Coarse Geometry of Topological Groups

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Preface

The present monograph is the product of my research initiated during an extended sabbatical financed in part by a fellowship by the Simons Foundation. While the basic discovery of an appropriate coarse structure on Polish groups really materialised around 2013, the initial seeds were sown in several prior works and, in my own case, in the two studies [**62**] and [**63**] where I developed various concepts of boundedness in topological groups.

The main objects of this study are *Polish groups*, that is, separable and completely metrisable topological groups. While these can of course be treated abstractly, I am principally interested in them as they appear in applications, namely as transformation groups of various mathematical structures and even as the additive groups of separable Banach spaces. This class of groups has received a substantial amount of attention over the last two decades and the results here are my attempt at grappling with possible geometric structures on them. In particular, this is a response to the question of how to apply the language and techniques of geometric group theory, abstract harmonic and functional analysis to their study.

Since this project has been a long time coming, my ideas on the subject have evolved over time and been influenced by a number of different people. Evidently, the work of Mikhail Gromov pervades all of geometric group theory and hence also the ideas presented here. But another specific reference is John Roe's lectures on coarse geometry [**60**] and whose framework of coarse spaces allowed me to extend the definition of a geometric structure to all topological groups and not just the admittedly more interesting subclass of locally bounded Polish groups.

While this book contains the first formal presentation of the theory, parts of it have already found its way into other publications. In particular, in [**65**] I made a systematic study of equivariant geometry of amenable Polish groups, including separable Banach spaces, and made use of some of the theory presented here. Similarly, in a collaboration with Kathryn Mann [**46**], we investigated the large scale geometry of homeomorphism groups of compact manifolds within the present framework, while other authors [**16, 83, 31**] have done so for other groups. Finally, [**66**] contains a characterisation of the small scale geometry of Polish groups and its connection to the large scale.

Over the years, I have greatly benefitted from all my conversations on this topic with a number of people. Though I am bound to exclude many, these include Uri Bader, Bruno Braga, Michael Cohen, Yves de Cornulier, Marc Culler, Alexander Dranishnikov, Alexander Furman, William Herndon, Kathryn Mann, Julien Melleray, Nicolas Monod, Justin Moore, Vladimir Pestov, Konstantin Slutsky, Sławomir Solecki, Andreas Thom, Simon Thomas, Todor Tsankov, Phillip Wesolek, Kevin Whyte and Joseph Zielinski.

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CHAPTER 1

Introduction

1. Motivation

Geometric group theory or large scale geometry of finitely generated discrete groups or compactly generated locally compact groups is by now a well-established theory (see [**55, 19**] for recent accounts). In the finitely generated case, the starting point is the elementary observation that the word metrics ρ_Σ on a discrete group Γ given by finite symmetric generating sets $\Sigma \subseteq \Gamma$ are mutually quasi-isometric and thus any such metric may be said to define the large scale geometry of Γ. This has led to a very rich theory weaving together combinatorial group theory, geometry, topology and functional analysis stimulated by the impetus of M. Gromov (see, e.g., [**30**]).

In the locally compact setting, matters have not progressed equally swiftly even though the basic tools have been available for quite some time. Indeed, by a result of R. Struble [**68**] dating back to 1951, every locally compact second countable group admits a compatible left-invariant *proper* metric, i.e., so that the closed balls are compact. Struble's theorem was based on an earlier well-known result due independently to G. Birkhoff [**10**] and S. Kakutani [**34**] characterising the metrisable topological groups as the first countable topological groups and, moreover, stating that every such group admits a compatible left-invariant metric. However, as is evident from the construction underlying the Birkhoff–Kakutani theorem, if one begins with a compact symmetric generating set Σ for a locally compact second countable group G, then one may obtain a compatible left-invariant metric d that is quasi-isometric to the word metric ρ_Σ induced by Σ. By applying the Baire category theorem and arguing as in the discrete case, one sees that any two such word-metrics ρ_{Σ_1} and ρ_{Σ_2} are quasi-isometric, which shows that the compatible left-invariant metric d is uniquely defined up to quasi-isometry by this procedure.

Thus far, there has been no satisfactory general method of studying large scale geometry of topological groups beyond the locally compact, though of course certain subclasses such as Banach spaces arrive with a naturally defined geometry. Largely, this state of affairs may be due to the presumed absence of canonical generating sets in general topological groups as opposed to the finitely or compactly generated ones. In certain cases, substitute questions have been considered such as the boundedness or unboundedness of specific metrics [**23**] or of all metrics [**62**], growth type and distortion of individual elements or subgroups [**58, 28**], equivariant geometry [**56**] and specific coarse structures [**54**].

In the present paper, we offer a solution to this problem, which in many cases allows one to isolate and compute a canonical word metric on a topological group G and thus to identify a unique quasi-isometry type of G. Moreover, this quasi-isometry type agrees with that obtained in the finitely or compactly generated settings and also verifies the main characterics encountered there, namely that it is a topological isomorphism invariant of G capturing all possible large scale behaviour of G. Furthermore, under mild additional assumptions on G, this quasi-isometry type may also be implemented by a compatible left-invariant metric on the group.

Though applicable to all topological groups, our main interest is in the class of *Polish groups*, i.e., separable completely metrisable topological groups. These include most interesting topological transformation groups, e.g., $\mathrm{Homeo}(\mathcal{M})$, $\mathrm{Diff}^k(\mathcal{M})$, for $\mathcal{M}$ a compact (smooth) manifold, and $\mathrm{Aut}(\mathbf{A})$, $\mathbf{A}$ a countable discrete structure, along with all separable Banach spaces and locally compact second countable groups. However, it should be stressed that the majority of our results are directly applicable in the greater generality of *European* groups, i.e., Baire topological groups, countably generated over every identity neighbourhood. This includes, for example, all σ-compact locally compact Hausdorff groups and all (potentially non-separable) Banach spaces.

One central technical tool is the notion of coarse structure due to J. Roe [**59, 60**], which may be viewed as the large scale counterpart to uniform spaces. Indeed, given an écart (aka. pre- or pseudometric) d on a group G, let $\mathcal{E}_d$ be the coarse structure on G generated by the entourages

$$E_\alpha = \{(x,y) \in G \times G \mid d(x,y) < \alpha\}$$

for $\alpha < \infty$. That is, $\mathcal{E}_d$ is the ideal of subsets of $G \times G$ generated by the E_α. In analogy with A. Weil's result [**78**] that the left-uniform structure $\mathcal{U}_L$ on a topological group G can be written as the union

$$\mathcal{U}_L = \bigcup_d \mathcal{U}_d$$

of the uniform structures $\mathcal{U}_d$ induced by the family of continuous left-invariant écarts d on G, we define the *left-coarse structure* $\mathcal{E}_L$ on G by

$$\mathcal{E}_L = \bigcap_d \mathcal{E}_d.$$

This definition equips every topological group with a left-invariant coarse structure, which, like a uniformity, may or may not be metrisable, i.e., be the coarse structure associated to a metric on the group. To explain when that happens, we say that a subset $A \subseteq G$ is *coarsely bounded in* G if A has finite diameter with respect to every continuous left-invariant écart on G. This may be viewed as an appropriate notion of "geometric compactness" in topological groups and, in the case of a Polish group G, has the following combinatorial reformulation. Namely, $A \subseteq G$ is coarsely bounded in G if, for every identity neighbourhood V, there are a finite set $F \subseteq G$ and a k so that $A \subseteq (FV)^k$.

THEOREM 1.1. *The following are equivalent for a Polish group G,*

(1) *the left-coarse structure $\mathcal{E}_L$ is metrisable,*
(2) *G is* locally bounded*, i.e., has a coarsely bounded identity neighbourhood,*
(3) *$\mathcal{E}_L$ is generated by a compatible left-invariant metric d, i.e., $\mathcal{E}_L = \mathcal{E}_d$,*
(4) *a sequence (g_n) eventually leaves every coarsely bounded set in G if and only if $d(g_n, 1) \xrightarrow[n]{} \infty$ for some compatible left-invariant metric d on G.*

In analogy with proper metrics on locally compact groups, the metrics appearing in condition (3) above are said to be *coarsely proper*. Indeed, these are exactly the compatible left-invariant metrics all of whose bounded sets are coarsely bounded. Moreover, by Struble's result, on a locally compact group these are the proper metrics.

