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AUTOMATIC CONTINUITY OF HOMOMORPHISMS AND FIXED POINTS ON METRIC COMPACTA

BY

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ABSTRACT

We prove that arbitrary homomorphisms from one of the groups

$$\mathrm{Homeo}(2^{\mathbb{N}}),\ \mathrm{Homeo}(2^{\mathbb{N}})^{\mathbb{N}},\ \mathrm{Aut}(\mathbb{Q},<),\ \mathrm{Homeo}(\mathbb{R}) \text{ or } \mathrm{Homeo}(S^1)$$

into a separable group are automatically continuous. This has consequences for the representations of these groups as discrete groups. For example, it follows, in combination with a result of V. G. Pestov, that any action of the discrete group $\mathrm{Homeo}_+(\mathbb{R})$ by homeomorphisms on a compact metric space has a fixed point.

1. Introduction

The classical theorem of Pettis [9, Theorem 9.10] says that any **Baire measurable** homomorphism from a Polish group to a separable group is continuous. Measure theoretic counterparts of this result are also known. Recently, it was proved [17] that if G is an amenable at 1 Polish group, then any **universally measurable** homomorphism from G to a separable group is continuous. All locally compact Polish groups and abelian Polish groups are amenable at 1 as are, for example, countable products of amenable locally compact Polish groups. Therefore, the classical measure theoretic automatic continuity results of Weil (locally compact groups) and Christensen (abelian groups) are contained in this theorem. Going beyond homomorphisms with regularity assumptions, of Baire category type or of measurable type, in automatic continuity results requires the

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domain group to be complicated. It was discovered recently [11] that each homomorphism whose domain is a Polish group with ample generic elements and whose range is separable is continuous. A Polish group G **has ample generics** if for each finite $n \geq 1$ the diagonal conjugation action of G on G^n given by

$$g \cdot (h_1, \ldots, h_n) = (gh_1g^{-1}, \ldots, gh_ng^{-1})$$

has a comeagre orbit. Examples of groups with ample generics can be found among permutation groups; most importantly $\mathcal{S}_\infty$, the group of all permutations of $\mathbb{N}$, is such a group (see [11]).

The principal goal of this paper is to exhibit groups which are less complicated than those with ample generics but which still have this very strong automatic continuity property. However, our results have broader consequences: for extremely amenable groups and for questions concerning representing groups as subgroups of $\mathcal{S}_\infty$ or as linear groups.

The main tool in our proofs of automatic continuity is a version of the classical fact due to Steinhaus that if $A \subseteq \mathbb{R}$ is a measurable set of positive Lebesgue measure, then $0 \in \mathrm{Int}(A - A)$. (Analogous lemmas were proved later by Pettis for non-meager subsets A of Polish groups with A having the Baire property and by Weil for non-Haar zero Haar measurable subsets of locally compact groups.)

Definition 1: Let G be a topological group. We say that G is **Steinhaus** if there is $k \geq 1$ such that for any symmetric countably syndetic set $W \subseteq G$ (i.e., covering G by countably many left-translates), W^k contains an open neighbourhood of 1_G. To emphasise the exponent k, we will sometimes say that G is Steinhaus with exponent k.

The first examples of Steinhaus groups come from [11] where it was shown that Polish groups with ample generics are Steinhaus with exponent 10.

The proposition below makes a connection between the Steinhaus property and automatic continuity. Its proof is analogous to the derivation of continuity of Baire measurable homomorphisms from Pettis' lemma.

Proposition 2: *Let G be a Steinhaus topological group and $\pi\colon G \to H$ a homomorphism into a separable group. Then π is continuous.*

Proof: We need only show that π is continuous at 1_G. So suppose $U \subseteq H$ is an open neighbourhood of 1_H and find some symmetric open V such that $1_H \in V \subseteq V^{2k} \subseteq U \subseteq H$. As H is separable, V covers H by countably many translates $\{h_nV\}$. So for each h_nV intersecting $\pi[G]$ take some $g_n \in G$

such that $\pi(g_n) \in h_n V$. Then $h_n V \subseteq \pi(g_n)V^{-1}V = \pi(g_n)V^2$, and hence the $\pi(g_n)V^2$ cover $\pi[G]$. Now, if $g \in G$, find n such that $\pi(g) \in \pi(g_n)V^2$, whence $\pi(g_n^{-1}g) \in V^2$ and $g_n^{-1}g \in \pi^{-1}[V^2]$, i.e., the $g_n\pi^{-1}[V^2]$ cover G.

So $W = \pi^{-1}[V^2]$ is symmetric and countably syndetic and hence $1_G \in \mathrm{Int}(W^k)$ for some $k \geq 1$. But then $\pi[W^k] \subseteq V^{2k} \subseteq U$, and therefore, $1_G \in \mathrm{Int}(\pi^{-1}[U])$ and π is continuous at 1_G. ∎

We see that the exact condition we need to impose on H in the above proof is that any non-empty open set covers H by countably many translates. This condition is known in the literature as $\aleph_0$-boundedness and is equivalent to embedding as a subgroup into a direct product of second countable groups (see Guran [6]). Thus, by the definition of the product topology, to show that any homomorphism with range in an $\aleph_0$-bounded group is continuous, it is enough to show that any homomorphism into a second countable group is continuous. So from our perspective the three notions of second countable, separable, and $\aleph_0$-bounded are equivalent.

Let us mention an immediate corollary.

Corollary 3: *Let G and H be Polish groups. If G is Steinhaus and H is an image of G by a homomorphism, then H is Steinhaus.*

Proof: The homomorphism between G and H is continuous, by Proposition 2, and therefore, since it is surjective and since G and H are Polish, it is open. Now it follows that H is Steinhaus by a straightforward computation. ∎

The main result of this paper is the following.

Theorem 4: *The following groups are Steinhaus:*

$$\mathrm{Homeo}(2^{\mathbb{N}}), \mathrm{Homeo}(2^{\mathbb{N}})^{\mathbb{N}}, \mathrm{Aut}(\mathbb{Q},<), \mathrm{Homeo}_+(\mathbb{R}), \text{ and } \mathrm{Homeo}_+(S^1).$$

Here, $\mathrm{Homeo}_+(\mathbb{R})$ is the group of increasing homeomorphisms of $\mathbb{R}$, and similarly $\mathrm{Homeo}_+(S^1)$ is the group of orientation preserving homeomorphisms of the unit circle, both of them with the topology of uniform (or equivalently, pointwise) convergence.

A crucial role in our arguments will be played by the existence of comeager conjugacy classes in $\mathrm{Aut}(\mathbb{Q},<)$ [19], [13], and in $\mathrm{Homeo}(2^{\mathbb{N}})$ and $\mathrm{Homeo}_+(\mathbb{R})$ [11]. Of course, having ample generics implies having a comeager conjugacy class, however, it is known that $\mathrm{Aut}(\mathbb{Q},<)$ and $\mathrm{Homeo}(\mathbb{R})$ do not have ample generics; whether $\mathrm{Homeo}(2^{\mathbb{N}})$ has ample generics is open. All conjugacy classes of $\mathrm{Homeo}_+(S^1)$ are meager.

Corollary 5: *An arbitrary homomorphism from either*

$$\mathrm{Homeo}(2^{\mathbb{N}}),\ \mathrm{Homeo}(2^{\mathbb{N}})^{\mathbb{N}},\ \mathrm{Aut}(\mathbb{Q},<),\ \mathrm{Homeo}(\mathbb{R}),\ \textit{or}\ \mathrm{Homeo}(S^1)$$

into a separable group is continuous.

Proof: The result is clear for the three first groups. And for the last, notice that any homomorphism π: $\mathrm{Homeo}(\mathbb{R}) \to H$ into a separable group restricts to a continuous homomorphism from $\mathrm{Homeo}_+(\mathbb{R})$ into H, whence π is also continuous as $\mathrm{Homeo}_+(\mathbb{R})$ is open in $\mathrm{Homeo}(\mathbb{R})$. Similarly for $\mathrm{Hom}(S^1)$. ∎

Methods used to prove Theorem 4 yield also the following result on the small index property.

Theorem 6: $\mathrm{Homeo}_+(\mathbb{R})$ *is the only proper subgroup of* $\mathrm{Homeo}(\mathbb{R})$ *of index* $< 2^{\aleph_0}$. *Similarly,* $\mathrm{Homeo}_+(S^1)$ *is the only proper subgroup of* $\mathrm{Homeo}(S^1)$ *of index* $< 2^{\aleph_0}$.

