NON-UNITARISABLE REPRESENTATIONS AND MAXIMAL SYMMETRY

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ABSTRACT. We investigate questions of maximal symmetry in Banach spaces and the structure of certain bounded non-unitarisable groups on Hilbert space. In particular, we provide structural information about bounded groups with an essentially unique invariant complemented subspace. This is subsequently combined with rigidity results for the unitary representation of $\mathrm{Aut}(T)$ on $\ell_2(T)$, where T is the countably infinite regular tree, to describe the possible bounded subgroups of $\mathrm{GL}(\mathcal{H})$ extending a well-known non-unitarisable representation of $\mathbb{F}_\infty$.

As a related result, we also show that a transitive norm on a separable Banach space must be strictly convex.

1. INTRODUCTION

The research of the present paper aims to expand on a circle of ideas involving maximal symmetry in Banach spaces and non-unitarisable representations in Hilbert space. Let us recall that a subgroup G of the general linear group $\mathrm{GL}(X)$ of all continuous linear automorphisms of a Banach space X is said to be *bounded* if G is a uniformly bounded family of operators, i.e., $\sup_{T\in G}\|T\| < \infty$. In this case, X admits an equivalent G-invariant norm, namely, $|\!|\!|x|\!|\!| = \sup_{T\in G}\|Tx\|$. Thus, boundedness simply means that G is a group of isometries for some equivalent norm on X. Also, $G \leqslant \mathrm{GL}(X)$ is *maximal bounded* if it is not properly contained in another bounded subgroup of $\mathrm{GL}(X)$. Maximal bounded groups naturally correspond to maximally symmetric norms on X, in the sense that, if the isometry group of a specific norm is maximal bounded, in which case we say the norm is *maximal*, then there is no manner of renorming X to obtain a strictly larger set of isometries.

When X is finite-dimensional, every bounded $G \leqslant \mathrm{GL}(X)$ is contained in a maximal bounded subgroup, namely, the unitary group of a G-invariant inner product on X. This may be seen as an analogue of the Cartan–Iwasawa–Malcev theorem, i.e., the existence of maximal compact subgroups in connected Lie groups. Also, in many of the classical spaces such as, e.g., ℓ_p, the Hilbert space, or the space $C([0,1])$ of complex continuous functions on $[0,1]$, the canonical norm is maximal [15, 21, 25]. However, not every Banach space admits an equivalent maximal norm, indeed, counter-examples may be found among super-reflexive spaces [13]. Furthermore, as shown by S. J. Dilworth and B. Randrianantoanina [9], even among classical spaces such as ℓ_p, $1 < p < \infty$, $p \neq 2$, the general linear group contains bounded subgroups not contained in a maximal bounded subgroup. The following problem, which is the main motivation for our study, also remains stubbornly open.

Problem 1. *Is every maximal norm on a Hilbert space $\mathcal{H}$ euclidean, i.e., generated by an inner product?*

Ferenczi was supported by FAPESP, grant 2013/11390-4. Rosendal was partially supported by a Simons Foundation Fellowship (Grant #229959) and also recognises support from the NSF (DMS 1201295).

This problem is tightly related to two other issues in functional analysis, namely, the existence of non-unitarisable bounded representations and S. Mazur's rotation problem. Here a bounded representation $\lambda\colon \Gamma \to \mathrm{GL}(\mathcal{H})$ of a group Γ on a complex Hilbert space $\mathcal{H}$ is said to be *unitarisable* if there is an equivalent $\lambda(\Gamma)$-invariant inner product on $\mathcal{H}$, or, equivalently, if λ is conjugate to a unitary representation on $\mathcal{H}$. So Problem 1 is equivalent to asking whether every maximal bounded subgroup of $\mathrm{GL}(\mathcal{H})$ is unitarisable.

As was shown by M. Day [7] and J. Dixmier [10], strongly continuous bounded representations of amenable groups are always unitarisable. On the other hand, L. Ehrenpreis and F. I. Mautner [11] constructed the first example of a non-unitarisable bounded representation of a countable group Γ. In this connection, Dixmier posed the still central problem of whether unitarisability of all bounded representations characterises amenable groups among countable discrete groups. Now, by the Ehrenpreis–Mautner example, there are bounded subgroups $G \leqslant \mathrm{GL}(\mathcal{H})$ of complex separable Hilbert space not preserving any euclidean norm, but it remains an open question whether there are such G which are maximal.

Note that while the isometry group of a maximal norm on a space X is maximal bounded by definition, there may be several essentially distinct norms, i.e., not scalar multiplies of each other, with this same isometry group. One case where the norm is uniquely defined by its isometry group is when the latter acts transitively on every sphere, in which case, the norm is said to be *transitive*. This happens, for example, for Hilbert space $\mathcal{H}$ and ultrapowers of L^p spaces with non-atomic measures. However, in the separable setting, it is not known whether Hilbert space is the only such example either isomorphically or isometrically. This is known as Mazur's rotation problem [1, 19].

That the norm is uniquely defined (up to multiplicative constants) by its isometry group actually follows from a weaker property of the isometry group, called *almost transitivity*. A Banach space is almost transitive when the isometry group has dense orbits on spheres (and a bounded subgroup G of automorphisms is almost transitive when every G-invariant renorming is almost transitive). Classical examples are the L_p-spaces with non-atomic measures, $1 \leqslant p < \infty$. Another is Gurarij's space, where the isometry group acts transitively on the smooth points of the sphere [17].

Of course, if some maximal non-euclidean norm on Hilbert space were obtained, the next task would be to determine whether this norm is almost transitive or even transitive. So Mazur's rotation problem and the question of extension of non-unitarisable representations on Hilbert space are also related.

To study Problem 1, we shall be considering the structure of bounded groups containing the image of one specific widely studied non-unitarisable representation associated to actions on trees (see, e.g., [20, 22, 23]). For this, suppose that $\lambda\colon \Gamma \to \mathcal{U}(\mathcal{H})$ is a unitary presentation. A *bounded derivation* associated to λ is a uniformly bounded map $d\colon \Gamma \to \mathcal{B}(\mathcal{H})$ so that $d(gf) = \lambda(g)d(f) + d(g)\lambda(f)$ for all $g, f \in \Gamma$. This is simply equivalent to requiring that

$$\lambda_d(g) = \begin{pmatrix} \lambda(g) & d(g) \\ 0 & \lambda(g) \end{pmatrix}$$

defines a bounded representation of Γ on $\mathcal{H} \oplus \mathcal{H}$. The representation λ_d is unitarisable exactly when d is *inner*, i.e., $d(g) = \lambda(g)A - A\lambda(g)$ for some bounded linear operator A on $\mathcal{H}$ (see Section 3 for details).

The principal aim of the present paper is to elucidate bounded groups $G \leqslant \mathrm{GL}(\mathcal{H}\oplus\mathcal{H})$ containing $\lambda_d[\Gamma]$ for λ and d as above, which are potential examples of maximal non-unitarisable groups. Since $\lambda_d[\Gamma]$ leaves the first copy of $\mathcal{H}$ in the decompostion $\mathcal{H}\oplus\mathcal{H}$ invariant, in this context, it is natural to study G with the same property and we shall do this in a broader setting. We now proceed to describe the main outcomes of our study.

In Section 2, we investigate the structure of bounded subgroups of $\mathrm{GL}(X)$, where X is separable reflexive, which have a distinguished invariant subspace Y. More specifically, supposing that $X = Y \oplus Z$ is a separable reflexive Banach space and $G \leqslant \mathrm{GL}(Y \oplus Z)$ is a bounded group of upper triangular block matrices

$$\begin{pmatrix} u & w \\ 0 & v \end{pmatrix},$$

we first observe that, in this case, w is actually a function δ (or *derivation*) of u, v. However, under stronger assumptions, we show that the diagonal entries u and v are also in a one-to-one correspondence and so every element of G is uniquely determined by just the entry u and similarly by v. Interestingly, derivations $\delta(u, v)$ involving a nonlinear homogeneous map ψ show up naturally in this context.

Theorem 2. *Let $X = Y \oplus Z$ be separable reflexive and $G \leqslant \mathrm{GL}(X)$ a bounded subgroup leaving Y invariant. Assume that there are no closed linear G-invariant subspaces $\{0\} \subsetneq W \subsetneq Y$ nor superspaces $Y \subsetneq W \subsetneq X$ and there is no closed linear G-invariant complement of Y in X. Then the mappings*

$$\begin{pmatrix} u & w \\ 0 & v \end{pmatrix} \mapsto u \quad \textit{and} \quad \begin{pmatrix} u & w \\ 0 & v \end{pmatrix} \mapsto v$$

are $\mathtt{sot}$*-isomorphisms between G and the respective images in $\mathrm{GL}(Y)$ and $\mathrm{GL}(Z)$.*

In Section 3, we apply Theorem 2 when $G \leqslant \mathrm{GL}(\mathcal{H}\oplus\mathcal{H})$ is a bounded subgroup leaving the first copy of $\mathcal{H}$ invariant and containing the image $\lambda_d[\Gamma]$, where λ is an irreducible unitary representation and d is an associated non-inner derivation.

Corollary 3. *Suppose that $\lambda\colon \Gamma \to \mathcal{U}(\mathcal{H})$ is an irreducible unitary representation of a group Γ on a separable Hilbert space $\mathcal{H}$ and $d\colon \Gamma \to \mathcal{B}(\mathcal{H})$ is an associated non-inner bounded derivation. Suppose that $G \leqslant \mathrm{GL}(\mathcal{H}\oplus\mathcal{H})$ is a bounded subgroup leaving the first copy of $\mathcal{H}$ invariant and containing $\lambda_d[\Gamma]$. Then the mappings $G \to \mathrm{GL}(\mathcal{H})$ defined by*

$$\begin{pmatrix} u & w \\ 0 & v \end{pmatrix} \mapsto u \quad \textit{and} \quad \begin{pmatrix} u & w \\ 0 & v \end{pmatrix} \mapsto v$$

are $\mathtt{sot}$*-isomorphisms between G and the respective images in $\mathrm{GL}(\mathcal{H})$.*

As additional information, in Proposition 15, we show that, if $\lambda_d[\Gamma]$ is contained in some almost transitive bounded subgroup of $\mathrm{GL}(\mathcal{H}\oplus\mathcal{H})$, then there is a homogeneous Lipschitz map $\psi\colon \mathcal{H} \to \mathcal{H}$ defining the derivation by $d(a) = \lambda(a)\psi - \psi\lambda(a)$.

In Section 4, we turn to study one specific representation. For this, let T denote the $\aleph_0$-regular tree, i.e., the Cayley graph of the free group $\mathbb{F}_\infty$ on denumerably many generators with respect to its free generating set. Let also $\mathrm{Aut}(T)$ denote its group of automorphisms. We consider the representation $\lambda\colon \mathrm{Aut}(T) \curvearrowright \mathbb{C}^T$, i.e., the canonical shift action on the vector space of $\mathbb{C}$-valued functions on T, as well as the restriction of λ to a unitary representation on $\ell_2(T)$.

We observe that each of the subspaces $\ell_p(T) \subseteq \mathbb{C}^T$ are λ-invariant, and prove rigidity properties of this representation. While it is fairly easy to see that the unitary representation $\lambda\colon \mathrm{Aut}(T) \to \mathcal{U}(\ell_2(T))$ is irreducible and *uniquely unitarisable*, i.e., up to a scalar multiple preserves a unique inner product equivalent with the usual one, we may show significantly stronger results. Namely, in Theorem 20, we show that the usual inner product $\langle\cdot|\cdot\rangle$, up to a scalar multiple, is the only inner product (not necessarily equivalent to $\langle\cdot|\cdot\rangle$) preserved by λ. Also, strengthening irreducibility, we have the following.