The category of coarse spaces may best be understood by its morphisms, namely, the bornologous maps. In the case $\phi\colon (X,d_X)\to(Y,d_Y)$ is a map between pseudometric spaces, ϕ is *bornologous* if there is an increasing modulus $\theta\colon \mathbb{R}_+\to\mathbb{R}_+$ so that, for all $x,x'\in X$,

$$d_Y\big(\phi(x),\phi(x')\big)\leqslant \theta\big(d_X(x,x')\big).$$

Using this, we may quasiorder the continuous left-invariant écarts on G by setting $\partial \lll d$ if the identity map $(G,d)\to(G,\partial)$ is bornologous. One then shows that a metric is coarsely proper when it is the maximum element of this ordering. Though seemingly most familiar groups are locally bounded, counter-examples exist such as the infinite direct product of countably infinite groups, e.g., $\mathbb{Z}^{\mathbb{N}}$.

However, just as the word metric on a finitely generated group is well-defined up to quasi-isometry, we may obtain a similar canonicity provided that the group is actually *generated by a coarsely bounded*, i.e., algebraically generated by a coarsely bounded subset. In order to do this, we refine the quasiordering $\lll$ on continuous left-invariant écarts on G above by letting $\partial \ll d$ if there are constants K,C so that $\partial\leqslant K\cdot d+C$. Again, if d is maximum in this ordering, we say that d is *maximal*. Obviously, two maximal écarts are quasi-isometric, whence these induce a canonical *quasi-isometry type* on G. Moreover, as it turns out, the maximal écarts are exactly those that are quasi-isometric to the word metric

$$\rho_\Sigma(x,y)=\min(k\mid \exists z_1,\ldots,z_k\in\Sigma\colon x=yz_1\cdots z_k)$$

given by a coarsely bounded generating set $\Sigma\subseteq G$.

Theorem 1.2. *The following are equivalent for a Polish group G,*

(1) *G admits a compatible left-invariant maximal metric,*
(2) *G is generated by a coarsely bounded set,*
(3) *G is locally bounded and not the union of a countable chain of proper open subgroups.*

A reassuring fact about our definition of coarse structure and quasi-isometry type is that it is a conservative extension of the existing theory. Namely, as the coarsely bounded sets in a σ-compact locally compact group coincides with the relatively compact sets, one sees that our definition of the quasi-isometry type of a compactly generated locally compact group coincides with the classical one given in terms of word metrics for compact generating sets. The same argument applies to the category of finitely generated groups when these are viewed as discrete toplogical groups. Moreover, as will be shown, if $(X,\|\cdot\|)$ is a Banach space, then the norm-metric will be maximal on the underlying additive group $(X,+)$, whereby $(X,+)$ will have a well-defined quasi-isometry type, namely, that of $(X,\|\cdot\|)$. But even in the case of homeomorphism groups of compact manifolds M, as shown in [**50, 46**], the maximal metric on the group $\mathrm{Homeo}_0(M)$ of isotopically trivial homeomorphisms of M is quasi-isometric to the fragmentation metric originating in the work of R. D. Edwards and R. C. Kirby [**22**].

2. A word on the terminology

Some of the basic results of the present paper have previously been included in the preprint [**64**], which now is fully superseded by this. Under the impetus of T. Tsankov, we have changed the terminology from [**64**] to become less specific and more in line with the general language of geometric group theory. Thus, the coarsely bounded sets were originally called *relatively (OB) sets* to keep in line with the terminology from [**62**]. Similarly, locally bounded groups were denoted *locally (OB)* and groups generated by coarsely bounded sets were called *(OB) generated*. For this reason, other papers based on [**64**] such as [**82**], [**65**], [**46**] and [**16**], also use the language of relatively (OB) sets. The translation between the two is straightforward and involves no change in theory.

3. Summary of findings

To aid the reader in the navigation of the new concepts appearing here, we include a diagram of the main classes of Polish groups and a few simple representative examples from some of these. Observe that in the diagram the classes increase going up and from left to right.

		$\mathrm{Isom}(\mathbb{U})$	$\mathrm{Isom}(\mathbb{U}) \times\mathbb{F}_\infty$	$\prod_n \mathbb{Z}$
Bounded geometry	$\mathrm{Homeo}(\mathbb{S}^n)$	$\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$	$\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}) \times\mathbb{F}_\infty$	
$\cup$				
Locally compact	Compact gps.	Compactly generated	$\mathbb{R}\times\mathbb{F}_\infty$	
$\cup$				
Discrete	Finite gps.	Finitely generated	$\mathbb{F}_\infty$	
	Coarsely bounded $\subset$	Generated by bounded set $\subset$	Locally bounded	

Note that the shaded areas reflect the fact that every Polish group of bounded geometry is automatically locally bounded and that coarsely bounded groups trivially have bounded geometry.

3.1. Coarse structure and metrisability. Chapter 2 introduces the basic machinery of coarse structures with its associated morphisms of bornologous maps and analyses these in the setting of topological groups. We introduce the canonical left-invariant coarse structure $\mathcal{E}_L$ with its ideal of coarsely bounded sets and compare this with other coarse structures such as the group-compact coarse structure $\mathcal{E}_{\mathcal{K}}$.

The main results of the chapter concern the identification of coarsely proper and maximal metrics along with Theorems 1.1 and 1.2 characterising the existence of these. This also leads to a version of the Milnor–Schwarz Lemma [**52, 69**] adapted to our setting, which is the central tool in the computation of actual quasi-isometry types of groups.

3.2. Basic structure theory. In Chapter 3 we provide some of the basic tools for the geometric study of Polish groups and present a number of computations of the geometry of specific groups. The simplest class to consider is the "metrically compact" groups, i.e., those quasi-isometric to a one-point space. These are exactly those coarsely bounded in themselves. This class of groups was extensively studied in [**62**] and include a large number of topological transformation groups of highly homogeneous mathematical structures such as homeomorphism groups of spheres and the unitary group.

Another particularly interesting class are the locally Roelcke precompact groups. This includes examples such as the automorphism group of the countably regular tree $\mathrm{Aut}(T_\infty)$ and the isometry group of the Urysohn metric space $\mathrm{Isom}(\mathbb{U})$ that turn out to be quasi-isometric to the tree T_∞ and the Urysohn space $\mathbb{U}$ respectively. Since by a recent result of J. Zielinski [**83**] the locally Roelcke precompact groups have locally compact Roelcke completions, they also provide us with an important tool for the analysis of Polish groups of bounded geometry in Chapter 5.

Indeed, a closed subgroup H of a Polish group G is said to be *coarsely embedded* if the inclusion map is a coarse embedding or equivalently a subset $A \subseteq H$ is coarsely bounded in H if and only if it is coarsely bounded in G. Since in a locally compact group the coarsely bounded sets are simply the relatively compact sets, every closed subgroup is coarsely embedded; though not necessarily quasi-isometrically embedded in the compactly generated case. However, this fails dramatically for Polish groups. Indeed, every Polish group is isomorphic to a closed subgroup of the coarsely bounded group $\mathrm{Homeo}([0,1]^{\mathbb{N}})$. So this subgroup is coarsely embedded only if coarsely bounded itself. This difference along with the potential non-metrisability of the coarse structure account for a great deal of the additional difficulties arising when investigating general Polish groups.

THEOREM 1.3. *Every locally bounded Polish group G is isomorphic to a coarsely embedded closed subgroup of the locally Roelcke precompact group* $\mathrm{Isom}(\mathbb{U})$.

Via this embedding, every locally bounded Polish group can be seen to act continuously on a locally compact space preserving its geometric structure.

The main structural theory of Chapter 3 is a byproduct of the analysis the coarse geometry of product groups. Indeed, we show that a subset A of a product $\prod_i G_i$ is coarsely bounded if and only if each projection $\mathrm{proj}_i(A)$ is coarsely bounded in G_i. Via this, we obtain a universal representation of all Polish group.

