MATH 531 Written Homework 4 Radford DUE FRIDAY 11/16/07

In the following exercises F is a field and L is a finite-dimensional Lie algebra over F. We follow the notation of the text and that used in class. You may use results from the course materials on the class homepage, and the text, up through 8.5. In any case you can not quote exercises, unless you prove them.

In this execise set we begin a rather detailed study of $sl(n, F)$.

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1. Let $L = gl(n, F)$ and $\{e_{i\,j}\}_{1 \leq i,j \leq n}$ be the standard basis for the underlying vector space $\mathrm{M}(n, F)$ for L. Thus

$$e_{i\,j}e_{k\,\ell} = \delta_{j,k}e_{i,\ell}$$

for all $1 \leq i, j, k, \ell \leq n$.

(a) Show that

$$(\mathrm{ad}\, e_{i\,j} \circ \mathrm{ad}\, e_{k\,\ell})(e_{u\,v}) = \delta_{\ell,u}\delta_{j,k}e_{i\,v} - \delta_{\ell,u}\delta_{i,v}e_{k\,j} - \delta_{k,v}\delta_{j,u}e_{i\,\ell} + \delta_{k,v}\delta_{i,\ell}e_{u\,j}$$

for all $1 \leq i, j, k, \ell, u, v \leq n$.

(b) Show that the coefficient of $e_{u\,v}$ in the expression in part a) is

$$\delta_{\ell,u}\delta_{j,k}\delta_{i,u} - \delta_{\ell,u}\delta_{i,v}\delta_{k,u}\delta_{j,v} - \delta_{k,v}\delta_{j,u}\delta_{i,u}\delta_{\ell,v} + \delta_{k,v}\delta_{i,\ell}\delta_{j,v}.$$

(c) Show that the Killing form κ for L is given by

$$\kappa(e_{i\,j}, e_{k\,\ell}) = \mathsf{Tr}(\mathrm{ad}\, e_{i\,j} \circ \mathrm{ad}\, e_{k\,\ell}) = 2n\delta_{i,\ell}\delta_{j,k} - 2\delta_{i,j}\delta_{k,\ell}$$

for all $1 \leq i, j, k, \ell \leq n$.

(d) Suppose that the characteristic of F is zero. Show that $\mathrm{Rad}\,\kappa = FI_n$.

2. Suppose that F has characteristic zero and let $L = sl(n, F)$. Show that the Killing form κ for L is non-degenerate. (Thus L is semisimple when F is also algebraically closed.) [Hint: Since L is an ideal of $gl(n, F)$ it follows that $\kappa = \kappa_{gl(n,F)}|_{L \times L}$. Note that $gl(n, F) = sl(n, F) \oplus FI_n$ and compare $\mathrm{Rad}\,\kappa_{gl(n,F)}$ with $\mathrm{Rad}\,\kappa_{sl(n,F)}$.]

From this point on F is algebraically closed and has characteristic zero.

3. Let $L = sl(n, F)$, $H = d(n, F) \cap L$, and let $\{e_{i\,j}\}_{1 \leq i,j}$ be the standard basis for $\mathrm{M}(n, F)$. For $1 \leq k, \ell \leq n$, where k and ℓ are distinct, define $\alpha_{k,\ell} \in H^*$ by

$$\alpha_{k,\ell}(h) = \lambda_k - \lambda_\ell$$

for all $h = \sum_{i=1}^{n} \lambda e_{i\,i} \in H$. Show that:

(a) H is a maximal toral subalgebra of L;

(b) the set of $\alpha_{k\,\ell}$'s is a root system $\mathbf{\Phi}$ of L relative to H; and

(c) $L_{\alpha_{k\,\ell}} = Fe_{k\,\ell}$ for all $1 \leq k, \ell \leq n$ and $k \neq \ell$.

[Hint: Observe that $H + \sum_{\alpha\in\mathbf{\Phi}} L_\alpha = L$. For ideas see the class notes found on the course homepage.]

4. Let L, H, and $\mathbf{\Phi}$ be as in Problem 3. For $\alpha, \beta \in \mathbf{\Phi}$, and $\beta \neq \pm\alpha$, find all α-strings through β. [Hint: Let $\beta - r\alpha, \ldots, \beta + q\alpha$ be the α-strings through β. What is a highest weight vector for the simple $sl(2, F) = S_\alpha$-module $V = L_{\beta-r\alpha} \oplus \cdots \oplus L_{\beta+q\alpha}$ and how does $\operatorname{ad} y_\alpha$ generate a basis for V?]