Theorem 4. *The commutant of $\lambda[\mathrm{Aut}(T)]$ in the space of linear operators from $\ell_p(T)$ to $\mathbb{C}^T$, $1 < p \leqslant \infty$, is just $\mathbb{C}\cdot\mathrm{Id}$.*

One unsolved issue (Problem 21) is whether there are bounded subgroups of $\mathrm{GL}(\mathcal{H})$ containing $\lambda\big[\mathrm{Aut}(T)\big]$ and which are not unitarisable.

In Section 5, we describe a classical twisting of the representation λ producing a bounded non-unitarizable representation λ_d of $\mathbb{F}_\infty$, noting that it extends naturally to a representation of $\mathrm{Aut}(T)$. To construct the derivation, fix a root $e \in T$ and set $\hat{e} = e$, while for $s \in T$, $s \neq e$, we let $\hat{s}$ denote the penultimate vertex on the geodesic in T from e to s. Also, let $L\colon \ell_1(T) \to \ell_1(T)$ be the bounded linear operator satisfying $L(\mathbf{1}_s) = \mathbf{1}_{\hat{s}}$ for $s \neq e$ and $L(\mathbf{1}_e) = 0$. Then, if L^* denotes the adjoint operator on $\ell_\infty(T)$, for every $g \in \mathrm{Aut}(T)$, $d(g) = L^*\lambda(g) - \lambda(g)L^*$ restricts to a linear operator on $\ell_2(T)$ of norm $\leqslant 2$ and agrees with the operator $\lambda(g)L - L\lambda(g)$ on $\ell_1(T)$. It follows that d defines a bounded derivation associated to λ, which, however, is not inner. Moreover, with the aid of Theorem 4, we show that this definition of d is extremely rigid.

Theorem 5. *Let d be the derivation defined above and suppose $A\colon \ell_2(T) \to \mathbb{C}^T$ is a globally defined linear operator so that $d(g) = A\lambda(g) - \lambda(g)A$ for all $g \in \mathrm{Aut}(T)$. Then $A = L^* + \vartheta\mathrm{Id}$ for some $\vartheta \in \mathbb{C}$.*

Finally, we may combine the previous analysis of bounded subgroups with the specific nature of the given derivation to obtain the following rigidity of structure result.

Theorem 6. *Let d be the derivation defined above and suppose that $G \leqslant \mathrm{GL}(\ell_2(T) \oplus \ell_2(T))$ is a bounded subgroup leaving the first copy of $\ell_2(T)$ invariant and containing $\lambda_d[\mathrm{Aut}(T)]$. Then there is a homogeneous map $\psi\colon \ell_2(T) \to \ell_2(T)$, uniformly continuous on bounded sets, for which*

$$L^* + \psi\colon \ell_2(T) \to \ell_\infty(T) \quad \textit{and} \quad L - \psi\colon \ell_1(T) \to \ell_2(T)$$

commute with $\lambda(g)$ for $g \in \mathrm{Aut}(T)$ and so that every element of G is of the form

$$\begin{pmatrix} u & u\psi - \psi v \\ 0 & v \end{pmatrix}$$

for some $u, v \in \mathrm{GL}(\ell_2(T))$.

Finally, the mappings

$$\begin{pmatrix} u & u\psi - \psi v \\ 0 & v \end{pmatrix} \mapsto u \qquad \textit{and} \qquad \begin{pmatrix} u & u\psi - \psi v \\ 0 & v \end{pmatrix} \mapsto v$$

are `sot`*-isomorphisms between G and their respective images in $\mathrm{GL}(\ell_2(T))$.*

We subsequently use this result to compute some simple values of ψ, which could be useful for extracting information about possible G. However, the following problem remains open.

Problem 7. *Let d be the derivation defined above. Is $\lambda_d[\mathrm{Aut}(T)]$ contained in some maximal bounded subgroup of $\mathrm{GL}(\ell_2(T) \oplus \ell_2(T))$? Or even in some almost transitive bounded subgroup of $\mathrm{GL}(\ell_2(T) \oplus \ell_2(T))$?*

The results of the Section 6 are independent of the rest of the paper and concern Mazur's rotation problem. Though much information has been obtained on almost transitive Banach spaces under additional geometric assumptions such as reflexivity [5, 2], we are not aware of any results that necessitate actual transitivity. Related to the present study, we show in Theorem 28 that, if $(X, \|\cdot\|)$ is a separable real transitive Banach space, then X is strictly convex and $\|\cdot\|$ is Gâteaux differentiable. Let us remark that this result fails if X is only assumed to be almost transitive, as can be seen by considering $L^1([0,1])$.

2. On bounded representations with invariant subspaces

In the following, we consider a separable reflexive Banach space X and a bounded subgroup $G \leqslant \mathrm{GL}(X)$ along with a G-invariant closed linear subspace $Y \subseteq X$. We let $\pi\colon X \to X/Y$ denote the canonical quotient map and write $\dot{x}$ for $\pi(x) = x + Y \in X/Y$. Note also that every $T \in G$ induces an operator $\dot{T} \in \mathrm{GL}(X/Y)$ defined by

$$\dot{T}(\dot{x}) = \big(Tx\big)^{\bullet},$$

i.e., $\dot{T}(x+Y) = Tx + Y$. Moreover, as $\|\dot{T}(\dot{x})\| = \big\|\big(Tx\big)^{\bullet}\big\| \leqslant \|Tx\| \leqslant \|T\| \cdot \|x\|$ for all $x \in X$, we see that $\|\dot{T}\| \leqslant \|T\|$. In particular, $\dot{G} = \{\dot{T} \in \mathrm{GL}(X/Y) \mid T \in G\}$ is a bounded subgroup of $\mathrm{GL}(X/Y)$.

We recall a few facts about the nearest point map in Banach spaces. If X is reflexive with a strictly convex norm, then, for any non-empty closed convex subset C of X, the *nearest point map* $c\colon X \to C$ given by

$$c(x) = \text{ the unique point } y \in C \text{ closest to } x$$

is well-defined (see Exercise 7.46 [12]). If in addition the norm is locally uniformly convex (LUR), then c is continuous (Exercise 7.47 [12]), and, if it is uniformly convex, then c is uniformly continuous on bounded neighbourhoods of C (Lemma 2.5 [3]). If the modulus of convexity of the norm has power type p, then the associated modulus of continuity satisfies $\omega(\epsilon) \leqslant c\epsilon^{1/p}$. Moreover, this may be improved to $\omega(\epsilon) \leqslant c\epsilon^{q/p}$ if the modulus of smoothness of the norm has power type q (Theorem 2.8 [3]).

We note also for future reference that, if $\|\cdot\|$ is a uniformly convex norm on X, then the G-invariant equivalent norm $|||x||| = \sup_{T\in G}\|Tx\|$ is also uniformly convex. Furthermore, if $\|\cdot\|$ has modulus of convexity of power type p, then so will $|||\cdot|||$ (see, e.g., Lemma 1.1 [6]).

Lemma 8. *Let X be a separable reflexive Banach space, $G \leqslant \mathrm{GL}(X)$ a bounded subgroup and suppose that $Y \subseteq X$ is a G-invariant closed linear subspace. Then there is a continuous, homogeneous and thus bounded G-equivariant lifting $\phi\colon X/Y \to X$ of the quotient map π, that is, $\pi \circ \phi = \mathrm{Id}_{X/Y}$ and $\phi\dot{T} = T\phi$ for all $T \in G$.*

Proof. First, by results of G. Lancien [16], since X is separable reflexive and $G \leqslant \mathrm{GL}(X)$ is bounded, there is an equivalent G-invariant LUR norm $\|\cdot\|$ on X. In other words, G is a subgroup of the group $\mathrm{Isom}(X, \|\cdot\|)$ of linear isometries of $X, \|\cdot\|$.

Now, since X is reflexive and $\|\cdot\|$ is LUR, the Y-nearest point map $c\colon X \to Y$ is well-defined and continuous. Note then that, for every $x \in X$, $x - c(x)$ is the unique point in $x + Y$ of minimal norm. Let $b\colon X/Y \to X$ be a Bartle–Graves selector (see Corollary 7.56 [12]), that is, b is a continuous lifting of the quotient mapping $\pi\colon X \to X/Y$. We then let $\phi\colon X/Y \to X$ be defined by $\phi(z) = b(z) - c(b(z))$ and note that, for $x \in X$, $\phi(\dot{x})$ is the unique point of minimal norm in the affine subspace $x + Y \subseteq X$.

Suppose that $x \in X$ and $T \in G$. Then, since $T[Y] = Y$ and T is a linear isometry of X,

$$\begin{aligned}\phi(\dot{T}\dot{x}) &= \phi\big((Tx)^\bullet\big)\\ &= \text{ the unique point in } Tx + Y \text{ of minimal norm}\\ &= \text{ the unique point in } T[x + Y] \text{ of minimal norm}\\ &= T\big(\text{the unique point in } x + Y \text{ of minimal norm}\big)\\ &= T\phi(\dot{x}).\end{aligned}$$

Thus, $\phi\dot{T} = T\phi$ for all $T \in G$, i.e., ϕ is a continuous G-equivariant lifting of the quotient map. Similarly, for $x \in X$ and λ a scalar,

$$\begin{aligned}\phi(\dot{x}) = x &\Leftrightarrow \forall y \in Y\ \|x\| \leqslant \|x + y\|\\ &\Leftrightarrow \forall y \in Y\ \|\lambda x\| \leqslant \|\lambda(x + y)\|\\ &\Leftrightarrow \forall y \in Y\ \|\lambda x\| \leqslant \|\lambda x + y\|\\ &\Leftrightarrow \phi(\lambda\dot{x}) = \lambda x,\end{aligned}$$

whence ϕ is homogeneous. Finally, let us also note that $\|\phi(\dot{x})\| = \|\dot{x}\|$ for all $x \in X$. □

We observe that in Lemma 8, if $\|\cdot\|$ denotes the original norm on X, then ϕ may be chosen of norm at most $(1+\epsilon)\sup_{T\in G}\|T\|$, for any choice of $\epsilon > 0$. This easily follows from the construction of a G-invariant LUR norm on X. Indeed, define $\|\!|x|\!\| = \sup_{T\in G}\|Tx\|$, and let $\|\!|\cdot|\!\|'$ be an equivalent LUR norm for which

$$\mathrm{Isom}(X, \|\!|\cdot|\!\|) \leqslant \mathrm{Isom}(X, \|\!|\cdot|\!\|').$$

By density of the LUR property in the space of G-invariant norms (Proposition 4.5 [13]), $\|\!|\cdot|\!\|'$ may be chosen so that $\|\!|\cdot|\!\| \leqslant \|\!|\cdot|\!\|' \leqslant (1+\epsilon)\|\!|\cdot|\!\|$. Letting ϕ denote the lifting of Lemma 8, we have $\|\!|\phi(\dot{x})|\!\|' = \|\!|\dot{x}|\!\|'$ and thus

$$\|\phi(\dot{x})\| \leqslant \|\!|\phi(\dot{x})|\!\| \leqslant \|\!|\phi(\dot{x})|\!\|' = \|\!|\dot{x}|\!\|' \leqslant (1+\epsilon)\|\!|\dot{x}|\!\| \leqslant (1+\epsilon)\big(\sup_{T\in G}\|T\|\big)\|\dot{x}\|.$$

for all $x \in X$. This estimate may be improved by dropping the continuity property.