THEOREM 1.4. *Every Polish group G is isomorphic to a coarsely embedded closed subgroup of the countable product* $\prod_n \mathrm{Isom}(\mathbb{U})$.

This can be viewed as providing a product resolution of the coarse structure on non-locally bounded Polish groups.

3.3. Coarse geometry of group extensions. In Chapter 4 we address the fundamental and familiar problem of determining the coarse geometry of a group G from those of a closed normal subgroup K and the quotient G/K. While certain things can be said about the general situation, we mainly focus on a more restrictive setting, which includes that of central extensions. Namely, we suppose that K is a closed normal subgroup of a Polish group G which is generated by K and the centraliser $C_G(K)$, i.e. $G = K \cdot C_G(K)$, whence

$$G/K = C_G(K)/Z(K).$$

Then, if K is coarsely embedded in G and $G/K \xrightarrow{\phi} C_G(K)$ is a section for the quotient map so that $\phi\colon G/K \to G$ is bornologous, the map $(k,h) \mapsto k\phi(h)$ defines a coarse equivalence between $K \times G/K$ and G.

A common instance of this is when G is generated by the discrete normal subgroup $K = \Gamma$ and a connected closed subgroup F.

Theorem 1.5. *Suppose G is a Polish group generated by a discrete normal subgroup Γ and a connected closed subgroup F. Assume also that $\Gamma \cap F$ is coarsely embedded in F and that $G/\Gamma \xrightarrow{\phi} F$ is a bornologous section for the quotient map. Then G is coarsely equivalent with $G/\Gamma \times \Gamma$.*

In this connection, several fundamental issues emerge.

- When is K coarsely embedded in G?
- When does the quotient map $G \xrightarrow{\pi} G/K$ admit a bornologous section $G/K \xrightarrow{\phi} G$?
- Is G locally bounded provided K and G/K are?

Indeed, to determine whether K is coarsely embedded in G and whether a section $\phi\colon G/K \to G$ is bornologous requires some advance knowledge of the coarse structure on G itself. To circumvent this, we study the associated cocycles. Indeed, given a $G/K \xrightarrow{\phi} C_G(K)$ section for the quotient map, one obtains an associated cocycle $\omega_\phi\colon G/K \times G/K \to Z(K)$ by the formula

$$\omega_\phi(h_1,h_2) = \phi(h_1h_2)^{-1}\phi(h_1)\phi(h_2).$$

Assuming that ϕ is Borel and G/K locally bounded, the coarse qualities of the map $\phi\colon G/K \to G$ and whether K is coarsely embedded in G now become intimately tied to the coarse qualities of ω_ϕ. Let us state this for the case of central extensions.

Theorem 1.6. *Suppose K is a closed central subgroup of a Polish group G so that G/K is locally bounded and that $G/K \xrightarrow{\phi} G$ is a Borel measurable section of the quotient map. Assume also that, for every coarsely bounded set $B \subseteq G/K$, the image*

$$\omega_\phi\big[G/K \times B\big]$$

is coarsely bounded in K. Then G is coarsely equivalent to $K \times G/K$.

The main feature here is, of course, that the assumptions make no reference to to the coarse structure of G, only to those of K and G/K.

We then apply our analysis to covering maps of manifolds or more general locally compact spaces, which builds on a specific subcase from our joint work with K. Mann [**46**]. Our initial setup is a proper, free and cocompact action

$$\Gamma \curvearrowright X$$

of a finitely generated group Γ on a path-connected, locally path-connected and semilocally simply connected, locally compact metrisable space X. Then the normaliser $N_{\mathrm{Homeo}(X)}(\Gamma)$ of Γ in the homeomorphism group $\mathrm{Homeo}(X)$ is the group of all lifts of homeomorphisms of $M = X/\Gamma$ to X, while the centraliser $C_{\mathrm{Homeo}(X)}(\Gamma)$ is an open subgroup of $N_{\mathrm{Homeo}(X)}(\Gamma)$. Let

$$N_{\mathrm{Homeo}(X)}(\Gamma) \xrightarrow{\pi} \mathrm{Homeo}(M)$$

be the corresponding quotient map and let

$$Q_0 = \pi\big[C_{\mathrm{Homeo}(X)}(\Gamma)\big]$$

be the subgroup of $\mathrm{Homeo}(M)$ consisting of homeomorphisms admitting lifts in $C_{\mathrm{Homeo}(X)}(\Gamma)$. We show that Q_0 is open in $\mathrm{Homeo}(M)$. Also, assume H is a subgroup of Q_0 that is Polish in a finer group topology, say H is the transformation group of some additional structure on M, e.g., a diffeomorphism or symplectic group. Then the group of lifts $G = \pi^{-1}(H) \leqslant N_{\mathrm{Homeo}(X)}(\Gamma)$ carries a canonical lifted Polish group topology and is related to H via the exact sequence

$$1 \to \Gamma \to G \xrightarrow{\pi} H \to 1.$$

Using only assumptions on the structure of Γ, we can relate the geometry of G to those of H and Γ.

THEOREM 1.7. *Suppose* $\Gamma/Z(\Gamma) \xrightarrow{\psi} \Gamma$ *is a bornologous section for the quotient map,* $H \leqslant Q_0$ *is Polish in some finer group topology and* $G = \pi^{-1}(H)$*. Then* G *is coarsely equivalent to* $H \times \Gamma$*.*

Observe here that ψ is a section for the quotient map from the discrete group Γ to it quotient by the centre, which a priori has little to do with H and G. Nevertheless, a main feature of the proof is the existence of a bornologous section $H \xrightarrow{\phi} C_G(\Gamma)$ for the quotient map π, which is extracted from ψ.

Also, applying our result to the universal cover $X = \tilde{M}$ of a compact manifold M, we arrive at the following result.

THEOREM 1.8. *Suppose* M *is a compact manifold,* H *is a subgroup of* $\mathrm{Homeo}_0(M)$*, which is Polish in some finer group topology, and let* G *be the group of all lifts of elements in* H *to homeomorphisms of the universal cover* $\tilde{M}$*. Assume that the quotient map*

$$\pi_1(M) \longrightarrow \pi_1(M)/Z\big(\pi_1(M)\big)$$

admits a bornologous section. Then G *is coarsely equivalent to* $\pi_1(M) \times H$*.*

3.4. Polish groups of bounded geometry. Chapter 5 concerns perhaps the geometrically most well-behaved class of Polish groups beyond the locally compact, namely those of bounded geometry. Here a metric space (X, d) is said to have bounded geometry if there is α with the property that, for every β, there is $k = k(\beta)$ so that every set of diameter β can be covered by k sets of diameter α. J. Roe [**60**] extended this definition to all coarse spaces and we may therefore investigate the Polish groups of bounded geometry. As it turns out, these are all locally bounded and hence, when equipped with a coarsely proper metric, are just metric spaces of bounded geometry. Furthermore, relying on work of J. Zielinski [**83**] on locally Roelcke precompact groups, we obtain a dynamical characterisation of these.

THEOREM 1.9. *The following conditions are equivalent for a Polish group* G*.*

(1) *G has bounded geometry,*
(2) *G is coarsely equivalent to a metric space of bounded geometry,*
(3) *G is coarsely equivalent to a proper metric space,*
(4) *G admits a continuous, coarsely proper, modest and cocompact action $G \curvearrowright X$ on a locally compact metrisable space X.*

Here a continuous action $G \curvearrowright X$ on a locally compact metrisable space X is *coarsely proper* if

$$\{g \in G \mid gK \cap K \neq \emptyset\}$$

for every compact set $K \subseteq X$ and *modest* if $\overline{B \cdot K}$ is compact for all coarsely bounded $B \subseteq G$ and compact $K \subseteq X$.