2. Applications

2.1 Connections with extreme amenability. A phenomenon that has recently received considerable attention in topological dynamics is extreme amenability [16]. A topological group is called **extremely amenable** if all of its continuous actions on compact Hausdorff spaces have fixed points. Such groups are also said to have the **fixed point on compacta property**. The first examples of these groups date back to work of Herer and Christensen [7]. Of special interest to us are the results of Pestov [15] stating that $\mathrm{Aut}(\mathbb{Q},<)$ and $\mathrm{Homeo}_+(\mathbb{R})$ are extremely amenable.

All these examples belong necessarily to the topological setting as Ellis [4] proved that any abstract (that is, discrete) group acts freely on a compact space. (This was generalised later by Veech [21] to locally compact groups.) Therefore, non-trivial discrete groups are not extremely amenable. However, using our results on automatic continuity, we will see that when one restricts the attention to actions on metrisable compacta, extreme amenability type phenomena occur for abstract groups.

Corollary 7: *An arbitrary action by homeomorphisms of* $\mathrm{Aut}(\mathbb{Q},<)$ *or* $\mathrm{Homeo}_+(\mathbb{R})$ *on a compact metrisable space has a fixed point.*

Proof: It is enough to notice that any action by homeomorphisms on a compact metric space X corresponds to a homomorphism into the homeomorphism group of X, which is Polish. By automatic continuity the homomorphism is continuous, whence the action is continuous. So, by the results of Pestov, there is a fixed point on X. ∎

A continuous action of a Hausdorff topological group G on compact Hausdorff spaces is called a **G-flow**. Such a flow is called **minimal** if each of its orbits is dense and it is a general fact that each flow contains a minimal subflow. For any (Hausdorff) group G there is a **universal minimal flow**, that is, a continuous minimal action of G on a compact (Hausdorff) space X such that any minimal G-flow is a homomorphic continuous image of this one. So, extremely amenable groups are precisely those groups whose minimal flows are one point spaces. There are Polish groups that are not extremely amenable, but for which one nevertheless can compute the universal minimal flow, and for some groups these flows turn out to be metrisable. For example, Pestov showed that the universal minimal flow of $\mathrm{Homeo}_+(S^1)$ is simply the canonical action of $\mathrm{Homeo}_+(S^1)$ on S^1. Glasner and Weiss [5] computed the universal minimal flow of $\mathrm{Homeo}(2^{\mathbb{N}})$ to be equal to the canonical action on the space of maximal chains of compact subsets of $2^{\mathbb{N}}$, a space which was first introduced and studied by Uspenskij [20]. In general, there is no reason for a topological Hausdorff group to have a universal minimal **metrisable** flow, that is, a metrisable flow which maps homomorphically onto any metrisable flow. However, using the above results of Pestov and Glasner–Weiss and our automatic continuity result, we obtain the existence of such universal minimal metrisable flows for groups whose universal minimal flows are non-metrisable.

Corollary 8: *The discrete groups* $\mathrm{Homeo}_+(S^1)$ *and* $\mathrm{Homeo}(2^{\mathbb{N}})$ *have universal minimal* **metrisable** *flows, namely the canonical action on* S^1 *and the canonical action on the space of maximal chains of compact subsets of* $2^{\mathbb{N}}$*, respectively.*

Note that the universal minimal flows of the discrete groups considered in the corollary above are non-metrisable, as Kechris, Pestov and Todorcevic proved in [10] that the universal minimal flow of any locally compact, non-compact group is non-metrisable.

2.2 Connections with representations. There is some interest in representing groups by (faithful) actions on countable sets which corresponds to embedding the groups into $\mathcal{S}_\infty$. The maximal size of such groups is $2^{\aleph_0}$, and it is well-known that not every group of size at most $2^{\aleph_0}$ is embeddable into $\mathcal{S}_\infty$. For example, the quotient of $\mathcal{S}_\infty$ by its subgroup of finitary permutations is not so embeddable. Vladimir Pestov recently informed us that John D. Dixon had asked for a characterisation of the separable (Hausdorff) topological groups that were abstractly embeddable into $\mathcal{S}_\infty$, and, in particular, he asked if there was any counter-example at all. Theorem 6 shows that this is so. For clearly, if $\mathrm{Homeo}_+(\mathbb{R})$ acts on a set X of size $\kappa < 2^{\aleph_0}$, then the isotropy subgroup of $\mathrm{Homeo}_+(\mathbb{R})$ at some $x \in X$ is of index $\leq \kappa$ and hence has to be the whole group. Thus $\mathrm{Homeo}_+(\mathbb{R})$ has no non-trivial representations by permutations on a set of size $< 2^{\aleph_0}$ whatsoever.

Consider now a more general type of representations namely linear representations. Megrelishvili [14], proving a conjecture of Pestov, has shown that $\mathrm{Homeo}_+(\mathbb{R})$ has no non-trivial strongly continuous representations by linear isometries on a reflexive Banach space. Thus by Theorem 4, we see that that $\mathrm{Homeo}_+(\mathbb{R})$ has no representations by linear isometries on a reflexive separable Banach space.

Corollary 9: $\mathrm{Homeo}_+(\mathbb{R})$, *as an abstract group, has no non-trivial representations by permutations on a set of size* $< 2^{\aleph_0}$. *Moreover, it has no representations by linear isometries on a reflexive separable Banach space.*

2.3 Homomorphisms into locally compact Polish groups. Pestov's theorem that there are no fixed point free continuous actions of $\mathrm{Aut}(\mathbb{Q},<)$ or $\mathrm{Homeo}_+(\mathbb{R})$ on compact spaces and Veech's theorem that each locally compact group acts freely and continuously on a compact space in connection with our automatic continuity result (and the remarks following Proposition 2) imply that there is no abstract non-trivial homomorphism from these groups to locally compact σ-compact groups. (Note that for locally compact groups $\aleph_0$-boundedness coincides with σ-compactness.) Similarly, a non-trivial homomorphism from a group G into a locally compact group H, induces a representation of G by linear isometries on L^2 of the Haar measure of H, which is separable if H is second countable. So the slightly weaker result that $\mathrm{Homeo}_+(\mathbb{R})$ has no non-trivial homomorphism into a second countable locally compact group, also follows from Corollary 9.

We show now that the same result holds for $\mathrm{Homeo}(2^{\mathbb{N}})$. In fact, the argument for it is direct and applies also to $\mathrm{Aut}(\mathbb{Q},<)$ and $\mathrm{Homeo}_+(\mathbb{R})$. We will use the following property of an abstract group F first studied by Bergman [1]:

Whenever $W_0 \subseteq W_1 \subseteq \cdots \subseteq F$ is an exhaustive sequence of subsets, then for some n and k, $F = W_n^k$.

That this condition holds for $\mathrm{Homeo}(2^{\mathbb{N}})$ is due to Droste and Göbel [2]. It also holds for $\mathrm{Aut}(\mathbb{Q},<)$ and $\mathrm{Homeo}(\mathbb{R})$ as proved by Droste and Holland [3].

THEOREM 10: *Let G be a Polish group which has the above property and has a comeager conjugacy class. Then there is no non-trivial abstract homomorphism from G into a locally compact σ-compact group.*

Proof: Suppose $\pi: G \to H$ is a homomorphism into a locally compact Polish group. We claim that $\overline{\pi[G]}$ is compact. To see this take some increasing exhaustive sequence of compact subsets of H:

$$K_0 \subseteq K_1 \subseteq \cdots \subseteq H.$$

Then

$$(K_0 \cap \pi[G]) \subseteq (K_1 \cap \pi[G]) \subseteq \cdots \subseteq \pi[G]$$

is also exhaustive, so, since G is Bergman, there are n and k such that $\pi[G] = (K_n \cap \pi[G])^k \subseteq K_n^k$. But then $\pi[G]$ is relatively compact, whence $\overline{\pi[G]}$ is compact.

Secondly, we claim that the group $K = \overline{\pi[G]}$ has a dense conjugacy class. For, let $C \subseteq G$ be the comeagre conjugacy class of G and suppose $V \subseteq K$ is some non-empty open set. We claim that $V \cap \pi[C] \neq \emptyset$. This suffices as $\pi[C]$ is contained in a single conjugacy class of K. First notice that as K is compact and $\pi[G]$ is dense, there are $\{g_i\}_{i\leq n} \subseteq G$ such that $K = \bigcup_{i\leq n} \pi(g_i)V$, whence $G = \bigcup_{i\leq n} g_i\pi^{-1}(V)$. As C is comeagre, $\bigcap_{i\leq n} g_iC \neq \emptyset$, so take some $h \in \bigcap_{i\leq n} g_iC \neq \emptyset$ and let m be such that $h \in g_m\pi^{-1}(V)$. Then clearly, $g_m^{-1}h \in \pi^{-1}(V) \cap C$, and thus $\pi(g_m^{-1}h) \in V \cap \pi[C]$.