Lemma 9. *Let X be a separable reflexive Banach space, $G \leqslant \mathrm{GL}(X)$ a bounded subgroup and suppose that $Y \subseteq X$ is a G-invariant closed linear subspace. Then there exists a homogeneous G-equivariant lifting $\phi\colon X/Y \to X$ of the quotient map π with norm at most* $\sup_{T\in G}\|T\|$.

Proof. By the above remark, let ϕ_n be a homogeneous G-equivariant lifting associated to a choice of equivalent G-invariant LUR renorming $\|\cdot\|_n$ of norm at most $(1+\frac{1}{n})\sup_{T\in G}\|T\|$. Use reflexivity to define, for all $\dot{x} \in X/Y$, $\phi(\dot{x})$ as a weak limit along a non-trivial ultrafilter,

$$\phi(\dot{x}) = w - \lim_{n\to\mathcal{U}} \phi_n(\dot{x}).$$

It is easily checked that ϕ is a homogeneous G-equivariant lifting of π of norm at most $\sup_{T\in G}\|T\|$. □

Note that, if ϕ is the lifting defined by Lemma 8, then $p\colon X \to Y$ given by $p(x) = x - \phi(\dot{x})$ is a continuous homogeneous (potentially non-linear) projection of X onto its subspace Y. Moreover, in this case, we can define a homogeneous homeomorphism between X and $Y \oplus X/Y$ via $x \mapsto (p(x), \dot{x})$ with homogeneous inverse $(y, z) \mapsto y + \phi(z)$. By the G-equivariance of ϕ, we also have

$$Tx \mapsto (Tx - \phi(\dot{T}\dot{x}), \dot{T}\dot{x}) = (Tx - T\phi(\dot{x}), \dot{T}\dot{x}) = \big((T|_Y)(p(x)), \dot{T}\dot{x}\big),$$

which shows that the action of G on X is conjugate by the above homeomorphism to the G-action on $Y \oplus X/Y$ given by the block diagonal representation

$$T \mapsto \begin{pmatrix} T|_Y & 0 \\ 0 & \dot{T} \end{pmatrix}.$$

Lemma 10. *Let X be a separable reflexive Banach space, $G \leqslant \mathrm{GL}(X)$ a bounded subgroup and suppose that $Y \subseteq X$ is a G-invariant closed linear subspace. Then the mapping $T \mapsto \begin{pmatrix} T|_Y & 0 \\ 0 & \dot{T} \end{pmatrix}$ is an injective homomorphism of G into $\mathrm{GL}(Y \oplus X/Y)$.*

Proof. Assume $T \in G$ satisfies $T|_Y = \mathrm{Id}_Y$ and $\dot{T} = \mathrm{Id}_{X/Y}$. Then T acts as the identity on $Y \oplus X/Y$ and, since the action of G on X is conjugate by homeomorphism to the action of G on $Y \oplus X/Y$, we deduce that $T = \mathrm{Id}$. □

Theorem 11. *Let X be a separable reflexive Banach space and $G \leqslant \mathrm{GL}(X)$ be a bounded subgroup. Suppose that $Y \subseteq X$ is a G-invariant closed linear subspace so that*

(i) *there is no closed linear G-invariant subspace $\{0\} \subsetneq W \subsetneq Y$,*
(ii) *there is no closed linear G-invariant complement of Y in X.*

Then the mapping $T \mapsto \dot{T}$ is an isomorphism of the topological groups $(G, \mathtt{sot})$ and $(\dot{G}, \mathtt{sot})$.

Proof. Let $\phi\colon X/Y \to X$ be the lifting given by Lemma 8. Define $\Delta\colon X/Y \times X/Y \to X$ by $\Delta(\dot{x}_1, \dot{x}_2) = \phi(\dot{x}_1) + \phi(\dot{x}_2) - \phi(\dot{x}_1 + \dot{x}_2)$ and observe that, since $\phi(\dot{x}_1) \in x_1 + Y$, $\phi(\dot{x}_2) \in x_2 + Y$ and $\phi(\dot{x}_1 + \dot{x}_2) \in (x_1 + x_2) + Y$, we have $\Delta(\dot{x}_1, \dot{x}_2) \in Y$. Moreover, by G-equivariance of ϕ, we find that

$$\begin{aligned} T\Delta(\dot{x}_1, \dot{x}_2) &= T\phi(\dot{x}_1) + T\phi(\dot{x}_2) - T\phi(\dot{x}_1 + \dot{x}_2) \\ &= \phi(\dot{T}\dot{x}_1) + \phi(\dot{T}\dot{x}_2) - \phi(\dot{T}\dot{x}_1 + \dot{T}\dot{x}_2) \\ &= \Delta(\dot{T}\dot{x}_1, \dot{T}\dot{x}_2) \end{aligned}$$

for all $T \in G$ and $x_1, x_2 \in X$.

We claim that $\Delta(\dot{x}_1, \dot{x}_2) \neq 0$ for some $x_1, x_2 \in X$. Indeed, suppose not. Then ϕ is a bounded linear G-equivariant map, whereby the composition $P = \phi \circ \pi$ is a bounded linear projection with $\ker P = Y$ satisfying

$$PT(x) = \phi\pi(Tx) = \phi\big(\dot{T}\dot{x}\big) = T\phi(\dot{x}) = TP(x)$$

for all $x \in X$, i.e., $PT = TP$. So $W = P[X]$ is a G-invariant closed linear complement of Y in X, contradicting our assumption.

Thus, as $0 \neq \Delta(\dot{x}_1, \dot{x}_2) \in Y$ and there are no non-trivial G-invariant closed linear subspaces of Y, we see that $\overline{\mathrm{span}}\big(G \cdot \Delta(\dot{x}_1, \dot{x}_2)\big) = Y$.

We claim that, for all $T_n, T \in G$, we have

$$T_n \underset{\mathtt{sot}}{\longrightarrow} T \quad \Leftrightarrow \quad \dot{T}_n \underset{\mathtt{sot}}{\longrightarrow} \dot{T}.$$

The implication from left to right is obvious. For the other direction, assume that $\dot{T}_n \underset{\mathtt{sot}}{\longrightarrow} \dot{T}$. Suppose first that $S \in G$ is given. Then, since ϕ and hence also Δ are continuous, we have

$$\lim_n T_n S\Delta(\dot{x}_1, \dot{x}_2) = \lim_n \Delta(\dot{T}_n \dot{S}\dot{x}_1, \dot{T}_n \dot{S}\dot{x}_2) = \Delta(\dot{T}\dot{S}\dot{x}_1, \dot{T}\dot{S}\dot{x}_2) = TS\Delta(\dot{x}_1, \dot{x}_2).$$

As $\overline{\mathrm{span}}\big(G \cdot \Delta(\dot{x}_1, \dot{x}_2)\big) = Y$ and G is a group of isometries, this shows that $T_n y \to Ty$ for all $y \in Y$.

Let now $x \in X$ be given and write $x = \phi(\dot{x}) + y$ for some $y \in Y$. Then, since ϕ is continuous and G-equivariant, we have

$$T_n x = T_n\phi(\dot{x}) + T_n y = \phi(\dot{T}_n\dot{x}) + T_n y \underset{n}{\longrightarrow} \phi(\dot{T}\dot{x}) + Ty = T\phi(\dot{x}) + Ty = Tx,$$

which shows that $T_n \underset{\mathtt{sot}}{\longrightarrow} T$.

Since $T \to \dot{T}$ is clearly a group homomorphism, this shows that it is an isomorphism of the topological groups $(G, \mathtt{sot})$ and $(\dot{G}, \mathtt{sot})$. □

Corollary 12. *Let X be a separable reflexive Banach space and $G \leqslant \mathrm{GL}(X)$ be a bounded subgroup. Suppose that $Y \subseteq X$ is a G-invariant closed linear subspace so that*

(i) *there is no closed linear G-invariant superspace $Y \subsetneq W \subsetneq X$,*
(ii) *there is no closed linear G-invariant complement of Y in X.*

Then the mapping $T \mapsto T|_Y$ is an isomorphism of the topological groups $(G, \mathtt{sot})$ and $(G|_Y, \mathtt{sot})$, where $G|_Y = \{T|_Y \in \mathrm{GL}(Y) \mid T \in G\}$.

Proof. As in the proof of Theorem 11, we may assume that G is a group of isometries of X.

Note that the short exact sequence

$$0 \to Y \to X \to X/Y \to 0$$

gives rise to the short exact sequence

$$0 \to Y^\perp \to X^* \to X^*/Y^\perp \to 0$$

by duality.

We set $G^* = \{T^* \in \mathrm{GL}(X^*) \mid T \in G\}$. Then any G^*-invariant subspace $\{0\} \subsetneq V \subsetneq Y^\perp$ would induce a G-invariant subspace $Y \subsetneq W \subsetneq X$ by $W = V_\perp$. Similarly, a G^*-invariant complement V of $Y^\perp$ in X^* would induce a G-invariant complement $W = V_\perp$ of Y in X. Therefore, we see that G^* and X^* satisfy the conditions of Theorem 11, which means that the map $T^* \in G^* \mapsto (T^*)^{\boldsymbol{\cdot}} \in \mathrm{Isom}(X^*/Y^\perp)$ is an isomorphism of the topological group $(G^*, \mathtt{sot})$ with its image in $\big(\mathrm{Isom}(X^*/Y^\perp), \mathtt{sot}\big)$.

Now, since X is reflexive, the map $T \mapsto T^*$ is an isomorphism of $\big(\mathrm{Isom}(X), \mathtt{sot}\big)$ with $\big(\mathrm{Isom}(X^*), \mathtt{sot}\big)$. Similarly, as $X^*/Y^\perp$ can be identified with Y^* via Hahn–Banach, we again have an isomorphism between $\big(\mathrm{Isom}(X^*/Y^\perp), \mathtt{sot}\big)$ and $\big(\mathrm{Isom}(Y), \mathtt{sot}\big)$. Moreover, the composition of these three maps shows that $T \mapsto T|_Y$ is an isomorphism between $(G, \mathtt{sot})$ and $(G|_Y, \mathtt{sot})$. □

Corollary 13. *Let X be a separable reflexive Banach space and $G \leqslant \mathrm{GL}(X)$ be a bounded subgroup. Suppose that $Y \subseteq X$ is a G-invariant closed linear subspace so that*

(i) *there are no closed linear G-invariant subspaces $\{0\} \subsetneq W \subsetneq Y$ nor superspaces $Y \subsetneq W \subsetneq X$,*
(ii) *there is no closed linear G-invariant complement of Y in X.*

Then the mapping $T|_Y \mapsto \dot{T}$ is well-defined and provides an isomorphism between the topological groups $(G|_Y, \mathtt{sot})$ and $(\dot{G}, \mathtt{sot})$.

Now, returning to our original assumptions, we suppose that X is a separable reflexive Banach space and $G \leqslant \mathrm{GL}(X)$ is a bounded subgroup preserving a closed linear subspace $Y \subseteq X$. Suppose furthermore that Y is complemented in X, i.e., that we may write $X = Y \oplus Z$ for some closed linear subspace $Z \subseteq X$. Since Y is G-invariant, with respect to the decomposition $X = Y \oplus Z$, every element $T \in G$ may be represented by a block matrix

$$\begin{pmatrix} u_T & w_T \\ 0 & v_T \end{pmatrix},$$

where $u_T \in \mathrm{GL}(Y)$, $v_T \in \mathrm{GL}(Z)$ and w_T is a bounded linear operator from Z to Y. Also, u_T is simply the restriction $T|_Y$. Moreover, the quotient map $\pi\colon X \to X/Y$ restricts to an isomorphism between Z and X/Y and we note that the operator $\dot{T} \in \mathrm{GL}(X/Y)$ is conjugate to v_T via this isomorphism. So henceforth, we shall simply identify X/Y with Z and $\dot{T}$ with v_T.