Of course every locally compact Polish group has bounded geometry, but beyond that the primordial example is that of $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$, which is the group of all homeomorphism of $\mathbb{R}$ commuting with integral translations. One reason for its importance is appearance in dynamics and topology, namely as the group of lifts of orientation preserving homeomorphisms of the circle $\mathbb{S}^1$ to the universal cover $\mathbb{R}$ and thus its inclusion in the exact sequence

$$\mathbb{Z} \to \mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}) \to \mathrm{Homeo}_{+}(\mathbb{S}^1).$$

Other examples can be obtained in a similar manner or be built from these. For example, if we define a cocycle $\omega\colon \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ by $\omega\big((x_1, x_2), (y_1, y_2)\big) = \tau_{x_1 y_2}$, where τ_n is the translation by n, then we obtain an extension

$$\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}) \times_\omega \mathbb{Z}^2$$

of $\mathbb{Z}^2$ by $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ that is quasi-isometric to the Heisenberg group $H_3(\mathbb{Z})$.

Having established a dynamical criterion for bounded geometry of Polish groups, we turn the attention to a seminal result of geometric group theory due to M. Gromov. Namely, in Theorem $0.2.C_2'$ in [**30**], Gromov provides the following dynamical reformulation of quasi-isometry of finitely generated groups.

Theorem 1.10 (M. Gromov). *Two finitely generated groups Γ and Λ are quasi-isometric if and only if they admit commuting, continuous, proper and cocompact actions*

$$\Gamma \curvearrowright X \curvearrowleft \Lambda$$

on a locally compact Hausdorff space X.

This result also provides a model for other well-known notions of equivalence of groups such as measure equivalence involving measure-preserving actions on infinite measure spaces. While Gromov's result easily generalises to arbitrary countable discrete groups, only recently U. Bader and the author [**3**] established the theorem for locally compact groups. Combining this analysis with the mechanics entering in the proof of Theorem 1.9, we succeed in finding the widest possible generalisation.

Theorem 1.11. *Two Polish groups of bounded geometry G and H are coarsely equivalent if and only if they admit commuting, continuous, coarsely proper, modest and cocompact actions*

$$G \curvearrowright X \curvearrowleft H$$

on a locally compact Hausdorff space X.

The commuting actions $G \curvearrowright X \curvearrowleft H$ above are said to be a topological coupling of G and H. An almost tautological example is given by $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ in the coupling

$$\mathbb{Z} \curvearrowright \mathbb{R} \curvearrowleft \mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}).$$

The preceding results indicate the proximity of Polish groups of bounded geometry to the class of locally compact groups. However, these classes display significant differences regarding their harmonic analytical properties. For example, by virtue of [**48**], $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ admit no non-trivial continuous linear isometric representations on reflexive Banach spaces, while, on the other hand, N. Brown and E. Guentner [**12**] have shown that every locally compact Polish group has a proper continuous affine isometric action on a separable reflexive space. We obtain their result for bounded geometry groups under the added assumption of amenability.

THEOREM 1.12. *Let G be an amenable Polish group of bounded geometry. Then G admits a coarsely proper continuous affine isometric action on a reflexive Banach space.*

Similarly, while, by a result of W. Veech [**76**], every locally compact group acts freely on its universal minimal flow, i.e., a compact G-flow of which every other compact G-flow is a factor, this fails dramatically for general Polish. In fact, there are examples of so called extremely amenable Polish groups, that is, whose universal minimal flow reduces to a single point. Utilising recent work of Ben Yaacov, Melleray and Tsankov [**7**], we show that Veech's theorem has purely geometric content by establishing general non-metrisability of the universal minimal flow.

PROPOSITION 1.13. *Let G be a Polish group of bounded geometry with metrisable universal minimal flow. Then G is coarsely bounded.*

The Polish groups of bounded geometry also furnishes the first example of a reasonably complete classification. Namely, a well-known result of C. T. C. Wall states that every finitely generated group quasi-isometric to $\mathbb{R}$ contains a finite index cyclic subgroup. We extend this to groups of bounded geometry.

THEOREM 1.14. *Let G be a Polish group coarsely equivalent to $\mathbb{R}$. Then there is an open subgroup H of index at most 2 and a coarsely bounded set $A \subseteq H$ so that every $h \in H \setminus A$ generates a cobounded undistorted infinite cyclic subgroup.*

3.5. The geometry of automorphism groups. In Chapter 6 we turn our attention to the class of non-Archimedean Polish groups or, equivalently, the automorphism groups $\mathrm{Aut}(\mathbf{M})$ of countable first-order structures $\mathbf{M}$. These are of particular interest in mathematical logic and provide interesting links to model theory.

Given a structure $\mathbf{M}$, the automorphism group $\mathrm{Aut}(\mathbf{M})$ acts naturally on finite tuples $\overline{a} = (a_1, \ldots, a_n)$ in $\mathbf{M}$ via

$$g \cdot (a_1, \ldots, a_n) = (ga_1, \ldots, ga_n)$$

and with this notation the pointwise stabiliser subgroups

$$V_{\overline{a}} = \{g \in \mathrm{Aut}(\mathbf{M}) \mid g \cdot \overline{a} = \overline{a}\},$$

where $\overline{a}$ ranges over all finite tuples in $\mathbf{M}$, form a neighbourhood basis at the identity in $\mathrm{Aut}(\mathbf{M})$. So, if $A \subseteq \mathbf{M}$ is the finite set enumerated by $\overline{a}$ and $\mathbf{A} \subseteq \mathbf{M}$ is the substructure generated by A, we have $V_{\mathbf{A}} = V_A = V_{\overline{a}}$. An *orbital type* $\mathcal{O}$ in $\mathbf{M}$ is simply the orbit $\mathcal{O}(\overline{a}) = \mathrm{Aut}(\mathbf{M})\cdot\overline{a}$ of some tuple $\overline{a}$. In case $\mathbf{M}$ is ω-homogeneous $\mathcal{O}(\overline{a})$ is thus the set of realisations of the type $\mathrm{tp}^{\mathbf{M}}(\overline{a})$ in $\mathbf{M}$.

A geometric tool that will be used throughout this chapter is the graph $\mathbb{X}_{\overline{a},\mathcal{S}}$ associated to a tuple $\overline{a}$ and a finite set $\mathcal{S}$ of orbital types. Here the vertex set of $\mathbb{X}_{\overline{a},\mathcal{S}}$ is just $\mathcal{O}(\overline{a})$, while

$$(\overline{b},\overline{c}) \in \mathrm{Edge}\,\mathbb{X}_{\overline{a},\mathcal{S}} \;\Leftrightarrow\; \overline{b}\neq\overline{c} \;\&\; \Big(\mathcal{O}(\overline{b},\overline{c})\in\mathcal{S} \;\text{ or }\; \mathcal{O}(\overline{c},\overline{b})\in\mathcal{S}\Big).$$

In particular, if $\mathbf{M}$ is atomic, then the edge relation is type $\emptyset$-definable, that is, type definable without parameters. We equip $\mathbb{X}_{\overline{a},\mathcal{S}}$ with the shortest-path metric $\rho_{\overline{a},\mathcal{S}}$ (which may take value ∞ if $\mathbb{X}_{\overline{a},\mathcal{S}}$ is not connected).

Our first results provides a necessary and sufficient criterion for when an automorphism group has a well-defined quasi-isometric type and associated tools allowing for concrete computations of this same type.

Theorem 1.15. *Let* $\mathbf{M}$ *be a countable structure in a countable language. Then* $\mathrm{Aut}(\mathbf{M})$ *admits a maximal metric if and only if there is a tuple* $\overline{a}$ *in* $\mathbf{M}$ *satisfying the following two requirements.*

(1) *For every* $\overline{b}$*, there is a finite family* $\mathcal{S}$ *of orbital types for which the set*
$$\{(\overline{a},\overline{c}) \mid (\overline{a},\overline{c})\in\mathcal{O}(\overline{a},\overline{b})\}$$
has finite $\mathbb{X}_{(\overline{a},\overline{b}),\mathcal{S}}$*-diameter,*

(2) *there is a finite family* $\mathcal{R}$ *of orbital types so that* $\mathbb{X}_{\overline{a},\mathcal{R}}$ *is connected.*

Moreover, if $\overline{a}$ *and* $\mathcal{R}$ *are as in* (2)*, then the mapping*

$$g\in\mathrm{Aut}(\mathbf{M})\mapsto g\cdot\overline{a}\in\mathbb{X}_{\overline{a},\mathcal{R}}$$

is a quasi-isometry between $\mathrm{Aut}(\mathbf{M})$ *and* $(\mathbb{X}_{\overline{a},\mathcal{R}},\rho_{\overline{a},\mathcal{R}})$.

