppose that $u \in \mathrm{M}(n, F)$ is invertible and $u^{-1} = u^t$. (This is to say that u is orthogonal when $F = \mathbf{R}$.) Show that $\mathcal{L}_s \simeq \mathcal{L}_{usu^{-1}}$ as Lie algebras.

(d) For which positive integers n is it the case that $sl(n, F) = \mathcal{L}_s$ for some s? [Hint: Write $s = \sum_{1 \leq i,j \leq n} s_{i\,j} e_{i\,j}$ where $s_{i\,j} \in F$.]