Therefore $\overline{\pi[G]}$ is compact with a dense conjugacy class. But any conjugacy class in a compact Hausdorff group is closed, so $\overline{\pi[G]} = \{1\}$ as the conjugacy class of 1 is $\{1\}$. ∎

The result above implies that groups with Bergman's property and with a comeager conjugacy class cannot have non-trivial representations by automorphisms of locally finite graphs. Similarly, they cannot act non-trivially by isometries on compact metric spaces.

3. $\mathrm{Homeo}(2^{\mathbb{N}})$ and $\mathrm{Homeo}(2^{\mathbb{N}})^{\mathbb{N}}$

Before we begin our proofs, let us first mention the following elementary fact, which will be used repeatedly.

LEMMA 11: *Suppose $n = 1, 2, \ldots, \aleph_0$ and $\{G_i\}_{i\leq n}$ are Polish groups with comeagre conjugacy classes. Then $G = \prod_{i\leq n} G_i$ has a comeagre conjugacy class.*

Proof: Let, for each $i \le n$, $\mathcal{O}_i \subseteq G_i$ be the comeagre conjugacy class. Then obviously, $\prod_{i \le n} \mathcal{O}_i \subseteq G$ is a conjugacy class of G. Moreover, as

$$\prod_{i \le n} \mathcal{O}_i = \bigcap_{i \le n} \Big[\Big(\prod_{j \ne i} G_j \Big) \times \mathcal{O}_i \Big]$$

by Kuratowski-Ulam this class is comeagre in G. ∎

THEOREM 12: Homeo($2^{\mathbb{N}}$) *is Steinhaus.*

Proof: Let $G = \mathrm{Homeo}(2^{\mathbb{N}})$ and assume $W \subseteq G$ is symmetric and covers G by countably many left-translates $k_n W$, $k_n \in G$. In particular, W cannot be meagre and must therefore be dense in some non-empty open set in G. But then $W^{-1}W = W^2$ is dense in an open neighbourhood of the identity in G and we can therefore find some finite subalgebra $\mathsf{A} \subseteq \mathrm{clopen}(2^{\mathbb{N}})$ with atoms $A_1, \dots, A_p$ such that W^2 is dense in $G_{(\mathsf{A})} = \{g \in G \mid g[A_i] = A_i, i = 1, \dots, p\}$.

For each $i = 1, \dots, p$ choose a point $x_i \in A_i$ and let $B_n^i \subseteq A_i$ be a sequence of disjoint clopen sets converging in the Hausdorff metric to the set $\{x_i\}$. Moreover, let $B_n = B_n^1 \cup \dots \cup B_n^p$.

CLAIM 1: *For some n, B_n is full for W^2, i.e., if*

$$\gamma \in \{g \in \mathrm{Homeo}(B_n) : g[B_n^i] = B_n^i, i = 1, \dots p\},$$

then there is a $g \in W^2$ such that $g \restriction B_n = \gamma$.

Proof of Claim 1: It suffices to show that some B_n is full for $k_n W$ since then it is clearly also full for $W^2 = (k_n W)^{-1} k_n W$. Assuming otherwise, we can find for each n, some $\gamma_n \in \mathrm{Homeo}(B_n)$ such that for all $g \in k_n W$, $g \restriction B_n \ne \gamma_n$. Due to the convergence of the sets B_n to $\{x_1, \dots, x_p\}$, there is a $f \in G$ such that $f \restriction B_n = \gamma_n$ and

$$f \restriction \Big(2^{\mathbb{N}} \setminus \bigcup_n B_n \Big) = \mathrm{id}_{2^{\mathbb{N}} \setminus \bigcup_n B_n}.$$

But then by the choice of γ_n, $f \notin k_n W$ for any n, contradicting that the sets $k_n W$ cover G, and the claim is proved. ∎

So we can choose some $B = B_{n_0}$ that is full for W^2. Let now $G(B) = \{g \in G_{(\mathsf{A})} \mid \mathrm{supp}(g) \subseteq B\}$.

Claim 2: $G(B) \subseteq W^{12}$.

Proof of Claim 2: To see this, we notice first that $G(B)$ is topologically isomorphic to $\mathrm{Homeo}(2^{\mathbb{N}})^p$ and, therefore by [11] and Lemma 11, it has a comeagre conjugacy class. Find now some n_1 such that $k_{n_1}W \cap G(B)$ is non-meagre in $G(B)$. Then

$$\begin{aligned}(k_{n_1}W \cap G(B))^{-1} \cdot (k_{n_1}W \cap G(B)) &= (W^{-1}k_{n_1}^{-1} \cap G(B)) \cdot (k_{n_1}W \cap G(B)) \\ &\subseteq W^2 \cap G(B)\end{aligned}$$

is also non-meagre in $G(B)$, so we can find $f_0 \in W^2 \cap G(B)$, whose conjugacy class in $G(B)$ is comeagre in $G(B)$.

Take any $h \in G(B)$ and, by fullness of B for W^2 proved in Claim 1, find $g \in W^2$ such that $h \restriction B = g \restriction B$. Then as $f_0 \restriction \complement B = \mathrm{id}_{\complement B}$, we have

$$hf_0h^{-1} = gf_0g^{-1} \in W^6.$$

This means that the conjugacy class of f_0 in $G(B)$ is contained in W^6 and that therefore $W^6 \cap G(B)$ is comeagre in $G(B)$, whence, as the square of a comeagre set is everything, $W^{12} \cap G(B) = G(B)$, proving the claim. ∎

Since $W^2 \cap G_{(A)}$ is dense in $G_{(A)}$, we can pick an $h_0 \in W^2 \cap G_{(\mathsf{A})}$ such that $h_0[B] = \complement B$, i.e., such that $h_0[B \cap A_i] = \complement B \cap A_i$ for each $i = 1, \dots, p$. Then clearly,

$$\begin{aligned}G(\complement B) &= \{g \in G_{(\mathsf{A})} \mid \mathrm{supp}(g) \subseteq \complement B\} \\ &= h_0 G(B) h_0^{-1} \\ &\subseteq W^{16}\end{aligned}$$

and

$$G_{(\mathsf{B})} = G(B) \cdot G(\complement B) \subseteq W^{28}$$

where B is the subalgebra of $\mathrm{clopen}(2^{\mathbb{N}})$ generated by A and the set B. So W^{28} contains the open neighbourhood of the identity, $G_{(\mathsf{B})}$, and $\mathrm{Homeo}(2^{\mathbb{N}})$ is Steinhaus with exponent 28. ∎

We now show how techniques similar to the ones employed above give that the group $\mathrm{Homeo}(2^{\mathbb{N}})^{\mathbb{N}}$ is Steinhaus. This result implies the previous one by Corollary 3. Moreover, notice that it is not in general true that the countable product of Steinhaus groups is Steinhaus. The simplest counter-example is $(\mathbb{Z}/2)^{\mathbb{N}}$. Here an ultrafilter on $\mathbb{N}$ corresponds to a subgroup of index 2, which is open if and only if the ultrafilter is principal. So $(\mathbb{Z}/2)^{\mathbb{N}}$ is not Steinhaus, even though the discrete group $\mathbb{Z}/2$ is. However, it seems plausible that if W is a symmetric countably syndetic subset of a product $\Pi_i G_i$ of Steinhaus topological groups of some common exponent k, then W^k should contain a product $\Pi_i U_i$ of open subsets of the G_i. We have neither a proof nor a counter-example.

THEOREM 13: $\mathrm{Homeo}(2^{\mathbb{N}})^{\mathbb{N}}$ *is Steinhaus.*

Proof: The group $\mathrm{Homeo}(2^{\mathbb{N}})^{\mathbb{N}}$ is isomorphic to the subgroup G of $\mathrm{Homeo}(2^{\mathbb{N}}\times\mathbb{N})$ consisting of all $h\in\mathrm{Homeo}(2^{\mathbb{N}}\times\mathbb{N})$ with $h[2^{\mathbb{N}}\times\{i\}]=2^{\mathbb{N}}\times\{i\}$ for each $i\in\mathbb{N}$. Put $K_i=2^{\mathbb{N}}\times\{i\}$. Let $W\subseteq G$ be such that $G=\bigcup_n k_nW$ for some $k_n\in G$, $n\in\mathbb{N}$.