Furthermore, condition (1) alone gives a necessary and sufficient criterion for $\mathrm{Aut}(\mathbf{M})$ being locally bounded and thus having a coarsely proper metric.

With this at hand, we can subsequently relate the properties of the theory $T = \mathrm{Th}(\mathbf{M})$ of the model $\mathbf{M}$ with properties of its automorphism group. First, a metric d on a set X is said to be *stable* in the sense of B. Maurey and J.-L. Krivine [**44**] provided that for all bounded sequences (x_n), (y_n) and ultrafilters $\mathcal{U}$, $\mathcal{V}$,

$$\lim_{n\to\mathcal{U}}\lim_{m\to\mathcal{V}} d(x_n,y_m) = \lim_{m\to\mathcal{V}}\lim_{n\to\mathcal{U}} d(x_n,y_m).$$

We may then combine stability with a result from [**65**] to produce affine isometric actions on Banach spaces.

Theorem 1.16. *Suppose* $\mathbf{M}$ *is a countable atomic model of a stable theory* T *so that* $\mathrm{Aut}(\mathbf{M})$ *is locally bounded. Then* $\mathrm{Aut}(\mathbf{M})$ *admits a coarsely proper continuous affine isometric action on a reflexive Banach space.*

Though Theorem 1.15 furnishes an equivalent reformulation of admitting a metrically proper or maximal compatible left-invariant metric, it is often useful to have more concrete instances of this. A particular case of this, is when $\mathbf{M}$ admits an obital A-independence relation, $\mathop{\smile\hskip-0.9em\raise0.4ex\hbox{$|$}}_A$, that is, an independence relation over a finite subset $A\subseteq\mathbf{M}$ satisfying the usual properties of symmetry, monotonicity, existence and stationarity (see Definition 6.20 for a precise rendering). In particular, this applies to the Boolean algebra of clopen subsets of Cantor space with the dyadic probability measure and to the countably regular tree.

THEOREM 1.17. *Suppose A is a finite subset of a countable structure $\mathbf{M}$ and $\mathop{\smile\!\!\!\!\!\!\mid}_A$ an orbital A-independence relation. Then the pointwise stabiliser subgroup V_A is coarsely bounded. Thus, if $A = \emptyset$, the automorphism group* $\mathrm{Aut}(\mathbf{M})$ *is coarsely bounded and, if $A \neq \emptyset$,* $\mathrm{Aut}(\mathbf{M})$ *is locally bounded.*

Now, model theoretical independence relations arise, in particular, in models of ω-stable theories. Though stationarity of the independence relation may fail, we nevertheless arrive at the following result.

THEOREM 1.18. *Suppose that $\mathbf{M}$ is a saturated countable model of an ω-stable theory. Then* $\mathrm{Aut}(\mathbf{M})$ *is coarsely bounded.*

In this connection, we should mention an earlier observation by P. Cameron, namely, that automorphism groups of countable $\aleph_0$-categorical structures are Roelcke precompact and thus coarsely bounded.

A particular setting giving rise to orbital independence relations, which has earlier been studied by K. Tent and M. Ziegler [**71**], is Fraïssé classes admitting a canonical amalgamation construction. For our purposes, we need a stronger notion than that considered in [**71**] and say that a Fraïssé class $\mathcal{K}$ admits a functorial amalgamation over some $\mathbf{A} \in \mathcal{K}$ if there is a map Θ that to every pair of embeddings $\eta_1 \colon \mathbf{A} \hookrightarrow \mathbf{B}_1$ and $\eta_2 \colon \mathbf{A} \hookrightarrow \mathbf{B}_2$ with $\mathbf{B}_i \in \mathcal{K}$ produces an amalgamation of $\mathbf{B}_1$ and $\mathbf{B}_2$ over these embeddings so that Θ is symmetric in its arguments and commutes with embeddings (see Definition 6.28 for full details).

THEOREM 1.19. *Suppose $\mathcal{K}$ is a Fraïssé class with limit $\mathbf{K}$ admitting a functorial amalgamation over some $\mathbf{A} \in \mathcal{K}$. Then* $\mathrm{Aut}(\mathbf{K})$ *is locally bounded. Moreover, if $\mathbf{A}$ is generated by the empty set, then* $\mathrm{Aut}(\mathbf{K})$ *is coarsely bounded.*

3.6. Zappa–Szép products. Zappa–Szép products of Polish groups appear frequently throughout our study. These are groups G containing subgroups H and K so that $G = H \cdot K$ and $H \cap K = \{1\}$. For example, if H is a locally compact group, the homeomorphism group $\mathrm{Homeo}(H)$ admits a Zappa–Szép decomposition

$$\mathrm{Homeo}(H) = H \cdot K,$$

where H is identified with the group of left-translations and $K = \{g \in \mathrm{Homeo}(H) \mid g(1_H) = 1_H\}$ is the isotropy subgroup at the identity of H.

As familiar from the case of direct and semidirect products, in Appendix A we show that if a Polish group G is the Zappa–Szép product of two closed subgroups H and K, then the group multiplication $(h,k) \mapsto hk$ defines a homeomorphism $\phi \colon H \times k \to G$. We also embark on a detailed study of their coarse structure and, in particular, investigate when the multiplication map ϕ is a coarse equivalence between the cartesian product $H \times K$ and G.

CHAPTER 2

Coarse structure and metrisability

1. Coarse structures on groups

As a motivation for our definition of the coarse structure on a topological group, we must first consider A. Weil's concept of uniform spaces [**78**]. A *uniform space* is a set X equipped with a family $\mathcal{U}$ of subsets $E \subseteq X \times X$ called *entourages* verifying the following conditions.

(1) Every $E \in \mathcal{U}$ contains the diagonal $\Delta = \{(x,x) \mid x \in X\}$,
(2) $\mathcal{U}$ is closed under taking supersets and finite intersections,
(3) $\mathcal{U}$ is closed under taking inverses, $E \mapsto E^{-1} = \{(y,x) \mid (x,y) \in E\}$,
(4) for any $E \in \mathcal{U}$, there is $F \in \mathcal{U}$ so that

$$F \circ F := \{(x,z) \mid \exists y\ (x,y),(y,z) \in F\} \subseteq E.$$

Whereas the concept of uniform spaces captures the idea of being "uniformly close" in a topological space, J. Roe [**60**] (see also [**59**] for an earlier definition) provided a corresponding axiomatisation of being at "uniformly bounded distance", namely the notion of a coarse space.

Definition 2.1 (Coarse spaces). *A* coarse space *is a set X equipped with a collection $\mathcal{E}$ of subsets $E \subseteq X \times X$ called* entourages *satisfying the following conditions.*

(1) *The diagonal Δ belongs to $\mathcal{E}$,*
(2) *if $E \subseteq F \in \mathcal{E}$, then also $E \in \mathcal{E}$,*
(3) *if $E, F \in \mathcal{E}$, then $E \cup F, E^{-1}, E \circ F \in \mathcal{E}$.*

We also call $\mathcal{E}$ a *coarse structure* or a *system of entourages* on X.