By Lemma 10, every element of G is represented by a block matrix

$$\begin{pmatrix} u & \delta(u,v) \\ 0 & v \end{pmatrix},$$

where $\delta(u,v)\colon Z \to Y$ is a bounded linear operator uniquely determined as a function of u and v. As

$$\begin{pmatrix} u_1 & \delta(u_1,v_1) \\ 0 & v_1 \end{pmatrix}\begin{pmatrix} u_2 & \delta(u_2,v_2) \\ 0 & v_2 \end{pmatrix} = \begin{pmatrix} u_1u_2 & u_1\delta(u_2,v_2) + \delta(u_1,v_1)v_2 \\ 0 & v_1v_2 \end{pmatrix},$$

we find that

$$\delta(u_1u_2, v_1v_2) = u_1\delta(u_2,v_2) + \delta(u_1,v_1)v_2.$$

Lemma 14. *Let $X = Y \oplus Z$ be separable reflexive and $G \leqslant \mathrm{GL}(X)$ a bounded subgroup leaving Y invariant. Then there is a continuous homogeneous map $\psi\colon Z \to Y$ so that*

$$\delta(u,v) = u\psi - \psi v.$$

If X is superreflexive, then ψ may be chosen to be uniformly continuous on bounded sets.

Proof. Suppose that $\phi\colon Z \to X$ is the G-equivariant continuous homogeneous lifting of the canonical projection $\pi\colon Y \oplus Z \to Z$ given by Lemma 8. Then we may write

$$\phi(z) = \begin{pmatrix} -\psi(z) \\ z \end{pmatrix}$$

for some continuous homogeneous $\psi\colon Z \to Y$. Now, for

$$T = \begin{pmatrix} u & \delta(u,v) \\ 0 & v \end{pmatrix} \in G,$$

we have, by the G-equivariance of ϕ and the identification $\dot{T} = v$, that $\phi v = T\phi$, i.e.,

$$\begin{pmatrix} -\psi v(z) \\ v(z) \end{pmatrix} = \phi v(z) = T\phi(z) = \begin{pmatrix} u & \delta(u,v) \\ 0 & v \end{pmatrix}\begin{pmatrix} -\psi(z) \\ z \end{pmatrix} = \begin{pmatrix} -u\psi(z) + \delta(u,v)(z) \\ v(z) \end{pmatrix}$$

for all $z \in Z$. In other words, $-u\psi + \delta(u,v) = -\psi v$ or equivalently $\delta(u,v) = u\psi - \psi v$.

If X is superreflexive, then let $\|\cdot\|$ be a uniformly convex norm on X and recall that the G-invariant norm $|\!|\!|x|\!|\!| = \sup_{T\in G}\|Tx\|$ is uniformly convex on X. The nearest point map $c(x)$ to x in Y considered in the proof of Lemma 8 is then uniformly continuous on bounded sets. Also, since Y is complemented, the Bartle-Graves selector b used in

Lemma 8 may be simply chosen to be the identity (modulo the identification of X/Y with Z). Therefore, according to the definition of ϕ in Lemma 8, ϕ and therefore also ψ are uniformly continuous on bounded sets. $\square$

Now, if $G \leqslant \mathrm{GL}(Y \oplus Z)$ a bounded subgroup leaving Y invariant, we set

$$U = \{u \in \mathrm{GL}(Y) \mid \exists v \in \mathrm{GL}(Z) \begin{pmatrix} u & \delta(u,v) \\ 0 & v \end{pmatrix} \in G\}$$

and

$$V = \{v \in \mathrm{GL}(Z) \mid \exists u \in \mathrm{GL}(Y) \begin{pmatrix} u & \delta(u,v) \\ 0 & v \end{pmatrix} \in G\}$$

and note that these are bounded subgroups of $\mathrm{GL}(Y)$ and $\mathrm{GL}(Z)$ respectively.

Specialising Theorem 11 and Corollary 12 to the setting above, we obtain our first main result.

Theorem 2. *Let $X = Y \oplus Z$ be separable reflexive and $G \leqslant \mathrm{GL}(X)$ a bounded subgroup leaving Y invariant. Assume that*

(i) *there are no closed linear G-invariant subspaces $\{0\} \subsetneq W \subsetneq Y$ nor superspaces $Y \subsetneq W \subsetneq X$,*

(ii) *there is no closed linear G-invariant complement of Y in X.*

Then the mappings

$$\begin{pmatrix} u & \delta(u,v) \\ 0 & v \end{pmatrix} \mapsto u$$

and

$$\begin{pmatrix} u & \delta(u,v) \\ 0 & v \end{pmatrix} \mapsto v$$

are topological group isomorphisms between $(G, \mathtt{sot})$ and $(U, \mathtt{sot})$, respectively $(G, \mathtt{sot})$ and $(V, \mathtt{sot})$.

3. Derivations and non-unitarisable representations

In this section, we apply our results from Section 2 to the special case of Hilbert space, that is, we assume that $Y = Z = \mathcal{H}$, where $\mathcal{H}$ is the separable infinite-dimensional Hilbert space. For this, we shall briefly review how to twist a unitary representation to obtain a non-unitarisable bounded representation.

So suppose that $\lambda\colon \Gamma \to \mathcal{U}(\mathcal{H})$ is a unitary representation of a group Γ on a separable infinite-dimensional Hilbert space $\mathcal{H}$. A *derivation* associated to λ is a map $d\colon \Gamma \to \mathcal{B}(\mathcal{H})$, where $\mathcal{B}(\mathcal{H})$ is the algebra of bounded linear operators on $\mathcal{H}$, satisfying the cocycle equation

$$d(ab) = \lambda(a)d(b) + d(a)\lambda(b)$$

for all $a, b \in \Gamma$. Letting $\mathcal{H}_1$ and $\mathcal{H}_2$ denote two copies of $\mathcal{H}$, this equation is simply equivalent to the requirement that the map $\lambda_d\colon \Gamma \to \mathrm{GL}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ given by

$$\lambda_d(a) = \begin{pmatrix} \lambda(a) & d(a) \\ 0 & \lambda(a) \end{pmatrix}$$

defines a representation, i.e., $\lambda_d(ab) = \lambda_d(a)\lambda_d(b)$. Moreover, this representation is bounded if and only if d is *bounded*, i.e., $\sup_{a\in\Gamma}\|d(a)\| < \infty$.

Let us recall the equivalence of the following statements for a bounded derivation d associated to λ.

(i) $\lambda_d\colon \Gamma \to \mathrm{GL}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ is unitarisable,

(ii) d is *inner*, that is, there is a bounded linear operator $L \in \mathcal{B}(\mathcal{H})$ with $d(a) = \lambda(a)L - L\lambda(a)$,

(iii) there is a closed linear complement $\mathcal{K}$ of $\mathcal{H}_1$ in $\mathcal{H}_1 \oplus \mathcal{H}_2$ invariant under the representation $\lambda_d \colon \Gamma \to \mathrm{GL}(\mathcal{H}_1 \oplus \mathcal{H}_2)$.

To see this, suppose first that $\mathcal{K}$ is a closed linear λ_d-invariant complement of $\mathcal{H}_1$ in $\mathcal{H}_1 \oplus \mathcal{H}_2$ and let P be the projection onto $\mathcal{H}_1$ along $\mathcal{K}$. By the λ_d-invariance of $\mathcal{K}$, P commutes with $\lambda_d(a)$ for all $a \in \Gamma$. So, viewing $d(a)$ as an operator from $\mathcal{H}_2$ to $\mathcal{H}_1$, for all $x \in \mathcal{H}_2$, we have

$$d(a)x + P\lambda(a)x = Pd(a)x + P\lambda(a)x = P\lambda_d(a)x = \lambda_d(a)Px = \lambda(a)Px,$$

i.e., $d(a)x = \lambda(a)Px - P\lambda(a)x$. Letting L be the restriction of P to $\mathcal{H}_2$, we thus find that $d(a) = \lambda(a)L - L\lambda(a)$ for all $a \in \Gamma$ and therefore d is an inner derivation.

Also, if d is inner and thus $d(a) = \lambda(a)L - L\lambda(a)$ for some bounded operator L, then, for all $a \in \Gamma$, we have

$$\begin{pmatrix} \lambda(a) & d(a) \\ 0 & \lambda(a) \end{pmatrix} = \begin{pmatrix} \mathrm{Id} & -L \\ 0 & \mathrm{Id} \end{pmatrix} \begin{pmatrix} \lambda(a) & 0 \\ 0 & \lambda(a) \end{pmatrix} \begin{pmatrix} \mathrm{Id} & L \\ 0 & \mathrm{Id} \end{pmatrix},$$

which shows that $\lambda_d \colon \Gamma \to \mathrm{GL}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ is similar to a block diagonal unitary representation and hence is unitarisable.

Finally, by the complete reducibility of unitary representations, if the representation $\lambda_d \colon \Gamma \to \mathrm{GL}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ is unitarisable, the λ_d-invariant subspace $\mathcal{H}_1$ has a λ_d-invariant complement $\mathcal{K}$ in $\mathcal{H}_1 \oplus \mathcal{H}_2$.

We may now apply the results of Section 2 in this setting.

Corollary 3. *Suppose that $\lambda \colon \Gamma \to \mathcal{U}(\mathcal{H})$ is an irreducible unitary representation of a group Γ on a separable infinite-dimensional Hilbert space $\mathcal{H}$ and $d \colon \Gamma \to \mathcal{B}(\mathcal{H})$ is an associated non-inner bounded derivation. Let also $\mathcal{H}_1$ and $\mathcal{H}_2$ be distinct copies of $\mathcal{H}$ and suppose that $G \leqslant GL(\mathcal{H}_1 \oplus \mathcal{H}_2)$ is a bounded subgroup leaving $\mathcal{H}_1$ invariant and containing $\lambda_d[\Gamma]$. Then the mappings $G \to \mathrm{GL}(\mathcal{H})$ defined by*

$$\begin{pmatrix} u & \delta(u,v) \\ 0 & v \end{pmatrix} \mapsto u$$

and

$$\begin{pmatrix} u & \delta(u,v) \\ 0 & v \end{pmatrix} \mapsto v$$

are topological group isomorphisms between $(G, \mathtt{sot})$ and $(U, \mathtt{sot})$, respectively $(G, \mathtt{sot})$ and $(V, \mathtt{sot})$. Furthermore there is a homogeneous map $\psi \colon \mathcal{H}_2 \to \mathcal{H}_1$, uniformly continuous on bounded sets, such that

$$\delta(u,v) = u\psi - \psi v.$$

Proof. First, by irreducibility of λ, there is no closed linear λ_d-invariant subspace $\{0\} \subsetneq \mathcal{K} \subsetneq \mathcal{H}_1$. Also, since d is not inner, there is no λ_d-invariant closed linear complement of $\mathcal{H}_1$ in $\mathcal{H}_1 \oplus \mathcal{H}_2$.

We also claim that there is no closed linear λ_d-invariant superspace $\mathcal{H}_1 \subsetneq \mathcal{K} \subsetneq \mathcal{H}_1 \oplus \mathcal{H}_2$, as otherwise, $\mathcal{K} \cap \mathcal{H}_2$ would be λ-invariant, contradicting the irreducibility of λ. Indeed, suppose that $x \in \mathcal{K} \cap \mathcal{H}_2$. Then $\lambda_d(a)x = d(a)x + \lambda(a)x \in \mathcal{K}$, whereby, as $d(a)x \in \mathcal{H}_1 \subseteq \mathcal{K}$ and $\lambda(a)x \in \mathcal{H}_2$, also $\lambda(a)x \in \mathcal{K} \cap \mathcal{H}_2$.

