MATH 531 Written Homework 2 Radford DUE FRIDAY 10/05/07

In the following exercises F is a field and all algebras are over F. We follow the notation of the text and that used in class.

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1. Let A be a Lie algebra over F, let S be a subspace S of L, and define $S^{(0)}$, $S^{(1)}$, $S^{(2)}$, ... inductively by $S^{(0)} = S$ and $S^{(i+1)} = [S^{(i)}\, S^{(i)}]$ for $i \geq 0$.

(a) Show that $(S^{(i)})^{(j)} = S^{(i+j)}$ for all $i, j \geq 0$.

(b) Suppose that T is also a subspace of L and $S \subseteq T$. Show that $S^{(i)} \subseteq T^{(i)}$ for all $i \geq 0$.

(c) Let $f : L \longrightarrow L'$ be a map of Lie algebras. Show that $f(S^{(i)}) = f(S)^{(i)}$ for all $i \geq 0$.

(d) If S is an ideal of L show that $S^{(i)}$ is an ideal of L for all $i \geq 0$.

(e) If S is a subalgebra of L show that $S^{(i)}$ is a subalgebra of L for all $i \geq 0$.

2. A *graded algebra over* F is an algebra A over F with a designated direct sum decomposition of subspaces $A = \bigoplus_{i=0}^{\infty} A(i)$ such that $A(i)A(j) \subseteq A(i+j)$ for all $i, j \geq 0$. Let $L = \bigoplus_{i=0}^{\infty} L(i)$ be a graded Lie algebra over F and set $L_i = L(i) \oplus L(i+1) \oplus L(i+2) \oplus \cdots$ for all $i \geq 0$.

(a) Show that $L = L_0 \supseteq L_1 \supseteq L_2 \supset \cdots$ is a descending chain of ideals of L which satisfies $[L_i\, L_j] \subseteq L_{i+j}$ for all $i, j \geq 0$.

(b) Suppose that $L(0)$ is a abelian. Show that $L^{(i)} \subseteq L_{2^{i-1}}$ for all $i \geq 1$.

Suppose that $L(0)$ is abelian and $L(n) = L(n+1) = \cdots = (0)$ for some $n \geq 0$.

(c) Show that L is solvable.

(d) Show that $[L\, L]$ is nilpotent.

(e) Suppose that $[L(0)\, L(1)] = L(1)$ and is not zero. Show that L is not nilpotent.

3. We discuss two applications of Exercise 2. First, let $n \geq 1$, $L = t(n, F)$, and let $L(i)$ be the span of the $e_{\ell \ell'}$'s, where $1 \leq \ell, \ell' \leq n$ and $\ell' = \ell + i$.

(a) Show that the $L(i)$'s give $L = t(n, F)$ has the structure of graded Lie algebra, $L(0)$ is abelian, $L(n) = L(n+1) = \cdots = (0)$, and $[L(0)\, L(1)] = L(1)$.

(b) Use Exercise 2 to conclude that L is solvable, $[L\,L]$ is nilpotent, and L is not nilpotent.

Now let L be the Lie algebra with basis $\{x, y\}$ determined by $[x\,y] = y$.

(c) Show that L has the structure of a graded Lie algebra $L = L(0) \oplus L(1) \oplus \cdots$ such that $L(0)$ is abelian, $L(2) = L(3) = \cdots = (0)$, and $[L(0)\,L(1)] = L(1)$.

(d) Use Exercise 2 to conclude that L is solvable, $[L\,L]$ is nilpotent, and L is not nilpotent. [Note: The ideal $I = Fx$ and the quotient L/I are nilpotent but L is not.]

4. Let L be a Lie algebra over F. Show that the following are equivalent:

(a) There exists a descending sequence of subalgebras

$$L = L_0 \supseteq L_1 \supseteq L_2 \supseteq \cdots \supseteq L_n = (0)$$

for some $n \geq 0$ such that L_{i+1} is an ideal of L_i and the quotient L_i/L_{i+1} is abelian for all $0 \leq i < n$.

(b) L is solvable.

5. Let L be a finite-dimensional nilpotent Lie algebra over F.

(a) Show that L has a flag of ideals.

(b) Suppose that I is an ideal of L. Show that there is an increasing sequence of ideals

$$I = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_r = L$$

for some $r \geq 0$ such that $\operatorname{Dim} I_{i+1} = \operatorname{Dim} I_i + 1$ for all $0 \leq i < r$.