Example 2.2 (Pseudometric spaces). The canonical example of both a coarse and a uniform space is when (X,d) is a metric or, more generally, a pseudometric space. Recall here that a *pseudometric* space is a set X equipped with an *écart*, that is, a map $d\colon X \times X \to \mathbb{R}_+$ so that $d(x,y) = d(y,x)$, $d(x,x) = 0$ and $d(x,y) \leqslant d(x,z) + d(z,y)$. In this case, we may, for every $\alpha > 0$, construct an entourage by

$$E_\alpha = \{(x,y) \mid d(x,y) < \alpha\}$$

and define a uniformity $\mathcal{U}_d$ by

$$\mathcal{U}_d = \{E \subseteq X \times X \mid \exists \alpha > 0\ \ E_\alpha \subseteq E\}.$$

Similarly, a coarse structure $\mathcal{E}_d$ is obtained by

$$\mathcal{E}_d = \{E \subseteq X \times X \mid \exists \alpha < \infty\ \ E \subseteq E_\alpha\}.$$

As is evident from the definition, the intersection $\bigcap_{i \in I} \mathcal{E}_i$ of an arbitrary family $\{\mathcal{E}_i\}_{i \in I}$ of coarse structures on a set X is again a coarse structure. Thus, if $\mathcal{F}$ is a collection of subsets of $X \times X$, there is a smallest coarse structure $\mathcal{E}$ on X containing

$\mathcal{F}$, namely the intersection of all coarse structures containing $\mathcal{F}$. Concretely, $E \in \mathcal{E}$ if there are $F_1, F_2, \ldots, F_n$, each finite unions of elements $\{\Delta\} \cup \mathcal{F} \cup \mathcal{F}^{-1}$, so that $E \subseteq F_1 \circ \ldots \circ F_n$. We say that $\mathcal{E}$ is *generated* by $\mathcal{F}$ or that $\mathcal{F}$ is a *basis* for $\mathcal{E}$. Also, a family $\mathcal{F}$ is said to be *cofinal* in a coarse structure $\mathcal{E}$ if, for every $E \in \mathcal{E}$, there is some $F \in \mathcal{F}$ with $E \subseteq F$.

Example 2.3 (Left-uniform structure on a topological group). As is well-known, a topological group G has a number of naturally defined uniformities (cf. [**61**] for a deeper study). Of particular interest to us is the *left-uniformity* $\mathcal{U}_L$, which is that generated by the family of left-invariant entourages

$$E_V = \{(x,y) \in G \times G \mid x^{-1}y \in V\},$$

where V varies over identity neighbourhoods in G.

It is easy to see that the union of a directed system of uniform structures is again a uniform structure. And, in fact, Weil [**78**] showed that every uniform structure is the union of the directed family of uniform structures given by continuous écarts. In the context of a topological group G, the left-uniformity $\mathcal{U}_L$ is actually generated by the class of continuous *left-invariant* écarts, i.e., écarts d on G so that $d(xy, xz) = d(y, z)$ for all $x, y, z \in G$. In other words,

$$\mathcal{U}_L = \bigcup\{\mathcal{U}_d \mid d \text{ is a continuous left-invariant écart on } G\}.$$

This is the only uniformity on topological groups we shall consider and notions of uniform continuity will always refer to this structure.

By the previous example, $\mathcal{U}_L$ is the coarsest common refinement of all the $\mathcal{U}_d$, where d varies over continuous left-invariant écarts on G. Now, whereas for uniformities $\mathcal{U} \subseteq \mathcal{V}$, we say that $\mathcal{V}$ is finer than $\mathcal{U}$, for coarse structures, the opposite is true. That is, if $\mathcal{E} \subseteq \mathcal{F}$ are coarse structures, then $\mathcal{E}$ is finer than $\mathcal{F}$. So the common refinement of a class of coarse structures is given by their intersection rather than union.

Thus, in analogy with the description of the left-uniform structure given above, we define the left-coarse structure as follows.

Definition 2.4. *For a topological group G, we define its* left-coarse structure *$\mathcal{E}_L$ by*

$$\mathcal{E}_L = \bigcap\{\mathcal{E}_d \mid d \textit{ is a continuous left-invariant écart on } G\}.$$

Thus, a subset $E \subseteq G \times G$ belongs to $\mathcal{E}_L$ if, for every continuous left-invariant écart d on G, we have $\sup_{(x,y)\in E} d(x,y) < \infty$.

The definition of the left-coarse structure, though completely analogous to the description of the left-uniformity, is not immediately transparent and it is therefore useful to provide other approaches to it as follow below.

2. Coarsely bounded sets

With every coarse structure comes a notion of coarsely bounded sets, which, in the case of a metric space, would be the sets of finite diameter.

Definition 2.5. *A subset $A \subseteq X$ of a coarse space $(X, \mathcal{E})$ is said to be* coarsely bounded *if $A \times A \in \mathcal{E}$.*

Our next task is to provide an informative reformulation of the class of coarsely bounded subsets of a topological group equipped with its left-coarse structure. This will be based on the classical metrisation theorem of G. Birkhoff [**10**] and S. Kakutani [**34**] or, more precisely, on the following lemma underlying Birkhoff's construction in [**10**] (see also [**32**]).

LEMMA 2.6. *Let G be a topological group and $(V_n)_{n\in\mathbb{Z}}$ a increasing chain of symmetric open identity neighbourhoods satisfying $G=\bigcup_{n\in\mathbb{Z}}V_n$ and $V_n^3\subseteq V_{n+1}$. Define $\delta(g,f)=\inf\big(2^n \;\big|\; g^{-1}f\in V_n\big)$ and put*

$$d(g,f)=\inf\Big(\sum_{i=0}^{k-1}\delta(h_i,h_{i+1}) \;\Big|\; h_0=g, h_k=f\Big).$$

Then

$$\frac{1}{2}\delta(g,f)\leqslant d(g,f)\leqslant \delta(g,f)$$

and d is a continuous left-invariant écart on G.

The metrisation theorem of Birkhoff and Kakutani, which shall be used several times states that a topological group G is metrisable if and only if it is first countable and, moreover, in this case it admits a compatible left-invariant metric.

Using Lemma 2.6, we have the following equivalences.

PROPOSITION 2.7. *Let G be a topological group equipped with its left-coarse structure. Then the following conditions are equivalent for a subset $A\subseteq G$,*

(1) *A is coarsely bounded,*
(2) *for every continuous left-invariant écart d on G,*
$$\mathrm{diam}_d(A)<\infty,$$
(3) *for every continuous isometric action on a metric space, $G\curvearrowright(X,d)$, and every $x\in X$,*
$$\mathrm{diam}_d(A\cdot x)<\infty,$$
(4) *for every increasing exhaustive sequence $V_1\subseteq V_2\subseteq\ldots\subseteq G$ of open subsets with $V_n^2\subseteq V_{n+1}$, we have $A\subseteq V_n$ for some n.*

Moreover, suppose G is countably generated over every identity neighbourhood, *i.e., for every identity neighbourhood V there is a countable set $C\subseteq G$ so that $G=\langle V\cup C\rangle$. Then (1)-(4) are equivalent to*

(5) *for every identity neighbourhood V, there is a finite set $F\subseteq G$ and a $k\geqslant 1$ so that $A\subseteq(FV)^k$.*

PROOF. Observe that $A\times A\in\mathcal{E}_L$ if and only if, for every continuous left-invariant écart d we have $\sup_{(x,y)\in A\times A}d(x,y)<\infty$, i.e., $\mathrm{diam}_d(A)<\infty$. So (1) and (2) are equivalent.

Also, (2) and (3) are equivalent. For, if d is a continuous left-invariant écart on G, let X be the corresponding metric quotient of G. Then the left-shift action of G on itself factors through to a continuous transitive isometric action on X. So, if every A-orbit is bounded, then A is d-bounded in G. And, conversely, if $G\curvearrowright(X,d)$ is a continuous isometric action on some metric space, then, for any fixed $x\in X$, $\partial(g,f)=d(g\cdot x,f\cdot x)$ defines a continuous left-invariant écart on G. Moreover, if A is ∂-bounded, then the A-orbit $A\cdot x$ is d-bounded and the same is true for any other A-orbit on X.

Note that, if d is a continuous left-invariant écart on G, then $d(1,xy)\leqslant d(1,x)+d(x,xy)=d(1,x)+d(1,y)$ for all $x,y\in G$. It follows that $V_n=\{x\in G \mid d(1,x)<2^n\}$ defines an increasing exhaustive chain of open subsets of G so that $V_n^2\subseteq V_{n+1}$. Moreover, the d-bounded sets are exactly those contained in some V_n. Thus, if A satisfies (4), it has finite diameter with respect to every continuous left-invariant écart on G. So (4)⇒(2).