Now, since G contains $\lambda_d[\Gamma]$, it follows that there are no G-invariant subspaces of the above type. Therefore, G satisfies the conditions of Lemma 14 and Theorem 2, whereby our result follows. ∎

Note that, if G is as above, then $|||x||| = \sup_{g\in G}\|g(x)\|$ defines a G-invariant norm on $\mathcal{H}_1 \oplus \mathcal{H}_2$ with modulus of convexity of power type 2. Similarly, $|||x^*||| = \sup_{g\in G}\|g(x^*)\|$ defined on $\big(\mathcal{H}_1 \oplus \mathcal{H}_2\big)^*$ has modulus of convexity of power type 2, whereby its G-invariant dual norm has modulus of smoothness of power type 2. Whether there exists a G-invariant norm on $\mathcal{H}_1 \oplus \mathcal{H}_2$ combining the two properties is unclear and turns out to be tightly related to the structure of G:

Proposition 15. *Suppose that $\lambda\colon \Gamma \to \mathcal{U}(\mathcal{H})$ is a unitary representation of a group Γ on a separable infinite-dimensional Hilbert space $\mathcal{H}$ and let d be a bounded derivation associated to λ. Consider the assertions*

- (i) *there is an almost transitive bounded subgroup G of $\mathrm{GL}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ containing $\lambda_d[\Gamma]$,*
- (ii) *there is a $\lambda_d[\Gamma]$-invariant norm on $\mathcal{H}_1 \oplus \mathcal{H}_2$ with moduli of convexity and smoothness of power type 2,*
- (iii) *there is a $\lambda_d[\Gamma]$-invariant norm on $\mathcal{H}_1 \oplus \mathcal{H}_2$ such that the $\mathcal{H}_1$-nearest point map $\mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{H}_1$ is well-defined and Lipschitz,*
- (iv) *there is a homogeneous Lipschitz map $\psi : \mathcal{H}_2 \to \mathcal{H}_1$ such that $d(a) = \lambda(a)\psi - \psi\lambda(a)$,*
- (v) *the group $\lambda_d[\Gamma_z]$ is unitarisable for z outside of a Gauss null subset of $\mathcal{H}_2$, where $\Gamma_z = \{a \in \Gamma \mid \lambda(a)(z) = z\}$.*

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v).

Proof. (i)$\Rightarrow$(ii): As observed by F. Cabello-Sanchez (Corollary 1.3 [6], see also the paper of C. Finet [14]), since there exists a norm with modulus of convexity of power type 2 on $\mathcal{H}_1 \oplus \mathcal{H}_2$, any G-invariant norm has the same property. The same holds for the modulus of smoothness (Corollary 1.6 [6]). In particular there is a $\lambda_d[\Gamma]$-invariant norm on $\mathcal{H}_1 \oplus \mathcal{H}_2$ whose moduli of convexity and smoothness are both of power type 2.

(ii)$\Rightarrow$(iii): According to Theorem 2.8 [3] (see also the beginning of the proof of Theorem 2.9 [3]), if a norm has moduli of convexity and smoothness of power type equal to 2, then the C-nearest point map is Lipschitz on the set $\{x \in \mathcal{H}_1 \oplus \mathcal{H}_2 \mid d(x, C) \leqslant 1\}$, whenever C is a non-empty closed convex subset of $\mathcal{H}_1 \oplus \mathcal{H}_2$. If $C = \mathcal{H}_1$, then this map is also homogeneous and therefore Lipschitz on $\mathcal{H}_1 \oplus \mathcal{H}_2$.

(iii)$\Rightarrow$(iv): Since the $\mathcal{H}_1$-nearest point map c is Lipschitz and since the Bartle–Graves selector b may be chosen to be linear continuous, the $\lambda_d[\Gamma]$-equivariant lifting $\phi : \mathcal{H}_2 \to \mathcal{H}_1 \oplus \mathcal{H}_2$ defined in Lemma 8 is Lipschitz. It follows that the map $\psi : \mathcal{H}_2 \to \mathcal{H}_1$ defined in Lemma 14 such that $d(a) = \lambda(a)\psi - \psi\lambda(a)$ is Lipschitz.

(iv)$\Rightarrow$(v): If ψ is Lipschitz, it is Gâteaux differentiable outside of a Gauss null subset of $\mathcal{H}_2$, [3] Theorem 6.42 (recall that A is Gauss null if it has measure 0 for any non degenerate Gaussian measure on $\mathcal{H}_2$). Let z be a point of Gâteaux differentiability of ψ. Then the relation

$$d(a) = \lambda(a)\psi - \psi\lambda(a)$$

differentiates in z as

$$d(a) = \lambda(a)\psi'(z) - \psi'\big(\lambda(a)z\big)\lambda(a).$$

We deduce that the derivation associated to the restriction of λ to Γ_z is inner, i.e., the subgroup $\lambda_d(\Gamma_z)$ is unitarisable. □

An interesting and unsolved problem is whether linearisation techniques of Lipschitz maps could be used to show that a derivation d satisfying (iv) actually has to be inner.

It is an open question of Deville, Godefroy and Zizler [8] whether a Banach space X, which has an equivalent norm with modulus of convexity of type $p \geqslant 2$ and another equivalent norm with modulus of smoothness of type $1 \leqslant q \leqslant 2$, should have an equivalent norm with both of these properties. Proposition 15 suggests an approach for the similar question concerning G-invariant norms on the Hilbert space, for different choices of bounded subgroups G of $\mathrm{GL}(\mathcal{H})$ - of course, if G is unitarisable, then the answer is trivially yes.

4. THE REPRESENTATION OF $\mathrm{Aut}(T)$ ON $\ell_2(T)$

In the following, we let T denote the $\aleph_0$-regular tree, that is, the countable connected, symmetric, irreflexive graph without loops in which every vertex has infinite valence. One particular realisation of T is as the Cayley graph of the free group on a denumerable set of generators, $\mathbb{F}_\infty$, with respect to its free generating set. We also let λ denote the unitary representation of its automorphism group, $\mathrm{Aut}(T)$, on the vector space $\mathbb{C}^T$ of $\mathbb{C}$-valued functions on T given by

$$\lambda(g)(x) = x(g^{-1}\,\cdot),$$

for $g \in \mathrm{Aut}(T)$ and $x \in \mathbb{C}^T$, and note that the linear subspaces $\ell_p(T)$, $1 \leqslant p \leqslant \infty$, and $c_0(T) \subseteq \mathbb{C}^T$ are $\lambda[\mathrm{Aut}(T)]$-invariant. The same holds for the space $c_{00}(T)$ of finitely supported functions. Let also $G_t = \mathrm{Aut}(T, t)$ denote the isotropy subgroup of the vertex $t \in T$ and, for a subset $A \subseteq T$, set $G_A = \bigcap_{t\in A} G_t$ and let $\mathbb{C}^A$ and $\ell_p(A)$ denote the subspaces of vectors whose support is included in A. We set $\mathbf{1}_t \in \mathbb{C}^T$ to be the Dirac function at the vertex $t \in T$.

We begin by two elementary observations that will be significantly strengthened later on.

Proposition 16. *The unitary representation $\lambda\colon \mathrm{Aut}(T) \to \mathcal{U}(\ell_2(T))$ is irreducible.*

Proof. Note that, since every vertex $s \neq t$ has infinite orbit in T under the action of G_t, $\mathbb{C}\mathbf{1}_t \subseteq \ell_2(T)$ is the 1-dimensional subspace of $\lambda[G_t]$-invariant vectors. Now, if $\ell_2(T) = \mathcal{H} \oplus \mathcal{H}^\perp$ were a $\lambda[\mathrm{Aut}(T)]$-invariant decomposition of $\ell_2(T)$, the orthogonal projection P onto $\mathcal{H}$ would commute with $\lambda[\mathrm{Aut}(T)]$ and so, in particular, $\lambda(g)P\mathbf{1}_t = P\lambda(g)\mathbf{1}_t = P\mathbf{1}_t$ for all $g \in G_t$, i.e., $P\mathbf{1}_t$ is $\lambda[G_t]$-invariant. It follows that $P\mathbf{1}_t \in \mathbb{C}\mathbf{1}_t \cap \mathcal{H}$ and so either $\mathbf{1}_t \in \mathcal{H}^\perp$ or $\mathbf{1}_t \in \mathcal{H}$. But, as $\lambda[\mathrm{Aut}(T)]\mathbf{1}_t$ spans $\ell_2(T)$, we see by the invariance of $\mathcal{H}^\perp$ and $\mathcal{H}$ that either $\mathcal{H}^\perp = \ell_2(T)$ or $\mathcal{H} = \ell_2(T)$, showing irreducibility. □

Proposition 17. $\lambda\colon \mathrm{Aut}(T) \to \mathcal{U}(\ell_2(T))$ *is* uniquely unitarisable*, i.e., up to a scalar multiple, there is a unique λ-invariant inner product on $\ell_2(T)$ equivalent with the usual inner product.*

Proof. Fix $s \in T$ and enumerate the neighbours of s in T as $\{\ldots, t_1, t_0, t_1, \ldots\}$. Pick a sequence of automorphisms $g_1, g_2, g_3, \ldots \in G_s$ so that $g_n(t_i) = t_{i+n}$ for all $n \geqslant 1$ and $i \in \mathbb{Z}$. Then $\lambda(g_n) \underset{wot}{\longrightarrow} P_s$, where P_s denotes the usual orthogonal projection onto $\mathbb{C}\mathbf{1}_s$.

Note that $\ell_2(T\setminus\{s\})$ is a closed linear $\lambda[G_s]$-invariant complement of $\mathbb{C}\mathbf{1}_s$. On the other hand, if $\mathcal{H} \subseteq \ell_2(T)$ is any other closed linear $\lambda[G_s]$-invariant complement of $\mathbb{C}\mathbf{1}_s$, then $P_s x = w\text{–}\lim \lambda(g_s)x$ belongs to $\mathbb{C}\mathbf{1}_s \cap \mathcal{H} = \{0\}$ for all $x \in \mathcal{H}$, whereby $\mathcal{H} \subseteq \ell_2(T \setminus \{s\})$ and hence $\mathcal{H} = \ell_2(T \setminus \{s\})$. It follows that $\ell_2(T \setminus \{s\})$ is the unique $\lambda[G_s]$-invariant closed linear complement of $\mathbb{C}\mathbf{1}_s$ in $\ell_2(T)$.

Now, suppose $\langle\cdot|\cdot\rangle$ denotes the usual inner product on $\ell_2(T)$ and $\langle\cdot|\cdot\rangle'$ is another $\lambda[\mathrm{Aut}(T)]$-invariant equivalent inner product on $\ell_2(T)$. Then, for every $s \in T$, the orthogonal complement $(\mathbb{C}\mathbf{1}_s)^{\perp'}$ is a closed linear $\lambda[G_s]$-invariant complement of $\mathbb{C}\mathbf{1}_s$, so $(\mathbb{C}\mathbf{1}_s)^{\perp'} = \ell_2(T \setminus \{s\}) = (\mathbb{C}\mathbf{1}_s)^\perp$, whereby $\langle \mathbf{1}_s|\mathbf{1}_t\rangle' = 0$ for all $s \neq t$ in T.