We will borrow two things from the proof of Theorem 12. First note that the proof of Theorem 12 gives that for any $m\in\mathbb{N}$ a relatively open neighbourhood of the identity of the subgroup

$$\Big\{f\in G \;\Big|\; \mathrm{supp}(f)\subseteq\bigcup_{i<m}K_i\Big\}$$

is contained in W^{28}. For this reason, it will suffice to prove that there exists m, perhaps depending on W and the sequence (k_n), such that

$$\Big\{f\in G \;\Big|\; \mathrm{supp}(F)\subseteq\bigcup_{m\leq i}K_i\Big\}\subseteq W^{108}.$$

Second, note that the following claim can be proved just like Claim 2 in the proof of Theorem 12. The only difference is that in an appropriate place we need to use Lemma 11 with $n=\aleph_0$.

CLAIM 1: *Let* $U_i^n\subseteq K_n$ *for* $i\leq n$ *be pairwise disjoint clopen sets. There exists* n_0 *such that*

$$\Big\{f\in G \;\Big|\; \mathrm{supp}(f)\subseteq\bigcup_{n\geq n_0}U_{n_0}^n\Big\}\subseteq W^{12}.$$

Now, let $\mathcal{U},\mathcal{U}'$ be families of pairwise disjoint clopen subsets of $\bigcup_n K_n$. We say that $\mathcal{U}'$ **refines** $\mathcal{U}$ if $\bigcup\mathcal{U}=\bigcup\mathcal{U}'$ and each set in $\mathcal{U}'$ is included in a set from $\mathcal{U}$. If $\mathcal{U}'$ refines $\mathcal{U}$ and σ and τ are permutations of $\mathcal{U}$ and $\mathcal{U}'$, respectively, we say that τ **refines** σ if $\tau(U')\subseteq\sigma(U)$ whenever $U'\subseteq U$ for $U'\in\mathcal{U}'$ and $U\in\mathcal{U}$. Finally, if $\mathcal{U}$ refines $\{K_i \mid i\geq n\}$ for some $n\in\mathbb{N}$, let $\mathrm{Sym}_0(\mathcal{U})$ be the group of all permutations of $\mathcal{U}$ which refine the identity permutation of $\{K_i \mid i\geq n\}$.

Claim 2: *There exist* $n_0 \in \mathbb{N}$ *and a family of clopen sets* $\mathcal{U}$ *refining* $\{K_i \mid i \geq n_0\}$ *such that for any* $\mathcal{U}'$ *refining* $\mathcal{U}$ *and any* $\tau \in \mathrm{Sym}_0(\mathcal{U}')$ *refining* $\mathrm{id} \in \mathrm{Sym}_0(\mathcal{U})$ *we can find* $h \in W^2$ *with*

$$h[U] = \tau(U) \quad \textit{for any } U \in \mathcal{U}'.$$

Proof of Claim 2: It suffices to find $n_0 \in \mathbb{N}$, a family of clopen sets $\mathcal{U}$ refining $\{K_i \mid i \geq n_0\}$, and $\sigma \in \mathrm{Sym}_0(\mathcal{U})$ such that for any $\mathcal{U}'$ refining $\mathcal{U}$ and any $\tau \in \mathrm{Sym}_0(\mathcal{U}')$ refining σ we can find $h \in k_{n_0}W$ with

$$h[U] = \tau(U) \quad \text{for any } U \in \mathcal{U}'.$$

Indeed, the claim follows from the statement above since

$$W^2 = (k_{n_0}W)^{-1}k_{n_0}W \quad \text{and} \quad \sigma^{-1} \circ \sigma = \mathrm{id} \in \underset{0}{\mathrm{Sym}}(\mathcal{U}).$$

Assume towards a contradiction that the statement fails. Let $\mathcal{U}_0$ be $\{K_i \mid i \geq 0\}$ and let $\tau_0 \in \mathrm{Sym}_0(\mathcal{U}_0)$ be the identity. Assume we are given $\mathcal{U}_n$ refining $\{K_i \mid i \geq n\}$ and $\tau_n \in \mathrm{Sym}(\mathcal{U}_n)$ such that there is no $h \in k_{n-1}W$ with $h[U] = \tau(U)$ for all $U \in \mathcal{U}_n$. Consider

$$\mathcal{U}'_n = \Big\{U \in \mathcal{U}_n \;\Big|\; U \subseteq \bigcup_{i \geq n+1} K_i\Big\} \quad \text{and} \quad \tau'_n = \tau_n \restriction \mathcal{U}'_n.$$

By our assumption, we can find $\mathcal{U}_{n+1}$ refining $\mathcal{U}'_n$ and $\tau_{n+1} \in \mathrm{Sym}_0(\mathcal{U}_{n+1})$ refining τ'_n such that for no $h \in k_{n+1}W$ do we have $h[U] = \tau(U)$ for all $U \in \mathcal{U}_{n+1}$.

The inductive construction allows us to find $h_0 \in G$ such that for each $U \in \mathcal{U}_n$ with $U \subseteq K_n$ we have $h_0[U] = \tau_n(U)$. Note that, again by the construction, $h_0 \notin k_nW$ for each n. This yields a contradiction since $\bigcup_n k_nW = G$, and the claim follows.

Claim 3: *Let* $B, C \subseteq 2^{\mathbb{N}}$ *be clopen. Assume that* $C \cap B$ *and* $C \setminus B$ *are both non-empty. Let*

$$G_1 = \{f \in \mathrm{Homeo}(2^{\mathbb{N}}) \mid f[B] = B\} \textit{ and } G_2 = \{f \in \mathrm{Homeo}(2^{\mathbb{N}}) \mid \mathrm{supp}(f) \subseteq C\}.$$

Then $\mathrm{Homeo}(2^{\mathbb{N}}) = G_1G_2G_1G_2G_1$.

Proof of Claim 3: Let $f \in \mathrm{Homeo}(2^{\mathbb{N}})$. Consider the clopen sets $L_1 = f[B] \cap B$ and $L_2 = f[\complement B] \cap \complement B$. Assume both of them are non-empty. This assumption allows us to find $g_1 \in G_1$ such that

$$g_1[B \setminus L_1] \subsetneq C \cap B \quad \text{and} \quad g_1[\complement B \setminus L_2] \subsetneq C \cap \complement B.$$

Now there exists $g_2 \in G_2$ such that

$$g_2[g_1[B \setminus L_1]] \subseteq C \cap \complement B \quad \text{and} \quad g_2[g_1[\complement B \setminus L_2]] \subseteq C \cap B$$

and

$$g_2[C \setminus g_1[B \setminus L_1]] \subseteq C \cap B \quad \text{and} \quad g_2[C \setminus g_1[\complement B \setminus L_2]] \subseteq C \cap \complement B.$$

Note that $f(g_2g_1)^{-1} \in G_1$, so the conclusion follows.

Assume now that L_1 or L_2 is empty, say $L_1 = \emptyset$. Then note that $\complement B \setminus L_2 \neq \emptyset$. Let $g_1 \in G_1$ be such that

$$g_1[\complement B \setminus L_2] \cap C \neq \emptyset.$$

If now $g_2 \in G_2$ is such that $g_2[C \setminus B] = C \cap B$, then clearly the sets L_1 and L_2 computed for $f(g_2g_1)^{-1}$ are both non-empty, so we can apply the previous procedure to get the conclusion of the claim.