Conversely, suppose there is some increasing exhaustive chain of symmetric open subsets $W_n\subseteq G$ so that $W_n^2\subseteq W_{n+1}$, while $A\not\subseteq W_n$ for all n. Without loss of generality, we may suppose that $1\in W_0$. Pick also symmetric open identity neighbourhoods $V_k\subseteq W_0$ for all $k<0$ so that $V_k^3\subseteq V_{k+1}$ and set $V_k=W_{2k+2}$ for $k\geqslant 0$. Then the V_k, $k\in\mathbb{Z}$, satisfy the conditions of Lemma 2.6, so there is is a continuous left-invariant écart d on G whose open n-ball is contained in V_{2^n}. It follows that $\mathrm{diam}_d(A)=\infty$, showing (2)⇒(4).

That (5) implies (4) for any G is obvious. For, if (5) held and $V_1\subseteq V_2\subseteq\ldots\subseteq G$ are as in (4), then we may find some finite F and k so that $A\subseteq(FV_1)^k$, whence $A\subseteq(FV_1)^k\subseteq(V_mV_1)^k\subseteq V_{m+k+1}$ provided $F\subseteq V_m$.

Conversely, assume that G is countably generated over every identity neighbourhood, that A is coarsely bounded and V is an identity neighbourhood. Pick a countable set $C=\{x_n\}_n$ generating G over V and let $V_n=\big(V\cup\{x_1,\ldots,x_n\}\big)^{2^n}$. Then the V_n are as in (4), whence $A\subseteq V_n=\big(V\cup\{x_1,\ldots,x_n\}\big)^{2^n}$ for some n, showing (4)⇒(5). □

Note that, if G is countably generated over every identity neighbourhood, then, by condition (5), in order to detect the coarse boundedness of a subset A, we need only quantify over basic identity neighbourhoods V rather than over all continuous left-invariant écarts d on G. In practice, the first condition is often the easiest to consider. This also shows that, for a Polish group G, the ideal $\mathcal{OB}$ of closed coarsely bounded sets in G is Borel in the Effros–Borel space of closed subsets of G.

Corollary 2.8. *A subset A of a σ-compact locally compact group G is coarsely bounded in the left-coarse structure if and only if it is relatively compact*[1].

Proof. Suppose V is a compact identity neighbourhood in G. Then, since V covers any compact set by finitely many left-translates, we see that $A\subseteq G$ is relatively compact if and only if $A\subseteq(FV)^k$ for some finite F and $k\geqslant 1$.

Note now that a locally compact group G is σ-compact exacly when G is countably generated by every identity neighbourhood. The equivalence then follows from condition (5) of Proposition 2.7. □

Corollary 2.9. *A subset A of a Banach space $(X,\|\cdot\|)$ is coarsely bounded in the underlying additive group $(X,+)$ if and only if A is norm bounded.*

Proof. It suffices to show that a norm bounded set A is coarsely bounded. But, any identity neighbourhood in $(X,+)$ contains a ball ϵB_X for some $\epsilon>0$ and $A\subseteq n\epsilon B_X=\underbrace{\epsilon B_X+\ldots+\epsilon B_X}_{n\text{ times}}$ for some sufficiently large n, showing that A is coarsely bounded. □

[1] i.e., A has compact topological closure in G.

By reason of condition (3) of Proposition 2.7, we denote the collection of coarsely bounded subsets of a topological group G by $\mathcal{OB}$. Namely, $A \in \mathcal{OB}$ if *orbits* under A are *bounded* for every continuous isometric action of G.

PROPOSITION 2.10. *Let G be a topological group equipped with its left-coarse structure. Then the family of coarsely bounded sets form an ideal $\mathcal{OB}$ stable under the operations,*

$$A \mapsto \mathrm{cl}\, A, \quad A \mapsto A^{-1}, \quad (A, B) \mapsto AB.$$

PROOF. Stability of $\mathcal{OB}$ under union, subsets, topological closure and products $AB = \{ab \mid a \in A \ \&\ b \in B\}$ follow immediately from conditions (2) and (4) of Proposition 2.7. Also, if d a continuous left-invariant écart on G, then $d(a^{-1}, 1) = d(1, a)$ for all a and hence, if A has finite d-diameter, so does A^{-1}. $\square$

Let us note that while relative compactness of a subset A of a topological group G only depends on the closure $\mathrm{cl}\, A$ in G as a topological space, but not on the remaining part of the ambient topological group G, the situation is very different for coarse boundedness. Indeed, though A is coarsely bounded in G if and only if $\mathrm{cl}\, A$ is, it is possible that A is coarsely bounded in G, while failing to be coarsely bounded in some intermediate closed subgroup $A \subseteq H \leqslant G$. So, one must always stress in which ambient group G a subset A is coarsely bounded.

3. Comparison with other left-invariant coarse structures

In the same way as a left-invariant uniformity on a group can be described in terms of a certain filter, one may reformulate left-invariant coarse structure on groups as ideals of subsets. This connection is explored in greater detail by A. Nicas and D. Rosenthal in [**54**] and we shall content ourselves with some elementary observations here.

If G is a group, a subset $E \subseteq G \times G$ is said to the *left-invariant* if $(xy, xz) \in E$ whenever $(y, z) \in E$ and $x \in G$. Note that, if E is left-invariant, it can be recovered from the set

$$A_E = \{x \in G \mid (1, x) \in E\}$$

by noting that $E = \{(x, y) \in G \times G \mid x^{-1}y \in A_E\}$. Conversely, if $A \subseteq G$, then A may be recovered from the left-invariant set

$$E_A = \{(x, y) \in G \times G \mid x^{-1}y \in A\}$$

by observing that $A = \{x \in G \mid (1, x) \in E_A\}$. Thus, $A \mapsto E_A$ defines a bijection between subsets of G and left-invariant subsets of $G \times G$ with inverse $E \mapsto A_E$.

REMARK 2.11. It is worth pointing a few basic facts that will be used repeatedly. Namely, suppose $A, B \subseteq G$. Then

$$E_A^{-1} = E_{A^{-1}}, \quad E_A \circ E_B = E_{AB}$$

and

$$E_A[B] := \{x \in G \mid \exists b \in B\ (x, b) \in E_A\} = BA^{-1}.$$

Moreover, evidently, $A \subseteq B$ if and only if $E_A \subseteq E_B$.

Observe that, if E, F are left-invariant, then so are E^{-1}, $E \cup F$ and $E \circ F$. It follows that the coarse structure generated by a collection of left-invariant sets has a cofinal basis consisting of left-invariant sets. We say that a coarse structure $\mathcal{E}$ on a group G is *left-invariant* if it has a basis consisting of left-invariant sets (these are also called *compatible* in [**54**]). For example, being the intersection of a family of left-invariant coarse structures, the left-coarse structure $\mathcal{E}_L$ is itself left-invariant.

By the above considerations, the following proposition becomes evident.

PROPOSITION 2.12. *Let G be a group. Then*

$$\mathcal{E} \mapsto \mathcal{A}_{\mathcal{E}} = \{A \mid A \subseteq A_E \text{ for some } E \in \mathcal{E}\}$$

with inverse

$$\mathcal{A} \mapsto \mathcal{E}_{\mathcal{A}} = \{E \mid E \subseteq E_A \text{ for some } A \in \mathcal{A}\}$$

defines a bijection between the collection of left-invariant coarse structures $\mathcal{E}$ on G and the collection of ideals $\mathcal{A}$ on G, containing $\{1\}$ and closed under inversion $A \mapsto A^{-1}$ and products $(A, B) \mapsto AB$.

The ideal $\mathcal{A}_{\mathcal{E}}$ associated to a left-invariant coarse structure $\mathcal{E}$ is simply the ideal of coarsely bounded sets.

