Solutions for 3rd Midterm

MA 415, April 2008

Exercise 1

(a) (6 points)

The incidence matrix is $A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{pmatrix}$

(b) (6 points)

We have $R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $C = R^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

and

$$K = A^t C A = A^t A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

$$\vec{f} = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} = K\vec{u}, \quad \vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix}.$$

(c) (7 points)

Solving for $\vec{u}$:

$$\left(\begin{array}{ccc|c} 2 & -1 & -1 & 2 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & -2 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 1 & -2 & 2 \\ 0 & 3 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & 4/3 \\ 0 & 1 & -1 & 2/3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

so $\underline{\underline{u_1 = 4/3}}$, $\underline{\underline{u_2 = 2/3}}$.

(d) (6 points)

We have $\vec{v} = A\vec{u} = \begin{pmatrix} 2/3 \\ 2/3 \\ 4/3 \end{pmatrix}$ and $\vec{y} = C\vec{v} = \begin{pmatrix} 2/3 \\ 2/3 \\ 4/3 \end{pmatrix}$,

so $\underline{\underline{y_2 = 2/3}}$.

Exercise 2

(a) (10 points)

The matrix representation A of L wrt $\mathcal{E}$ is

$$A = \left([L(\vec{e}_1)]_{\mathcal{E}} , [L(\vec{e}_2)]_{\mathcal{E}} \right) = \underline{\begin{pmatrix} 2 & 2 \\ -1 & 3 \end{pmatrix}}$$

(b) (10 points)

The matrix representation R of L wrt $\mathcal{B}$ is

$$R = P_{\mathcal{E}}^{\mathcal{B}} \cdot A \cdot P_{\mathcal{B}}^{\mathcal{E}} = \left(P_{\mathcal{B}}^{\mathcal{E}}\right)^{-1} A \, P_{\mathcal{B}}^{\mathcal{E}} ,$$

where $P_{\mathcal{B}}^{\mathcal{E}} = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}$ and computing the inverse :

$$\left(\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 2 & -1 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|cc} 1 & 0 & -1 & 1 \\ 0 & 1 & -2 & 1 \end{array}\right)$$

so $\left(P_{\mathcal{B}}^{\mathcal{E}}\right)^{-1} = \begin{pmatrix} -1 & 1 \\ -2 & 1 \end{pmatrix}$ whence

$$R = \begin{pmatrix} -1 & 1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ -5 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}$$

$$= \underline{\begin{pmatrix} -1 & 2 \\ -7 & 6 \end{pmatrix}}$$

Exercise 3

(a) (6 points) $p(x,y) = x^2 + 4xy + 7y^2 - 6x - 4y + 4$

$$= (x\ y)\begin{pmatrix}1 & 2\\ 2 & 7\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} - 2(x\ y)\begin{pmatrix}3\\ 2\end{pmatrix} + 4$$

(b) (7 points) $K = \begin{pmatrix}1 & 2\\ 2 & 7\end{pmatrix} = \begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}\begin{pmatrix}1 & 2\\ 0 & 3\end{pmatrix} = \underbrace{\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}}_{L}\underbrace{\begin{pmatrix}1 & 0\\ 0 & 3\end{pmatrix}}_{D}\underbrace{\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}}_{L^t}$

So since D has > 0 diagonal entries, K is positive definite.

(c) (7 points) p attains it minimum value at the point $\begin{pmatrix}x_0\\ y_0\end{pmatrix}$ which is the unique solution to $K\begin{pmatrix}x_0\\ y_0\end{pmatrix} = \vec{f}$;

$$\left(\begin{array}{cc|c}1 & 2 & 3\\ 2 & 7 & 2\end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c}1 & 2 & 3\\ 0 & 3 & -4\end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c}1 & 0 & 17/3\\ 0 & 1 & -4/3\end{array}\right),$$

So $\begin{pmatrix}x_0\\ y_0\end{pmatrix} = \begin{pmatrix}17/3\\ -4/3\end{pmatrix}$.

Exercise 4

(a) (8 points) For a 3×3 matrix $A = (a_{ij})$,

$$p_A(\lambda) = -\lambda^3 + \mathrm{trace}A\,\lambda^2 + \Big(\sum_{1\le i<j\le 3} a_{ij}a_{ji} - a_{ii}a_{jj}\Big)\lambda + \det A$$

So in our case:

$$p_A(\lambda) = -\lambda^3 + 6\lambda^2 + (1-4-4-4)\lambda$$
$$+ 2(4) - 0 - 1(0+2)$$
$$= \underline{-\lambda^3 + 6\lambda^2 - 11\lambda + 6}.$$

(b) (9 points)

By inspection we see that $p_\lambda(1) = 0$,

so $\underline{\lambda = 1}$ is an eigenvalue.

So now, either we see $p_\lambda(2) = p_\lambda(3) = 0$,

or we do long division:

$$\begin{array}{r|l} -\lambda+1 & -\lambda^3 + 6\lambda^2 - 11\lambda + 6 \quad \lambda^2 - 5\lambda + 6 \\ & -\lambda^3 + \lambda^2 \\ & \qquad 5\lambda^2 - 11\lambda + 6 \\ & \qquad 5\lambda^2 - 5\lambda \\ & \qquad\qquad -6\lambda + 6 \\ & \qquad\qquad \underline{-6\lambda + 6} \\ & \qquad\qquad\qquad 0 \end{array}$$

Solving $\lambda^2 - 5\lambda + 6 = 0$, we see that $\underline{\lambda = 2}$ and $\underline{\lambda = 3}$ are the two other eigenvalues.

(c) (8 points) We find the corresponding eigenspaces

$$V_1 = \ker(A - 1\cdot I) = \ker\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \ker\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathbb{R}\cdot\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$V_2 = \ker\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \mathbb{R}\cdot\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$V_3 = \ker\begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix} = \ker\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathbb{R}\cdot\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

So the corresponding eigenvectors are $\underline{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}$.

Exercise 5

(a) (5 points) Suppose λ is an eigenvalue with corresponding eigenvector $\vec{x} \neq \vec{0}$. Then

$$\vec{0} \neq \vec{x}\cdot\vec{x} = A\vec{x}\cdot A\vec{x} = (\lambda\vec{x})\cdot(\lambda\vec{x}) = \lambda^2(\vec{x}\cdot\vec{x}),$$

so $\lambda^2 = 1$.

(b) (5 points) $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ is orthogonal and proper, since $\det D = 1$, and has eigenvalues 1 and -1. So $\underline{-1}$ is an eigenvalue.