We prove now the theorem from the three claims. Pick n_0 and $\mathcal{U}$ as in Claim 2. For $n \geq n_0$, pick pairwise disjoint non-empty clopen sets $V_i^n \subseteq K_n$ with $i \leq n$ in such a way that for each $U \in \mathcal{U}$ with $U \subseteq K_n$ and each $i \leq n$ we have $U \cap V_i^n \neq \emptyset$. We assume no V_i^n contains a $U \in \mathcal{U}$. By Claim 1, there is $n_1 \geq n_0$ such that

$$\left\{ f \in G \;\middle|\; \mathrm{supp}(f) \subseteq \bigcup_{n \geq n_1} V_{n_1}^n \right\} \subseteq W^{12}.$$

For $U \in \mathcal{U}$ with $U \subseteq K_n$ for some $n \geq n_1$, put

$$U^0 = V_{n_1}^n \cap U \quad \text{and} \quad U^1 = U \setminus V_{n_1}^n.$$

Let

$$B_0 = \bigcup \{ U^0 \mid U \in \mathcal{U} \text{ and } U \subseteq \bigcup_{n \geq n_1} K_n \}$$

and

$$B_1 = \bigcup \{ U^1 \mid U \in \mathcal{U} \text{ and } U \subseteq \bigcup_{n \geq n_1} K_n \}.$$

So we have

$$\{ f \in G \mid \mathrm{supp}(f) \subseteq B_0 \} \subseteq W^{12}. \tag{1}$$

Note that $\mathcal{U}' = \{U^0, U^1 \mid U \in \mathcal{U}\}$ refines $\mathcal{U}$ and $\tau \in \mathrm{Sym}_0(\mathcal{U}')$ given by $\tau(U^j) = U^{1-j}$ refines id. Therefore, by Claim 2, we have $h_0 \in W^2$ such that

$h_0[U^j] = \tau(U^j)$ for each $U^j \in \mathcal{U}'$. It follows that $h_0[B_0] = B_1$ and $h_0[B_1] = B_0$, whence from (1)

$$\begin{aligned}\{f \in G \mid \operatorname{supp}(f) \subseteq B_1\} &\subseteq h_0^{-1}\{f \in G \mid \operatorname{supp}(f) \subseteq B_0\}h_0 \\ &\subseteq W^2W^{12}W^2 = W^{16}.\end{aligned} \tag{2}$$

For $n \geq n_1$ and all $i \leq n$ pick pairwise disjoint clopen sets $C_i^n \subseteq K_n$ so that each C_i^n intersects both B_0 and B_1. Applying Claim 1, we see that there is $n_2 \geq n_1$ such that

$$\Big\{f \in G \mid \operatorname{supp}(f) \subseteq \bigcup_{n \geq n_2} C_{n_2}^n\Big\} \subseteq W^{12}. \tag{3}$$

Now using Claim 3 for each $n \geq n_2$ (with $B = B_0 \cap K_n$ and $C = C_{n_2}^n$) along with (1), (2), and (3), we get

$$\Big\{f \in G \mid \operatorname{supp}(f) \subseteq \bigcup_{n \geq n_2} K_n\Big\} \subseteq W^{28}W^{12}W^{28}W^{12}W^{28} = W^{108},$$

and the theorem follows. ∎

Let us now see how this result leads to automatic continuity for other groups containing $\mathrm{Homeo}(2^{\mathbb{N}})^{\mathbb{N}}$. Fix a denumerable model-theoretical structure A and suppose that $\mathrm{Aut}(\mathsf{A})$ is Steinhaus, or just that any homomorphism from $\mathrm{Aut}(\mathsf{A})$ into a separable group is continuous. We can assume that the domain of A is $\mathbb{N}$. Now let α be an action of $\mathrm{Aut}(\mathsf{A})$ on $\mathrm{Homeo}(2^{\mathbb{N}})^{\mathbb{N}}$ defined as follows: $\alpha(g, [n \mapsto h_n]) = [n \mapsto h_{g(n)}]$. Thus, we can form the topological semidirect product $\mathrm{Aut}(\mathsf{A}) \ltimes_\alpha \mathrm{Homeo}(2^{\mathbb{N}})^{\mathbb{N}}$. Recall that the topology on the semidirect product is the same as the product topology on $\mathrm{Aut}(\mathsf{A}) \times \mathrm{Homeo}(2^{\mathbb{N}})^{\mathbb{N}}$. Now, if K is a separable group and $\pi\colon \mathrm{Aut}(\mathsf{A}) \ltimes_\alpha \mathrm{Homeo}(2^{\mathbb{N}})^{\mathbb{N}} \to K$ is a homomorphism, then π restricts to a continuous homomorphism on each of the factors, whence π is continuous on the semi-direct product. When A is just the empty structure we have:

Corollary 14: *Let X be the topological space $\mathbb{N} \times 2^{\mathbb{N}}$ and E the equivalence relation on X given by $(n, \alpha)\mathsf{E}(m, \beta) \iff n = m$. Then any homomorphism from $\mathrm{Homeo}(X, \mathsf{E})$ (i.e., the group of homeomorphisms of X preserving the equivalence relation E) into a separable group is continuous.*

4. Aut$(\mathbb{Q},<)$

Theorem 15: Aut$(\mathbb{Q},<)$ *is Steinhaus.*

Our proof of this result relies on the combinatorics of Truss' proof from [18] that Aut$(\mathbb{Q},<)$ satisfies the so called small index property, that is, that every subgroup of index strictly less than the continuum is open. Let $G = \mathrm{Aut}(\mathbb{Q},<)$ and let $\mathbb{D}$ be the family of all subsets $X \subseteq \mathbb{Q}$ of the form

$$X = \bigcup_{n\in\mathbb{Z}}]x_{2n}, x_{2n+1}[,$$

where $(x_n)_{n\in\mathbb{Z}}$ is a sequence of irrationals satisfying $x_n < x_{n+1}$ and $x_n \to \pm\infty$ for $n \to \pm\infty$. Moreover, for $X \in \mathbb{D}$ we let

$$A(X) = \{g \in G \mid \mathrm{supp}(g) \subseteq X\}.$$

Since any element $g \in \mathrm{Aut}(\mathbb{Q},<)$ can be extended to a unique homeomorphism of $\mathbb{R}$, we will sometimes evaluate expressions $g(x)$ for $g \in G$ and x an irrational number.

The following lemma can be extracted from [18].

Lemma 16 (Truss):

$$G = \bigcup_{X,Y\in\mathbb{D}} A(X)\cdot A(Y).$$

Proof: Given $g \in G$, find a sequence $(x_n)_{n\in\mathbb{Z}}$ of irrationals such that $x_{n-1} < g(x_n) < x_{n+1}$ and $x_n \to \pm\infty$ for $n \to \pm\infty$. Now, put $I_n =]x_n, x_{n+1}[$ and notice by the choice of x_n that

$$g([x_{4n}, x_{4n+1}]) \subseteq]x_{4n-1}, x_{4n+2}[.$$

So we can define some $h \in G$ such that for each $n \in \mathbb{Z}$

$$h\restriction]x_{4n+2}, x_{4n+3}[= \mathrm{id}$$
$$h\restriction g(I_{4n}) = g^{-1}.$$

Then $hg\restriction I_{4n} = \mathrm{id}$ and $h^{-1}\restriction I_{4n+2} = \mathrm{id}$. Letting

$$Y = \bigcup_{n\in\mathbb{Z}} I_{4n+1}\cup I_{4n+2}\cup I_{4n+3} \quad\text{and}\quad X = \bigcup_{n\in\mathbb{Z}} I_{4n}\cup I_{4n+1}\cup I_{4n+3}$$

we have $g = h^{-1}\cdot hg \in A(X)\cdot A(Y)$. ∎

Proof of Theorem 15: Suppose $W \subseteq G$ is symmetric and countably syndetic. Then W cannot be meagre and hence W^2 must be dense in some open neighbourhood of the identity, $V = \{g \in G \mid g(q_1) = q_1, \dots, g(q_{p-1}) = q_{p-1}\}$, for some rational numbers $q_1 < \cdots < q_{p-1}$. Notice that V is topologically isomorphic to G^p. Fix some $k_n \in G$ such that the $G = \bigcup_n k_n W$.

We now let $\mathbb{E}$ be the family of all sets $X \subseteq \mathbb{Q}$ of the form

$$X = \Big(\bigcup_{n\in\mathbb{Z}}]x^1_{2n}, x^1_{2n+1}[\Big) \cup \cdots \cup \Big(\bigcup_{n\in\mathbb{Z}}]x^p_{2n}, x^p_{2n+1}[\Big),$$

where x^i_n are irrationals such that for $q_0 = -\infty$ and $q_p = +\infty$, we have

$$\begin{gathered} q_{i-1} < x^i_n < x^i_{n+1} < q_i, \\ x^i_n \to q_i \quad \text{for } n \to +\infty, \\ x^i_n \to q_{i-1} \quad \text{for } n \to -\infty. \end{gathered}$$

Moreover, for $X \in \mathbb{E}$, we let

$$A(X) = \{g \in G \mid \operatorname{supp}(g) \subseteq X\}.$$

Clearly, by Lemma 16, we have

$$V = \bigcup_{X,Y\in\mathbb{E}} A(X) \cdot A(Y).$$

So to prove that $\operatorname{Aut}(\mathbb{Q})$ is Steinhaus with exponent 96, it suffices to show the following claim.