Since $\lambda[\mathrm{Aut}(T)]$ acts transitively on $\{\mathbf{1}_s\}_{s\in T}$, we also see that $\langle \mathbf{1}_s|\mathbf{1}_s\rangle' = \langle \mathbf{1}_t|\mathbf{1}_t\rangle'$ for all $s,t\in T$. So, up to multiplication by a scalar, we have $\langle\cdot|\cdot\rangle = \langle\cdot|\cdot\rangle'$. □

We shall now significantly improve the preceding two results by removing any assumptions of continuity. For this, recall that an isometric linear representation $\alpha\colon \Gamma \curvearrowright X$ of a group Γ on a Banach space X is said to have *almost invariant unit vectors* if, for all $\epsilon>0$ and finite sets $F\subseteq\Gamma$, there is a non-zero $x\in X$ with

$$\max_{g\in F}\|x-\lambda(g)x\| < \epsilon\|x\|.$$

Lemma 18. *For all $t\in T$ and $1\leqslant p<\infty$, there are no almost $\lambda[G_t]$-invariant unit vectors in $\ell_p(T\setminus\{t\})$.*

Proof. Fix a countable non-amenable group Γ, a vertex $t\in T$ and enumerate the neighbours of t in T by the elements of Γ. Also, for every $a\in\Gamma$, let T_a denote the subtree of all vertices $s\in T$ whose geodesic to t passes through a. So the rooted trees $\{(T_a,a)\}_{a\in\Gamma}$ are all isomorphic to some fixed rooted tree (T',r) and we can therefore identify $T\setminus\{t\}$ with $T'\times\Gamma$ in such a way that each T_a is identified with $T'\times\{a\}$ via the aforementioned isomorphism. Moreover, if we define an action $\rho\colon\Gamma\curvearrowright\ell_p(T'\times\Gamma)$ by letting Γ shift the second coordinate, it suffices to show that this action does not have almost invariant unit vectors.

To see this, note that, as Γ is non-amenable, the left regular representation $\sigma\colon\Gamma\curvearrowright\ell_p(\Gamma)$ does not have almost invariant unit vectors. There are therefore $g_1,\ldots,g_k\in\Gamma$ and $\epsilon>0$ so that

$$\max_{1\leqslant i\leqslant k}\|x-\sigma(g_i)x\| > \epsilon\|x\| \tag{1}$$

for every non-zero vector $x\in\ell_p(\Gamma)$. For $s\in T'$, let $P_s\colon\ell_p(T'\times\Gamma)\to\ell_p(\{s\}\times\Gamma)$ denote the canonical projection and note that P_s commutes with the $\rho(g_i)$. Now, fix $0\neq x\in\ell_p(T'\times\Gamma)$, set

$$N_i=\big\{s\in T' \;\big|\; \|P_sx-P_s\rho(g_i)x\| = \|P_sx-\rho(g_i)P_sx\| > \epsilon\|P_sx\|\big\}$$

and note that, by (1), $T'=\bigcup_{1\leqslant i\leqslant k}N_i$. We pick i so that $\big(\sum_{s\in N_i}\|P_sx\|^p\big)^{\frac1p}\geqslant\frac1k\|x\|$ and see that

$$\|x-\rho(g_i)x\|^p \geqslant \sum_{s\in N_i}\|P_sx-P_s\rho(g_i)x\|^p > \sum_{s\in N_i}\epsilon^p\|P_sx\|^p \geqslant \frac{\epsilon^p}{k^p}\|x\|^p,$$

i.e., $\|x-\rho(g_i)x\|>\frac{\epsilon}{k}\|x\|$. Thus, no unit vector in $\ell_p(T'\times\Gamma)$ is $\big(\rho(g_1),\ldots,\rho(g_k);\frac{\epsilon}{k}\big)$-invariant. □

The operator R below occurs frequently in work on uniqueness of translation invariant functionals, e.g., [4].

Lemma 19. *For all $t\in T$ and $1<p\leqslant\infty$, every linear operator $S\colon\ell_p(T)\to\mathbb{C}^T$ in the commutant of $\lambda[G_t]$ maps $\ell_p(T\setminus\{t\})$ into $\mathbb{C}^{T\setminus\{t\}}$. It follows that $\ell_p(T\setminus\{t\})$ is the unique $\lambda[G_t]$-invariant linear complement of $\mathbb{C}\mathbf{1}_t$ in $\ell_p(T)$.*

Proof. Let $1\leqslant q<\infty$ be the conjugate index of p. Fix $g_1,\ldots,g_k\in G_t$ and $\epsilon>0$ so that

$$\max_{1\leqslant i\leqslant k}\|x-\lambda(g_i)x\| > \epsilon\|x\|$$

for any non-zero vector $x \in \ell_q(T \setminus \{t\})$. It follows that the operator

$$R\colon \ell_q\big(T \setminus \{t\}\big) \longrightarrow \underbrace{\ell_q\big(T \setminus \{t\}\big) \oplus \ldots \oplus \ell_q\big(T \setminus \{t\}\big)}_{k \text{ copies}}$$

defined by $Rx = \big(x - \lambda(g_1)x, \ldots, x - \lambda(g_k)x\big)$ is an isomorphism with a closed subspace and therefore the conjugate operator R^* is surjective. Thus, every element $x \in \ell_p\big(T \setminus \{t\}\big)$ can be written as

$$x = \sum_{i=1}^{k} y_i - \lambda(g_i)y_i,$$

for some $y_i \in \ell_p\big(T \setminus \{t\}\big)$.

In particular, if $S\colon \ell_p\big(T \setminus \{t\}\big) \to \mathbb{C}^T$ is any linear operator commuting with $\lambda[G_t]$, then

$$\mathbf{1}_t^*(Sx) = \sum_{i=1}^{k} \mathbf{1}_t^* S y_i - \mathbf{1}_t^* \lambda(g_i) S y_i = \sum_{i=1}^{k} \mathbf{1}_t^* S y_i - \mathbf{1}_t^* S y_i = 0,$$

as $\mathbf{1}_t^* \lambda(g_i) = \mathbf{1}_t^*$. That is, S maps $\ell_p(T \setminus \{t\})$ into $\mathbb{C}^{T \setminus \{t\}}$.

Thus, if $P\colon \ell_p(T) \to \mathbb{C}\mathbf{1}_t$ is a linear projection commuting with $\lambda[G_t]$, then $\ell_p(T \setminus \{t\}) \subseteq \ker P$, and, since $\ell_p(T \setminus \{t\})$ is also a linear complement of $\mathbb{C}\mathbf{1}_t$, it follows that $\ell_p(T \setminus \{t\}) = \ker P$, whereby P is the projection along the subspace $\ell_p(T \setminus \{t\})$. □

We can now obtain the following strengthening of Proposition 17.

Theorem 20. *The usual inner product is, up to a scalar multiple, the unique $\lambda[\mathrm{Aut}(T)]$-invariant inner product on $\ell_2(T)$.*

Proof. Note that, if $\langle\cdot|\cdot\rangle'$ is a $\lambda[\mathrm{Aut}(T)]$-invariant inner product on $\ell_2(T)$, then, for every $t \in T$, the orthogonal complement $(\mathbb{C}\mathbf{1}_t)^{\perp'}$ of $\mathbb{C}\mathbf{1}_t$ with respect to $\langle\cdot|\cdot\rangle'$ is a $\lambda(G_t)$-invariant linear complement. So, by Lemma 19, we have $(\mathbb{C}\mathbf{1}_t)^{\perp'} = \ell_2(T \setminus \{t\})$ and, in particular, $\langle \mathbf{1}_s|\mathbf{1}_t\rangle' = 0$ for all $s \neq t$ in T, whereby $\{\mathbf{1}_t\}_{t\in T}$ is an $\langle\cdot|\cdot\rangle'$-orthogonal sequence. Since $\mathrm{Aut}(T)$ acts transitively on T, we also see that $\langle \mathbf{1}_s|\mathbf{1}_s\rangle' = \langle \mathbf{1}_t|\mathbf{1}_t\rangle' > 0$ for all $s, t \in T$ and hence, by multiplying by a positive scalar, we may suppose that $\{\mathbf{1}_t\}_{t\in T}$ is actually orthonormal with respect to $\langle\cdot|\cdot\rangle'$, whence the usual inner product agrees with $\langle\cdot|\cdot\rangle'$ on $c_{00}(T)$.

Observe now that

$$c_{00}(T)^{\perp'} = \bigcap_{t\in T} \ell_2(T \setminus \{t\}) = \{0\},$$

showing that $c_{00}(T)$ is $\|\cdot\|'$-dense in $\ell_2(T)$, where $\|\cdot\|'$ is the norm induced by $\langle\cdot|\cdot\rangle'$. It then follows from Parseval's Equality applied to each of the inner products that any $x \in \ell_2(T)$ may be simultaneously approximated in the two norms by an element of $c_{00}(T)$, which, by Cauchy–Schwarz, implies that the two inner products agree on $\ell_2(T)$. □

Theorem 4. *The commutant of $\lambda[\mathrm{Aut}(T)]$ in the space of linear operators from $\ell_p(T)$ to $\mathbb{C}^T$, $1 < p \leqslant \infty$, is just $\mathbb{C}\cdot\mathrm{Id}$.*

Proof. Note that if S belongs to the commutant, then, by Lemma 19, S maps $\ell_p(T \setminus \{t\})$ into $\mathbb{C}^{T\setminus\{t\}}$ for every $t \in T$ and hence maps $\ell_p(A) = \bigcap_{t\notin A} \ell_p(T \setminus \{t\})$ into $\mathbb{C}^A = \bigcap_{t\notin A} \mathbb{C}^{T\setminus\{t\}}$ for all subsets $A \subseteq T$. So fix $t \in T$ and write $S\mathbf{1}_t = \alpha\mathbf{1}_t$ for some $\alpha \in \mathbb{C}$. Then, for any $g \in G$, we have

$$S\mathbf{1}_{g(t)} = S\lambda(g)\mathbf{1}_t = \lambda(g)S\mathbf{1}_t = \alpha\lambda(g)\mathbf{1}_t = \alpha\mathbf{1}_{g(t)}.$$

As $\mathrm{Aut}(T)$ acts transitively on T, it follows that $S\mathbf{1}_s = \alpha\mathbf{1}_s$ for all $s \in T$. Now, suppose that $x \in \ell_p(T)$ and $s \in T$ and write $x = y + \xi\mathbf{1}_s$, with $y \in \ell_p(T \setminus \{s\})$ and $\xi \in \mathbb{C}$. It then follows that

$$\mathbf{1}_s^*(Sx) = \mathbf{1}_s^*(Sy) + \mathbf{1}_s^*\big(S(\xi\mathbf{1}_s)\big) = \alpha\xi,$$

showing that $Sx = \alpha x$. Thus $S = \alpha{\cdot}\mathrm{Id}$. □

Let us also observe that Theorem 4 fails for $p = 1$. Indeed, if we define $N\colon \ell_1(T) \to \ell_\infty(T)$ by $N(\mathbf{1}_s) = \sum_{t \in \mathcal{N}_s} \mathbf{1}_t$, where $\mathcal{N}_s$ is the set of neighbours of s in T, then N clearly commutes with every $\lambda(g)$, $g \in \mathrm{Aut}(T)$.

Theorems 20 and 4 show strong rigidity properties of the representation $\lambda\colon \mathrm{Aut}(T) \to \mathcal{U}(\ell_2(T))$. In this connection, it is natural to ask whether, apart from determining the inner product, it also determines the norm on $\ell_2(T)$. Indeed, suppose $\lambda[\mathrm{Aut}(T)] \leqslant K \leqslant \mathrm{GL}(\ell_2(T))$ is a bounded subgroup. Then there is an equivalent K-invariant norm $\vert\!\vert\!\vert \cdot \vert\!\vert\!\vert$ on $\ell_2(T)$ that is uniformly convex and uniformly smooth. Moreover, for any finite subtree $A \subseteq T$, the space $\ell_2(T)^{\lambda[G_A]}$ of $\lambda[G_A]$-invariant vectors is just $\ell_2(A)$. So, by the Alaoglu–Birkhoff Theorem (see, e.g., Theorem 4.10 [13]), there is a projection P_A of $\ell_2(T)$ onto the subspace $\ell_2(A)$ with $\vert\!\vert\!\vert P_A \vert\!\vert\!\vert = 1$. Furthermore, this must be the usual orthogonal projection since it commutes with $\lambda[G_A]$. Note also that the same holds in the dual. Finally, by approximating by finite subtrees and passing to a `wot`-limit, one observes that $\vert\!\vert\!\vert P_A \vert\!\vert\!\vert = 1$ for all non-empty subtrees $A \subseteq T$. This puts serious restrictions on the norm $\vert\!\vert\!\vert \cdot \vert\!\vert\!\vert$ and thus also on K.

Problem 21. *Is every bounded subgroup $\lambda[\mathrm{Aut}(T)] \leqslant K \leqslant \mathrm{GL}(\ell_2(T))$ contained in $\mathcal{U}(\ell_2(T))$?*

Observe that this is equivalent to asking whether every such K is unitarisable, since then the K-invariant inner product must be the usual one and hence $K \leqslant \mathcal{U}(\ell_2(T))$.

5. A DERIVATION ASSOCIATED TO $\mathrm{Aut}(T)$

In the following, we shall study a well-known derivation giving rise to a non-unitarisable representation of $\mathbb{F}_\infty$ (see also [22, 20] for different presentations). For this, we introduce a bounded linear operator

$$L\colon \ell_1(T) \to \ell_1(T),$$

where T is the $\aleph_0$-regular tree as in Section 4. We begin by fixing a root $e \in T$ and let $\hat{\cdot}\colon T \to T$ be the map defined by $\hat{e} = e$ and $\hat{s} = s_{n-1}$, whenever $s \neq e$ and $s_0, s_1, s_2, \ldots, s_{n-1}, s_n$ is the geodesic from $s_0 = e$ to $s_n = s$. Also, for any $s \in T$, let $\mathcal{N}_s$ denote the set of neighbours of s in T.

We then let $L\colon \ell_1(T) \to \ell_1(T)$ be the unique bounded linear operator satisfying

$$L(\mathbf{1}_s) = \mathbf{1}_{\hat{s}}$$

for $s \neq e$ and

$$L(\mathbf{1}_e) = 0.$$

Observe then that the adjoint operator $L^*\colon \ell_\infty(T) \to \ell_\infty(T)$ satisfies

$$L^*(\mathbf{1}_s) = \Big(\sum_{t \in \mathcal{N}_s} \mathbf{1}_t\Big) - \mathbf{1}_{\hat{s}} = \sum_{\hat{t} = s} \mathbf{1}_t$$

for $s \neq e$ and

$$L^*(\mathbf{1}_e) = \sum_{t \in \mathcal{N}_e} \mathbf{1}_t.$$

In other words, if $N\colon \ell_1(T) \to \ell_\infty(T)$ is the bounded operator defined at the end of Section 4 by $N(\mathbf{1}_s) = \sum_{t\in\mathcal{N}_s} \mathbf{1}_t$, then $L^* + L = N$, which commutes with every $\lambda(g)$, $g \in \mathrm{Aut}(T)$. From this it follows that, for every $g \in \mathrm{Aut}(T)$,

$$\lambda(g)L - L\lambda(g) = L^*\lambda(g) - \lambda(g)L^*$$

as operators on $\ell_1(T)$. We may hence conclude that

$$d(g) = L^*\lambda(g) - \lambda(g)L^*$$

defines an operator on $\ell_\infty(T)$ of norm at most 2, which restricts to an operator on $\ell_1(T)$ of norm at most 2 and therefore, by the Riesz-Thorin interpolation Theorem, that $d(g)$ restricts to an operator on $\ell_2(T)$ of norm at most 2. As evidently

$$\begin{aligned} d(gf) &= L^*\lambda(gf) - \lambda(gf)L^* \\ &= (L^*\lambda(g) - \lambda(g)L^*)\lambda(f) + \lambda(g)(L^*\lambda(f) - \lambda(f)L^*) \\ &= d(g)\lambda(f) + \lambda(g)d(f), \end{aligned}$$

we see that $d\colon \mathrm{Aut}(T) \to \mathcal{B}(\ell_2(T))$ is a bounded derivation.

Now, if one identifies T with the Cayley graph of $\mathbb{F}_\infty$, it is known that even the restriction of d to $\mathbb{F}_\infty$ viewed as translations of T is non-inner (see, e.g., [22, 20]). However, allowing for all of $\mathrm{Aut}(T)$, we see that not only is d not defined by an element of $\mathcal{B}(\ell_2(T))$, but L^* is essentially the only linear operator from $\ell_2(T)$ to $\mathbb{C}^T$ defining d.

Theorem 22. *Suppose $A\colon \ell_2(T) \to \mathbb{C}^T$ is a globally defined linear operator so that $d(g) = A\lambda(g) - \lambda(g)A$ for all $g \in \mathrm{Aut}(T)$. Then $A = L^* + \vartheta\mathrm{Id}$ for some $\vartheta \in \mathbb{C}$.*

Proof. Assume that $A\colon \ell_2(T) \to \mathbb{C}^T$ is as above. Then, for all $g \in \mathrm{Aut}(T)$,

$$A\lambda(g) - \lambda(g)A = d(g) = L^*\lambda(g) - \lambda(g)L^*,$$

i.e., $\big(A - L^*\big)\lambda(g) = \lambda(g)\big(A - L^*\big)$. By Theorem 4, it follows that $A - L^* = \vartheta\mathrm{Id}$ for some $\vartheta \in \mathbb{C}$ and our theorem follows. □

Thus, to see that d is not inner or even that d cannot be written as $d(g) = A\lambda(g) - \lambda(g)A$ with $A\colon \ell_2(T) \to \ell_2(T)$ a globally defined linear operator, note that, in this case, $A = L^* + \vartheta\mathrm{Id}$ for some ϑ, whereby L^* would have to map $\ell_2(T)$ into $\ell_2(T)$, which it does not.

However, even though the derivation $d\colon \mathrm{Aut}(T) \to \mathcal{B}(\ell_2(T))$ is not inner, by Lemma 14, we see that there is a homogeneous map $\psi\colon \ell_2(T) \to \ell_2(T)$, uniformly continuous on bounded sets, so that

$$d(g) = \lambda(g)\psi - \psi\lambda(g)$$

for all $g \in \mathrm{Aut}(T)$.

In the following, we combine the results above with the analysis of Sections 2 and 3. So, to simplify notation, we let $\mathcal{H}_1$ and $\mathcal{H}_2$ denote two distinct copies of $\ell_2(T)$. Now, suppose that $G \leqslant \mathrm{GL}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ is a bounded subgroup leaving $\mathcal{H}_1$ invariant and containing $\lambda_d[\mathrm{Aut}(T)]$, i.e., containing the block matrices

$$\begin{pmatrix} \lambda(g) & d(g) \\ 0 & \lambda(g) \end{pmatrix} = \begin{pmatrix} \lambda(g) & L^*\lambda(g) - \lambda(g)L^* \\ 0 & \lambda(g) \end{pmatrix},$$

for all $g \in \mathrm{Aut}(T)$. As we have seen in Section 2, there is a partial map

$$\delta\colon \mathrm{GL}(\mathcal{H}_1) \times \mathrm{GL}(\mathcal{H}_2) \to \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$$

so that every element of G is of the form

$$\begin{pmatrix} u & \delta(u,v) \\ 0 & v \end{pmatrix}$$

for some $u \in \mathrm{GL}(\mathcal{H}_1)$ and $v \in \mathrm{GL}(\mathcal{H}_2)$. Also, by Lemma 14, there is a homogeneous map $\psi\colon \mathcal{H}_2 \to \mathcal{H}_1$, uniformly continuous on bounded sets, so that

$$\delta(u,v) = u\psi - \psi v$$

for all u, v.

Therefore, by the expressions for $d(g)$, we see that

$$\lambda(g)\big(L^* + \psi\big) = \big(L^* + \psi\big)\lambda(g),$$

when L^* and ψ are viewed as continuous maps $\ell_2(T) \to \ell_\infty(T)$, while

$$\lambda(g)\big(L - \psi\big) = \big(L - \psi\big)\lambda(g),$$

when L and ψ are viewed as continuous maps $\ell_1(T) \to \ell_2(T)$. In other words, $L^* + \psi$ and $L - \psi$ commute with $\lambda(g)$ for $g \in \mathrm{Aut}(T)$.

Now, for every subset $S \subseteq T$, let $G_S = \{g \in \mathrm{Aut}(T) \mid g(t) = t,\ \forall t \in S\}$ and note that $\lambda[G_S]$ acts trivially on $\ell_1(S)$. Since $L - \psi$ commutes with the $\lambda(g)$, we see that, for any $g \in G_S$ and $x \in \ell_1(S)$,

$$(L-\psi)x = (L-\psi)\lambda(g)x = \lambda(g)(L-\psi)x,$$

which means that $(L-\psi)x \in \ell_2(T)^{\lambda[G_S]}$, where the latter denotes the subspace of $\lambda[G_S]$-invariant vectors in $\ell_2(T)$. But, if S is a finite subtree, then

$$\ell_2(T)^{\lambda[G_S]} = \ell_2(S),$$

showing that $L - \psi$ maps $\ell_1(S)$ into $\ell_2(S)$. Approximating arbitrary subtrees by finite subtrees and extending by continuity, we conclude that $L - \psi$ maps $\ell_1(S)$ into $\ell_2(S)$ for all subtrees $S \subseteq T$. However, L maps $\ell_1(S)$ into $\ell_1(S \cup \hat{S})$, which shows that ψ maps $\ell_1(S)$ into $\ell_2(S \cup \hat{S})$. More precisely, assuming that $e \notin S$, if $s \in S$ denotes the vertex closest to e, we have, for all $x \in \ell_1(S)$,

$$\mathbf{1}^*_{\hat{s}}(Lx) = \mathbf{1}_s(x),$$

and so $\mathbf{1}^*_{\hat{s}}(\psi x) = \mathbf{1}_s(x)$.

Observe also that the continuous homogenous map $\Delta\colon \mathcal{H}_2 \times \mathcal{H}_2 \to \mathcal{H}_1$ given by

$$\Delta(x,y) = \psi(x) + \psi(y) - \psi(x+y)$$

satisfies $\Delta(x,y) = (\psi - L)x + (\psi - L)y - (\psi - L)(x+y)$ for $x, y \in \ell_1(T)$, as L is linear. Therefore, for any subtree $S \subseteq T$, Δ maps $\ell_1(S) \times \ell_1(S)$ into $\ell_2(S)$ and hence, by density of $\ell_1(S)$ in $\ell_2(S)$,

$$\Delta\colon \ell_2(S) \times \ell_2(S) \to \ell_2(S).$$

Also, as $\delta(u,v) = u\psi - \psi v$ is linear, one readily verifies that $\Delta(vx, vy) = u\Delta(x,y)$ for all x, y.

Finally, since by Proposition 16 the unitary representation $\lambda\colon \mathrm{Aut}(T) \to \mathcal{U}(\ell_2(T))$ is irreducible, it follows from Corollary 3 that the maps

$$\begin{pmatrix} u & \delta(u,v) \\ 0 & v \end{pmatrix} \mapsto u$$

and

$$\begin{pmatrix} u & \delta(u,v) \\ 0 & v \end{pmatrix} \mapsto v$$

are $\texttt{sot}$-isomorphisms between G and the respective images in $\mathrm{GL}(\mathcal{H}_1)$ and $\mathrm{GL}(\mathcal{H}_2)$.