Claim: *$A(X) \subseteq W^{48}$ for any $X \in \mathbb{E}$.*

Proof of Claim: Fix some $X \in \mathbb{E}$ and sequences x^i_n as above. Moreover, for each $i = 1, \dots, p$, let

$$I^i_n =]x^i_{2n}, x^i_{2n+1}[$$

and for each $\vec{a} = (a_1, \dots, a_p)$, where $a_i \subseteq \mathbb{Z}$ is bi-infinite, let

$$X_{\vec{a}} = (\bigcup_{n\in a_1} I^1_n) \cup \cdots \cup (\bigcup_{n\in a_p} I^p_n).$$

We stress the fact that the I^i_n name only every second interval of the $\mathbb{Z}$-ordered partition of $]q_{i-1}, q_i[$ into the intervals $]x^i_m, x^i_{m+1}[$. Thus, if $h \in A(X_{\vec{a}})$, then $h[I^i_n] = I^i_n$ for each $n \in a_i$, $i = 1, \dots, p$. Now pick a sequence of $\vec{a}_n$ such that the sets $X_{\vec{a}_n}$ are all disjoint, which is equivalent to demanding that the j-th terms of $\vec{a}_n$ and $\vec{a}_m$ are disjoint for $n \neq m$ and $j = 1, \dots, p$. From the remark about

h above it follows that if $g_n \in A(X_{\vec{a}_n})$, then $g : \mathbb{Q} \to \mathbb{Q}$ defined to be $g_n \restriction X_{\vec{a}_n}$ on $X_{\vec{a}_n}$ and the identity on $\mathbb{Q} \setminus \bigcup_n X_{\vec{a}_n}$ is an element of $G = \mathrm{Aut}(\mathbb{Q})$.

We claim that for some n_0, $X_{\vec{a}_{n_0}}$ is full for $k_{n_0}W$, i.e., that for any $g \in A(X_{\vec{a}_{n_0}})$, there is some $h \in k_{n_0}W$ such that $g \restriction X_{\vec{a}_{n_0}} = h \restriction X_{\vec{a}_{n_0}}$. If not, we could find for each n some $g_n \in A(X_{\vec{a}_n})$ such that for all $h \in k_n W$, we have $g_n \restriction X_{\vec{a}_n} \neq h \restriction X_{\vec{a}_n}$. As noticed above, we could then find one single $g \in G$ such that $g \restriction X_{\vec{a}_n} = g_n \restriction X_{\vec{a}_n}$ for every n. But this would contradict that the $k_n W$ cover G.

So suppose $X_{\vec{a}_{n_0}}$ is full for $k_{n_0}W$. Then $X_{\vec{a}_{n_0}}$ is also full for W^2. For simplicity, let $\vec{a} = (a_1, \ldots, a_p) = \vec{a}_{n_0}$.

Clearly, $A(X_{\vec{a}})$ is topologically isomorphic to $(G^{\mathbb{Z}})^p$, so it has a comeagre conjugacy class by [19] and Lemma 11. Find now some n_1 such that $k_{n_1}W \cap A(X_{\vec{a}})$ is non-meagre in $A(X_{\vec{a}})$, whence also $W^2 \cap A(X_{\vec{a}})$ is non-meagre in $A(X_{\vec{a}})$. Therefore there is some $f \in W^2$ belonging to the comeagre conjugacy class in $A(X_{\vec{a}})$. But if $h \in A(X_{\vec{a}})$, then there is a $g \in W^2$ agreeing with h on $X_{\vec{a}}$, whence $hfh^{-1} = gfg^{-1} \in W^6$. So W^6 contains the comeagre conjugacy class of $A(X_{\vec{a}})$ and as the product of two comeagre sets in a group is everything, $A(X_{\vec{a}}) \subseteq W^{12}$.

Let now $(\vec{a}^\alpha)_\alpha$ be a continuum size family of sequences $\vec{a}^\alpha = (a_1^\alpha, \ldots, a_p^\alpha)$ such that $a_i^\alpha \subseteq a_i$ is bi-infinite and $a_i^\alpha \cap a_i^\beta$ is finite for every $\alpha \neq \beta$. (See, e.g., Kunen [12], p. 48.)

For each α write also $\mathbb{Q} \setminus \{q_1, \ldots, q_{p-1}\}$ as a disjoint union of non-empty irrational intervals $J_{i,n}^\alpha$ ($n \in \mathbb{Z}$, $i = 0, \ldots, p$), such that

$$J_{0,n}^\alpha < J_{0,n+1}^\alpha < q_1 < J_{1,n}^\alpha < J_{1,n+1}^\alpha < q_2 < \cdots < q_{p-1} < J_{p,n}^\alpha < J_{p,n+1}^\alpha,$$

where

$$X_{\vec{a}^\alpha} = \bigcup_{\substack{i=0,\ldots,p \\ n \in \mathbb{Z}}} J_{i,2n}^\alpha,$$

$$\mathbb{Q} \setminus X_{\vec{a}^\alpha} = \bigcup_{\substack{i=0,\ldots,p \\ n \in \mathbb{Z}}} J_{i,2n+1}^\alpha.$$

We notice that this forces each $J_{i,2n}^\alpha$ to be equal to some $I_m^i =]x_{2m}^i, x_{2m+1}^i[$ for an $m \in a_i^\alpha$, while each $J_{i,2n+1}^\alpha$ must be of the form $J_{i,2n+1}^\alpha =]x_{2m+1}^i, x_{2l}^i[$ for some $m < l$ in a_i^α.

Now, find $g_\alpha \in V$ such that

$$g_\alpha[J_{i,n}^\alpha] = J_{i,n+1}^\alpha.$$

By the uncountability there is n_2 and distinct α and β such that $g_\alpha, g_\beta \in k_{n_2}W$, whence $g_\alpha^{-1}g_\beta, g_\beta^{-1}g_\alpha \in W^2$.

If $n \notin a_i^\alpha$, then $I_n^i \subseteq J_{i,2l+1}^\alpha$ for some l, whence

$$g_\alpha[I_n^i] \subseteq g_\alpha[J_{i,2l+1}^\alpha] = J_{i,2l+2}^\alpha = I_m^i$$

for some $m > n$ with $m \in a_i^\alpha$. Similarly, if $n \notin a_i^\beta$, then $g_\beta[I_n^i] \subseteq I_m^i$ for some $m > n$ with $m \in a_i^\beta$. This, along with the almost disjointness of a_i^α and a_i^β, allows us to find N big enough so that for all $i = 0, \ldots, p$ and $|n| \geq N$

$$n \notin a_i^\alpha \Longrightarrow g_\alpha[I_n^i] \subseteq I_m^i \quad (m \in a_i^\alpha \setminus a_i^\beta)$$
$$n \notin a_i^\beta \Longrightarrow g_\beta[I_n^i] \subseteq I_m^i \quad (m \in a_i^\beta \setminus a_i^\alpha).$$

From this, by a similar argument, we get

$$n \notin a_i^\alpha \Longrightarrow g_\beta^{-1}g_\alpha[I_n^i] \subseteq I_l^i \quad (l \in a_i^\beta \subseteq a_i)$$

$$n \notin a_i^\beta \Longrightarrow g_\alpha^{-1}g_\beta[I_n^i] \subseteq I_l^i \quad (l \in a_i^\alpha \subseteq a_i).$$

Suppose also that N has been chosen large enough to ensure that for all $|n| \geq N$ either $n \notin a_i^\alpha$ or $n \notin a_i^\beta$. Then for all $|n| \geq N$, either

$$g_\beta^{-1}g_\alpha[I_n^i] \subseteq X_{\vec{a}} \tag{4}$$

or

$$g_\alpha^{-1}g_\beta[I_n^i] \subseteq X_{\vec{a}}. \tag{5}$$

As W^2 is dense in V, we can choose $h \in W^2$ and $m_i \in a_i$ such that

$$h[I_{-N}^i \cup \cdots \cup I_N^i] \subseteq I_{m_i}^i,$$

for every $i = 0, \ldots, p$. Then for all $n \in \mathbb{Z}$, either (4) or (5) or

$$h[I_n^i] \subseteq X_{\vec{a}} \tag{6}$$

where, as noticed, $g_\beta^{-1}g_\alpha, g_\alpha^{-1}g_\beta, h \in W^2$ and $A(X_{\vec{a}}) \subseteq W^{12}$.