We sum up the above discussion in the following theorem.

Theorem 6. *Suppose that $G \leqslant \mathrm{GL}(\ell_2(T) \oplus \ell_2(T))$ is a bounded subgroup leaving the first copy of $\ell_2(T)$ invariant and containing $\lambda_d[\mathrm{Aut}(T)]$.*

Then there is a homogeneous map $\psi\colon \ell_2(T) \to \ell_2(T)$, uniformly continuous on bounded subsets, for which

$$L^* + \psi\colon \ell_2(T) \to \ell_\infty(T) \quad \textit{and} \quad L - \psi\colon \ell_1(T) \to \ell_2(T)$$

commute with $\lambda(g)$ for $g \in \mathrm{Aut}(T)$ and so that every element of G is of the form

$$\begin{pmatrix} u & u\psi - \psi v \\ 0 & v \end{pmatrix}$$

for some $u, v \in \mathrm{GL}(\ell_2(T))$.

Finally, the mappings

$$\begin{pmatrix} u & u\psi - \psi v \\ 0 & v \end{pmatrix} \mapsto u \qquad \textit{and} \qquad \begin{pmatrix} u & u\psi - \psi v \\ 0 & v \end{pmatrix} \mapsto v$$

are $\mathtt{sot}$-isomorphisms between G and their respective images in $\mathrm{GL}(\ell_2(T))$.

Note that it seems to remain open whether the two mappings in the conclusion of this theorem have to be identical, i.e., whether u is necessary equal to v in the notation of the theorem. Another unsolved question, directly related to Proposition 15, is whether the map ψ may be chosen to be Lipschitz.

Using the information given by Theorem 6 and its proof, one may compute some simple values of the function ψ associated to a bounded subgroup G.

Example 23. Since $\psi - L$ maps $\ell_1(\{e\}) = \mathbb{C}\mathbf{1}_e$ into $\ell_2(\{e\}) = \mathbb{C}\mathbf{1}_e$ and $L\mathbf{1}_e = 0$, we must have $\psi(\mathbf{1}_e) = \mu\mathbf{1}_e$ for some $\mu \in \mathbb{C}$. Thus, for any other vertex $s \in T \setminus \{e\}$, write $s = g(e)$ for some $g \in \mathrm{Aut}(T)$, whereby

$$\psi(\mathbf{1}_s) = L\mathbf{1}_s - (L-\psi)\lambda(g)\mathbf{1}_e = \mathbf{1}_{\hat{s}} - \lambda(g)(L-\psi)\mathbf{1}_e = \mathbf{1}_{\hat{s}} - \mu\mathbf{1}_s.$$

Example 24. Suppose that $s \neq e$ and $\hat{s} = e$. Then $(\psi - L)(\mathbf{1}_s + \mathbf{1}_e) = \mu\mathbf{1}_s + \nu\mathbf{1}_e$ for some $\mu, \nu \in \mathbb{C}$, whence $\psi(\mathbf{1}_s + \mathbf{1}_e) = \mu\mathbf{1}_s + (1+\nu)\mathbf{1}_e$ with μ, ν independent of s. Again, for any pair t and $\hat{t}$ of neighbouring vertices in $T \setminus \{e\}$, there is a $g \in \mathrm{Aut}(T)$ so that $g(s) = t$ and $g(e) = \hat{t}$, whereby

$$\psi(\mathbf{1}_t + \mathbf{1}_{\hat{t}}) = (\psi - L)(\mathbf{1}_t + \mathbf{1}_{\hat{t}}) + L(\mathbf{1}_t + \mathbf{1}_{\hat{t}}) = \mu\mathbf{1}_t + (1+\nu)\mathbf{1}_{\hat{t}} + \mathbf{1}_{\hat{\hat{t}}}.$$

Example 25. Consider now the special case when the map δ is defined by $\delta(u, v) = Au - vA$ for some globally defined linear operator $A\colon \ell_2(T) \to \mathbb{C}^T$ (note that this requires the v to be defined from $A[\ell_2(T)]$ into $\mathbb{C}^T$). Then, by Theorem 22, we have that $A = L^* + \vartheta\mathrm{Id}$ for some $\vartheta \in \mathbb{C}$, i.e.,

$$\delta(u, v) = L^*u - vL^* + \vartheta(u - v).$$

Moreover, as

$$\begin{pmatrix} \mathrm{Id} & \vartheta\mathrm{Id} \\ 0 & \mathrm{Id} \end{pmatrix}\begin{pmatrix} u & L^*u - vL^* + \vartheta(u-v) \\ 0 & v \end{pmatrix}\begin{pmatrix} \mathrm{Id} & -\vartheta\mathrm{Id} \\ 0 & \mathrm{Id} \end{pmatrix} = \begin{pmatrix} u & L^*u - vL^* \\ 0 & v \end{pmatrix},$$

we see that by conjugating G by the bounded operator $\begin{pmatrix} \mathrm{Id} & \vartheta\mathrm{Id} \\ 0 & \mathrm{Id} \end{pmatrix}$, we obtain another bounded subgroup G' with a corresponding map $\delta'(u, v) = L^*u - vL^*$.

6. STRICT CONVEXITY OF SEPARABLE TRANSITIVE BANACH SPACES

Let $(X, \|\cdot\|)$ be a fixed separable *transitive* Banach space, i.e., whose linear isometry group $\mathrm{Isom}(X, \|\cdot\|)$ acts transtively on the unit sphere $S_X = \{x \in X \mid \|x\| = 1\}$. By a classical theorem of S. Mazur (Theorem 8.2 [12]) the norm $\|\cdot\|$ is Gâteaux differentiable on a dense G_δ subset of S_X. So, by transitivity of the norm, this implies that $\|\cdot\|$ is actually Gâteaux differentiable at every point of S_X. Hence the following lemma.

Lemma 26. *Let $(X, \|\cdot\|)$ be a separable transitive Banach space. Then $\|\cdot\|$ is Gâteaux differentiable, i.e., for every $x \in S_X$, there is a unique support functional $\phi_x \in S_{X^*}$, that is, so that $\phi_x(x) = 1$.*

Now, Gâteaux differentiablity of norms and strict convexity are related via duality (see Corollary 7.23 [12]), in the sense that, e.g., Gâteaux differentiability of the dual norm $\|\cdot\|^*$ implying strict convexity of $\|\cdot\|$. However, little information can be gained directly from the Gâteaux differentiability of the norm on X. Nevertheless, using the Bishop–Phelps theorem, we shall see that the norm is actually strictly convex. For this, let us recall the statement of the Bishop–Phelps theorem: If C is a non-empty bounded closed convex subset of a real Banach space X, then the set

$$\{\phi \in X^* \mid \exists x \in C \ \sup_{y \in C} \phi(y) = \phi(x)\}$$

is norm-dense in X^*.

Lemma 27. *Let X be a separable real Banach space and $C \subseteq S_X$ a non-empty closed convex set so that the setwise stabiliser $\{T \in \mathrm{Isom}(X) \mid T[C] = C\}$ acts transitively on C. Then C consists of a single point.*

Proof. Since X is separable, we can pick a dense sequence (x_n) in C. Let also $\lambda_n > 0$ be so that $\sum_n \lambda_n = 1$ and define $x = \sum_n \lambda_n x_n \in C$ (note that since $\|x_n\| = 1$ the sum is absolutely convergent).

As was observed by Rolewicz [24], if $\phi \in X^*$ attains its maximum on C at x, then ϕ must be constant on C. Indeed, in this case,

$$\phi(x) = \phi\big(\sum_n \lambda_n x_n\big) = \sum_n \lambda_n \phi(x_n) \leqslant \sum_n \lambda_n \phi(x) = \phi(x),$$

so $\phi(x_n) = \phi(x)$ for all n, whence $\phi \equiv \phi(x)$ on C.

Now suppose for a contradiction that C contains distinct points y and z. We pick $\psi \in X^*$ of norm 1 so that $\psi(y - z) = \epsilon > 0$. Then, by the theorem of Bishop–Phelps, there is some $\phi \in X^*$ with $\|\psi - \phi\| < \frac{\epsilon}{2\|y-z\|}$ that attains it supremum on C at some point $v \in C$. Also,

$$|\phi(y - z)| \geqslant |\psi(y - z)| - |(\psi - \phi)(y - z)| \geqslant \epsilon - \|\psi - \phi\| \cdot \|y - z\| > \frac{\epsilon}{2} > 0,$$

so ϕ is not constant on C.

Choose some $T \in \mathrm{Isom}(X)$ with $T[C] = C$ so that $Tx = v$ and note that $T^*\phi$ attains it maximum on C at x and thus must be constant on C. However, this is absurd, since $T[C] = C$ and ϕ fails to be constant on C. $\square$

Theorem 28. *Let $(X, \|\cdot\|)$ be a separable real transitive Banach space. Then X is strictly convex and $\|\cdot\|$ is Gâteaux differentiable.*

Proof. We already know that $\|\cdot\|$ is Gâteaux differentiable and thus every $x \in S_X$ has a unique support functional $\phi_x \in S_{X^*}$. Now, for $x \in S_X$, consider the closed convex subset

$$C_x = \{z \in S_X \mid \phi_z = \phi_x\} = \{z \in S_X \mid \phi_x(z) = 1\},$$

where the second equality follows from the uniqueness of the support functional.

Then, for all $T \in \mathrm{Isom}(X)$, either $T[C_x] = C_x$ or $T[C_x] \cap C_x = \emptyset$. For, suppose $z, Tz \in C_x$. Then $T^*\phi_x(z) = \phi_x(Tz) = 1 = \phi_x(z)$, whence by uniqueness of support functionals we have $T^*\phi_x = \phi_x$ and hence

$$\phi_x(Ty) = T^*\phi_x(y) = 1$$

for all $y \in C_x$, i.e., $T[C_x] \subseteq C_x$. Similarly, $T^{-1}[C_x] \subseteq C_x$ and thus $T[C_x] = C_x$.

Therefore, as $\mathrm{Isom}(X)$ acts transitively on S_X, we see that $\{T \in \mathrm{Isom}(X) \mid T[C_x] = C_x\}$ acts transitively on C_x and hence, by the preceding lemma, $C_x = \{x\}$. It follows that the mapping $x \in S_x \mapsto \phi_x \in S_{X^*}$ is injective and that every functional $\phi \in S_{X^*}$ attains its norm in at most one point of S_X, that is, X is strictly convex. $\square$

We note that, by the theorem of Bishop-Phelps, the set of norm attaining functionals is norm dense in X^*, which, in our setting means that the ϕ_x for $x \in S_X$ are norm dense in S_{X^*}. Moreover, by the Gâteaux differentiability of the norm, the mapping $x \in S_X \mapsto \phi_x \in S_{X^*}$ is $\|\cdot\|$-to-w^* continuous. The action of $\mathrm{Isom}(X)$ on S_{X^*} is transitive on the set $\{\phi_x\}_{x \in S_X}$ and thus X^* is almost transitive. However, little geometric information can be obtained exclusively from almost transitivity, since it is known by a result of W. Lusky [18] that every (separable) Banach space X is isometric to a complemented subspace of an almost transitive (separable) Banach space.

Note also that if X has the Radon-Nikodym Property or is an Asplund space, then quite stronger results may be obtained. Indeed, any almost transitive norm on such a space is already uniformly convex (Corollary 1.3 and Theorem 2.1 [6]).

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