Now define sets

$$b_i = \{n \in \mathbb{Z} \mid g_\beta^{-1}g_\alpha[I_n^i] \subseteq X_{\vec{a}}\}$$
$$c_i = \{n \in \mathbb{Z} \mid g_\alpha^{-1}g_\beta[I_n^i] \subseteq X_{\vec{a}}\}$$
$$d_i = \{n \in \mathbb{Z} \mid h[I_n^i] \subseteq X_{\vec{a}}\}$$

and let $\vec{b} = (b_0, \ldots, b_p)$, $\vec{c} = (c_0, \ldots, c_p)$ and $\vec{d} = (d_0, \ldots, d_p)$. Since (4), (5), or (6) holds for each integer n, we get

$$A(X) = A(X_{\vec{b}}) \cdot A(X_{\vec{c}}) \cdot A(X_{\vec{d}}).$$

Since additionally, directly from the definitions of b_i, c_i and d_i, we have

$$\begin{aligned} A(X_{\vec{b}}) &\subseteq (g_\beta^{-1} g_\alpha)^{-1} A(X_{\vec{a}}) g_\beta^{-1} g_\alpha \\ A(X_{\vec{c}}) &\subseteq (g_\alpha^{-1} g_\beta)^{-1} A(X_{\vec{a}}) g_\alpha^{-1} g_\beta \\ A(X_{\vec{d}}) &\subseteq h^{-1} A(X_{\vec{a}}) h, \end{aligned}$$

we get $A(X) \subseteq (W^{16})^3 = W^{48}$, proving the claim and thereby the theorem. ∎

5. Homeo($\mathbb{R}$) and Homeo(S^1)

THEOREM 17: *Homeo$_+(\mathbb{R})$ is Steinhaus.*

Proof: Let us first recall the Polish group topology on Homeo($\mathbb{R}$). It has as basis the following sets

$$U(f; q_1, \ldots, q_{p-1}; \epsilon) = \{g \in \text{Homeo}(\mathbb{R}) \mid d(f(q_i), g(q_i)) < \epsilon, \forall i < p\}$$

where $f \in \text{Homeo}(\mathbb{R})$, $\epsilon > 0$, and $q_1 < \cdots < q_{p-1}$ belong to $\mathbb{R}$. A similar topology on Homeo($[0,1]$) gives a topologically isomorphic group. The structure of the subgroup Homeo$_+(\mathbb{R})$ of all increasing homeomorphisms is very similar to that of Aut($\mathbb{Q}, <$) except from the fact that the former is connected and the latter is totally disconnected. Nevertheless, the proof for Aut($\mathbb{Q}, <$) translates almost word for word into a proof for Homeo$_+(\mathbb{R})$. Let us just mention the changes needed:

The first thing to notice is that there is a comeagre conjugacy class in Homeo$_+(\mathbb{R})$, which is shown in [11]. Secondly, instead of working with irrational intervals of $\mathbb{Q}$ one replaces these by, say, half open intervals $]r, s] \subseteq \mathbb{R}$. One easily sees that Lemma 16 goes through as before. Supposing now that $W \subseteq \text{Homeo}_+(\mathbb{R})$ is symmetric and countably syndetic, we find some open neighbourhood of the identity

$$V = \{g \in \text{Homeo}_+(\mathbb{R}) \mid d(q_i, g(q_i)) < \epsilon, \forall i < p\}$$

in which W^2 is dense. We can suppose that

$$\epsilon < \min_i (d(q_i, q_{i+1}))/3.$$

Let also

$$U = \{g \in \mathrm{Homeo}_+(\mathbb{R}) \mid g(q_i) = q_i, \forall i < p\}$$

and notice that U is topologically isomorphic to $\mathrm{Homeo}_+(\mathbb{R})^p$. One sees that

$$U = \bigcup_{X,Y \in \mathbb{E}} A(X) \cdot A(Y),$$

where $\mathbb{E}$ is defined as in the proof of Theorem 15. We can prove now that $A(X) \subseteq W^{48}$ exactly as Claim was established in the proof of Theorem 15, noticing that we do not need W^2 to be dense in U but only in V. So we get that $U \subseteq W^{96}$. But U is not open in the Polish topology on $\mathrm{Homeo}_+(\mathbb{R})$. The following claim will show that $\mathrm{Homeo}_+(\mathbb{R})$ is Steinhaus with exponent 194.

Claim: $V \subseteq W^{194}$.

Proof of Claim 1: Suppose $f \in V$ and $f(q_i) = r_i$ for $i = 1, \ldots, p-1$. Then by the density of W^2 in V there is an $h \in W^2$ such that

$$d(r_i, h(q_i)) < 1/2d(r_i, q_i), \quad \text{for } i = 1, \ldots, p.$$

But then there is also some $g \in U \subseteq W^{96}$ satisfying $g(h(q_i)) = r_i$, whence $f = (gh)h^{-1}g^{-1}f \subseteq UW^2U \subseteq W^{194}$, since $h^{-1}g^{-1}f \in U$. ∎

Consider now the group of homeomorphisms of the unit circle $\mathrm{Homeo}(S^1)$ with the topology of uniform (or equivalently, pointwise) convergence. As in the case of $\mathrm{Homeo}(\mathbb{R})$, we let $\mathrm{Homeo}_+(S^1)$ be the index 2 subgroup of orientation preserving homeomorphisms. It is a well-known fact that $\mathrm{Homeo}_+(S^1)$ does not even have a non-meagre conjugacy class, as e.g. the rotation number (see Katok and Hasselblatt, [8, Chapter 11]) is a continuous conjugacy invariant. But, on the other hand, for any $x \in S^1$ the closed subgroup

$$\mathrm{Homeo}(S^1, x) = \{g \in \mathrm{Homeo}(S^1) \mid g(x) = x\}$$

is topologically isomorphic to $\mathrm{Homeo}(\mathbb{R})$.

Theorem 18: $\mathrm{Homeo}_+(S^1)$ *is Steinhaus.*

Proof: We will deduce this result from the result for $\mathrm{Homeo}_+(\mathbb{R})$. So assume that $W \subseteq \mathrm{Homeo}_+(S^1)$ is symmetric and countably syndetic. Then we can find some neighbourhood of the identity

$$V = \{g \in \mathrm{Homeo}_+(S^1) \mid d(q_i, g(q_i)) < \epsilon, \forall i \leq n\}$$

in which W^2 is dense. Let

$$H = \{g \in \text{Homeo}_+(S^1) \mid g(q_1) = q_1\}$$

then H is topologically isomorphic to $\text{Homeo}_+(\mathbb{R})$.

Claim 1: *$W^2 \cap H$ is symmetric and countably syndetic in H.*

Proof of Claim 1: Let $\{k_n W\}_{\mathbb{N}}$ cover $\text{Homeo}_+(S^1)$. For each n such that $k_n W \cap H \neq \emptyset$ pick some g_n in the intersection. Then $k_n \in g_n W^{-1} = g_n W$ and thus

$$\begin{aligned} H &\subseteq \Big(\bigcup_n k_n W\Big) \cap H \subseteq \Big(\bigcup_n g_n W^2\Big) \cap H \\ &= \bigcup_n g_n (W^2 \cap H) \end{aligned} \tag{7}$$

where the last equality holds as $g_n \in H$. The claim is proved.

Now, since $H \cong \text{Homeo}_+(\mathbb{R})$ is Steinhaus with exponent 194, there are $p_1, \ldots, p_m \in S^1$ and $\epsilon > \delta > 0$ such that

$$U = \{g \in \text{Homeo}_+(S^1) \mid d(p_i, g(p_i)) < \delta, \forall i \leq m \& g(q_1) = q_1\} \subseteq (W^2)^{194} = W^{388}.$$

In particular,

$$U' = \{g \in \text{Homeo}_+(S^1) \mid \forall \leq n g(q_i) = q_i \& \forall i \leq m g(p_i) = p_i\} \subseteq W^{388}$$

and W^2 is dense in the set

$$V' = \{g \in \text{Homeo}_+(S^1) \mid \forall \leq n d(g(q_i), q_i) < \delta \& \forall i \leq m d(g(p_i), p_i) < \delta\}.$$

As in the proof of Claim 1, we see that $V' \subseteq U' W^2 U' \subseteq W^{768}$, hence $\text{Homeo}_+(S^1)$ is Steinhaus with exponent 768. ∎

Theorem 19: $\text{Homeo}_+(\mathbb{R})$ *is the only proper subgroup of* $\text{Homeo}(\mathbb{R})$ *of index* $< 2^{\aleph_0}$.

Proof: Since $\text{Homeo}_+(\mathbb{R})$ is connected, it is enough to show that any subgroup $H \leq \text{Homeo}_+(\mathbb{R})$ of index $< 2^{\aleph_0}$ is open. This is done by repeating the proof above for $W = H$. However, there are a few things that have to be noticed before this can be done. Namely, since H is not necessarily countably syndetic, we have to see exactly where this is used and propose a substitute. First of all, we have to prove that H cannot be meagre in $\mathrm{Homeo}_+(\mathbb{R})$, and secondly, we have to show that some $X_{\vec{a}}$ is full for $H^2 = H$.

To see that H is not meagre in $\mathrm{Homeo}_+(\mathbb{R})$, notice that the map $(g, h) \mapsto gh^{-1}$ is continuous and open from $(\mathrm{Homeo}_+(\mathbb{R}))^2$ to $\mathrm{Homeo}_+(\mathbb{R})$. So if H is meagre then $E = \{(g, h) \in (\mathrm{Homeo}_+(\mathbb{R}))^2 : gh^{-1} \in H\}$ would be a meagre equivalence relation and therefore by Mycielski's Theorem (see [9, Theorem 19.1]) have a continuum of classes, contradicting $[\mathrm{Homeo}_+(\mathbb{R}) : H] < 2^{\aleph_0}$.

Now to see that some $X_{\vec{a}}$ is full for $H^2 = H$, we pick our sequence $\vec{a}_n$ as in the proof of Claim of Theorem 15 such that the sets $X_{\vec{a}_n}$ are all disjoint. Let now

$$N = \{g \in \mathrm{Homeo}_+(\mathbb{R}) \mid \forall n g[X_{\vec{a}_n}] = X_{\vec{a}_n}\} \leq \mathrm{Homeo}_+(\mathbb{R})$$

and notice that N is topologically isomorphic to $\prod_n A(X_{\vec{a}_n})$, which is itself isomorphic to $\mathrm{Homeo}_+(\mathbb{R})^{\mathbb{N}}$. Let now H_n be the projection of $H \cap N$ into $A(X_{\vec{a}_n})$. Then $H \cap N \leq \prod_n H_n$, whence

$$\prod_n [A(X_{\vec{a}_n}) : H_n] = [N : \prod_n H_n] \leq [N : H \cap N] \leq [\mathrm{Homeo}_+(\mathbb{R}) : H] < 2^{\aleph_0}.$$

Therefore at most finitely many $[A(X_{\vec{a}_n}) : H_n]$ can be different from 1, meaning that at least for some n, $H_n = A(X_{\vec{a}_n})$, i.e., $X_{\vec{a}_n}$ is full for H. ∎

Corollary 20: $\mathrm{Homeo}_+(S^1)$ *is the only proper subgroup of* $\mathrm{Hom}(S^1)$ *of index* $< 2^{\aleph_0}$.

Proof: Fix two points, e.g. i and $-i$, on the unit circle S^1 and suppose $H < \mathrm{Homeo}(S^1)$, $[\mathrm{Homeo}(S^1), H] < 2^{\aleph_0}$. For $x \in S^1$ let

$$\mathrm{Homeo}_+(S^1, x) = \{g \in \mathrm{Homeo}_+(S^1) \mid g(x) = x\}$$

which is a subgroup isomorphic to $\mathrm{Homeo}_+(\mathbb{R})$. Moreover, we have

$$[\mathrm{Homeo}_+(S^1, i) : H \cap \mathrm{Homeo}_+(S^1, i)] < 2^{\aleph_0}$$

and

$$[\mathrm{Homeo}_+(S^1, -i) : H \cap \mathrm{Homeo}_+(S^1, -i)] < 2^{\aleph_0}.$$

Thus as $\mathrm{Homeo}_+(\mathbb{R})$ has no proper subgroups of index $< 2^{\aleph_0}$,

$$\mathrm{Homeo}_+(S^1, i) \leq H \quad \text{and} \quad \mathrm{Homeo}_+(S^1, -i) \leq H.$$

But it is not hard to see that

$$\mathrm{Homeo}_+(S^1) = \mathrm{Homeo}_+(S^1, i) \cdot \mathrm{Homeo}_+(S^1, -i) \cdot \mathrm{Homeo}_+(S^1, i).$$

So $\mathrm{Homeo}_+(S^1) = H$. ∎

References

[1] G. M. Bergman, *Generating infinite symmetric groups,* The Bulletin of the London Mathematical Society **38** (2006), 429–440.

[2] M. Droste and R. Göbel, *Uncountable cofinalities of permutation groups,* Journal of the London Mathematical Society (2) **71** (2005), 335–344.

[3] M. Droste and W. C. Holland, *Generating automorphism groups of chains,* Forum Mathematica **17** (2005), 699–710.

[4] R. Ellis, *Lectures on Topological Dynamics,* W. A. Benjamin, Inc., New York, 1969, xv+211 pp.

[5] E. Glasner and B. Weiss, *The universal minimal system for the group of homeomorphisms of the Cantor set,* Fundamenta Mathematicae **176** (2003), 277–289.

[6] I. Guran, *On topological groups close to being Lindelöf,* Soviet Mathematics Doklady **23** (1981), 173–175.

[7] W. Herer and J. P. R. Christensen, *On the existence of pathological submeasures and the construction of exotic topological groups,* Mathematische Annalen **213** (1975), 203–210.

[8] A. Katok and B. Hasselblatt, *Introduction to the modern theory of dynamical systems,* With a supplementary chapter by Katok and Leonardo Mendoza, Encyclopedia of Mathematics and its Applications, 54, Cambridge University Press, Cambridge, 1995. xviii+802 pp.

[9] A. S. Kechris, *Classical Descriptive Set Theory,* Graduate Texts in Mathematics, 156, Springer-Verlag, New York, 1995, xviii+402 pp.

[10] A. S. Kechris, V. G. Pestov and S. Todorcevic, *Fraïssé limits, Ramsey theory, and topological dynamics of automorphism groups,* Geometric and Functional Analysis **15** (2005), 106–189.

[11] A. S. Kechris and C. Rosendal, *Turbulence, amalgamation and generic automorphisms of homogeneous structures,* Proceedings of the London Mathematical Society (3) **94** (2007), 302–350.

[12] K. Kunen, *Set theory. An introduction to independence proofs,* Reprint of the 1980 original. Studies in Logic and the Foundations of Mathematics, 102. North-Holland Publishing Co., Amsterdam, 1983, xvi+313 pp.

[13] D. Kuske and J. K. Truss, *Generic automorphisms of the universal partial order,* Proceedings of the American Mathematical Society **129** (2001), 1939–1948.

[14] M. G. Megrelishvili, *Every semitopological semigroup compactification of the group $H_+[0,1]$ is trivial,* Semigroup Forum **63** (2001), 357–370.

[15] V. G. Pestov, *On free actions, minimal flows, and a problem by Ellis,* Transactions of the American Mathematical Society **350** (1998), 4149–4165.

[16] V. G. Pestov, *Dynamics of infinite-dimensional groups and Ramsey-type phenomena,* Publicações Matemáticas do IMPA. [IMPA Mathematical Publications] 25° Colóquio Brasileiro de Matemática. [25th Brazilian Mathematics Colloquium] Instituto Nacional de Matemtica Pura e Aplicada (IMPA), Rio de Janeiro, 2005, iv+241 pp.

[17] S. Solecki, *Amenability, free subgroups, and Haar null sets in non-locally compact groups,* Proceedings of the London Mathematical Society (3) **93** (2006), 693–722.

[18] J. K. Truss, *Infinite permutation groups. II. Subgroups of small index,* Journal of Algebra **120** (1989), 494–515.

[19] J. K. Truss, *Generic automorphisms of homogeneous structures,* Proceedings of the London Mathematical Society (3) **65** (1992), 121–141.

[20] V. Uspenskij, *On universal minimal compact G-spaces,* Proceedings of the 2000 Topology and Dynamics Conference (San Antonio, TX), Topology Proceedings **25** (2000), Spring, 301–308.

[21] W. A. Veech, *Topological dynamics,* Bulletin of the American Mathematical Society **83** (1977), 775–830.