PROPOSITION 2.13. *For every topological group G, we have $\mathcal{E}_L = \mathcal{E}_{\mathcal{OB}}$.*

PROOF. Suppose that $E \in \mathcal{E}_{\mathcal{OB}}$ and find $A \in \mathcal{OB}$ so that $E \subseteq E_A$. Assume also that d is a continuous left-invariant écart on G and note that, by Proposition 2.7, A has finite d-diameter, say $d(1, x) < \alpha$ for all $x \in A$. Then $E \subseteq E_A \subseteq \{(x, y) \mid d(x, y) < \alpha\}$, i.e., $E \in \mathcal{E}_d$. As d was arbitrary, it follows that $E \in \mathcal{E}_L$.

Conversely, if $E \in \mathcal{E}_L$, then, as $\mathcal{E}_L$ is left-invariant, we see that the saturation $\tilde{E} = \{(zx, zy) \mid z \in G \;\&\; (x, y) \in E\}$ also belongs to $\mathcal{E}_L$. However, since $\tilde{E}$ is left-invariant, we may write it as $\tilde{E} = E_A$ for some $A \subseteq G$, whence A must have finite diameter with respect to every continuous left-invariant écart on G. It follows, by Proposition 2.7, that $A \in \mathcal{OB}$ and thus that $E \subseteq E_A \in \mathcal{E}_{\mathcal{OB}}$, i.e., $E \in \mathcal{E}_{\mathcal{OB}}$. □

Thus, as by Corollary 2.9, the coarsely bounded sets in a Banach space are the norm bounded sets, the following is immediate.

COROLLARY 2.14. *The left-coarse structure on the underlying additive group $(X, +)$ of a Banach space $(X, \|\cdot\|)$ is that induced by the norm.*

The study of Banach spaces under their coarse structure is a main pillar in geometric non-linear functional analysis, cf. the treatise [**8**] or the recent survey [**36**].

In addition to the left-coarse structure, we could consider the left-invariant coarse structures on a topological group G defined by the following ideals.

$$\mathcal{K} = \{A \subseteq G \mid A \text{ is relatively compact in } G\},$$
$$\mathcal{V} = \{A \subseteq G \mid \forall V \ni 1 \text{ open } \; \exists k \;\; A \subseteq V^k\},$$
$$\mathcal{F} = \{A \subseteq G \mid \forall V \ni 1 \text{ open } \; \exists k \; \exists F \subseteq G \text{ finite } \; A \subseteq (FV)^k\}.$$

As can be easily checked, all of these ideals are closed under inversion and products and hence define left-invariant coarse structures on G. Note also that, since for every set $A \subseteq G$ and identity neighbourhood V we have $\overline{A} \subseteq AV$, each of these ideals are stable under the topological closure operation $A \mapsto \overline{A}$. Also, the following inclusions are obvious.

$$\begin{array}{ccccc} \mathcal{K} & \subseteq & \mathcal{F} & \subseteq & \mathcal{OB} \\ & & \cup\!\!\!\mid & & \\ & & \mathcal{V} & & \end{array}$$

As can easily be seen, $\bigcup\mathcal{V}$ equals the intersection of all of open subgroups of G. From this, we readily verify that the following conditions are equivalent on a topological group G.

(1) G has no proper open subgroups,
(2) $G = \bigcup\mathcal{V}$,
(3) $\mathcal{V} = \mathcal{F}$,
(4) $\mathcal{V} = \mathcal{F} = \mathcal{OB}$,
(5) the associated coarse structure $\mathcal{E}_{\mathcal{V}}$ is *connected*, i.e., for all $x, y \in G$, there is $E \in \mathcal{E}_{\mathcal{V}}$ with $(x, y) \in E$.

Since the notion of coarse structure $\mathcal{E}$ on a space X is supposed to capture what it means for points to a bounded distance apart, it is natural to require that it is connected in the above sense. Also, the coarse structure $\mathcal{E}_{\mathcal{V}}$ essentially only lives on the intersection of all open subgroups of G and therefore only structures that part of the group. For these reasons, $\mathcal{K}$, $\mathcal{F}$ and $\mathcal{OB}$ are better adapted to illuminate the geometric features of a topological group than $\mathcal{V}$.

Inasmuch as our aim is to study geometry, as opposed to topology, of topological groups, metrisability or écartability of a coarse structure is highly desirable. From that perspective, Proposition 2.13, which shows that the coarse structure $\mathcal{E}_{\mathcal{OB}} = \mathcal{E}_L$ is the common refinement of the continuously écartable coarse structures, gives prominence to the left-coarse structure as opposed to $\mathcal{E}_{\mathcal{K}}$, $\mathcal{E}_{\mathcal{V}}$ and $\mathcal{E}_{\mathcal{F}}$. Moreover, in some of the classical cases, namely, finitely generated, countable discrete or σ-compact locally compact groups, Corollary 2.8 also shows that the left-coarse structure coincides with the one classically studied, i.e., $\mathcal{E}_{\mathcal{K}}$.

One may, for example, consult the monograph [**19**] by Y. de Cornulier and P. de la Harpe for the coarse geometry of locally compact σ-compact groups and the paper [**54**] for more information about the *group-compact coarse structure* $\mathcal{E}_{\mathcal{K}}$ on general topological groups. The left-coarse structure appears to be new, but has its roots in earlier work in [**62**].

However, another indication that the left-coarse structure is more suitable from a geometric perspective is that, in contradistinction to $\mathcal{E}_{\mathcal{K}}$, it induces the correct coarse structure on Banach spaces, namely, that given by the norm.

Remark 2.15. Suppose $\mathcal{A}$ is an ideal on a topological group G closed under inversion and products and containing $\{1\}$. Then G is *locally in* $\mathcal{A}$, i.e., G has an identity neighbourhood V belonging to $\mathcal{A}$, if and only if there is a set $E \subseteq G \times G$ which is simultaneously an entourage for the left-uniformity $\mathcal{U}_L$ on G and the coarse structure $\mathcal{E}_{\mathcal{A}}$, i.e., if $\mathcal{U}_L \cap \mathcal{E}_{\mathcal{A}} \neq \emptyset$.

Indeed, given such an E, there is an identity neighbourhood W so that $E_W \subseteq E$, whence $E_W \in \mathcal{E}_{\mathcal{A}}$ and thus $W \in \mathcal{A}$. Conversely, if G is locally in $\mathcal{A}$, as witnessed by some identity neighbourhood $V \in \mathcal{A}$, then $E_V \in \mathcal{U}_L \cap \mathcal{E}_{\mathcal{A}}$.

Our next examples indicate that there is no simple characterisation of when $\mathcal{K} = \mathcal{F}$ or $\mathcal{F} = \mathcal{OB}$.

Example 2.16. Let $\mathrm{Sym}(\mathbb{N})$ denote the group of all (not necessarily finitely supported) permutations of $\mathbb{N}$ and equip it with the discrete topology. Then we see that $\mathcal{K} = \mathcal{F}$ is the ideal of finite subsets of $\mathrm{Sym}(\mathbb{N})$, while, as $V = \{1\}$ is an identity neighbourhood, $\mathcal{V}$ has a single element, namely $\{1\}$.

On the other hand, it follows from the main result of G. M. Bergman's paper [**9**] that, for any exhaustive chain $V_1 \subseteq V_2 \subseteq \ldots \subseteq \mathrm{Sym}(\mathbb{N})$ of subsets with $V_n^2 \subseteq V_{n+1}$, we have $\mathrm{Sym}(\mathbb{N}) = V_k$ for some k. In other words, $\mathcal{OB}$ contains all subsets of $\mathrm{Sym}(\mathbb{N})$.

EXAMPLE 2.17. A subset A of a topological group G is said to be *bounded in the left-uniformity* $\mathcal{U}_L$ if, for every identity neighbourhood V, there is n and a finite set $F \subseteq G$ so that $A \subseteq FV^n$. As is well-known in the theory of uniform spaces, this is equivalent to demanding that every uniformly continuous function $\phi\colon G \to \mathbb{R}$ is bounded on A. Clearly, the class of sets bounded in the left-uniformity contains $\mathcal{V}$ and is contained in $\mathcal{F}$. Thus, when G has no proper open subgroup, this class coincides with $\mathcal{OB}$.

On the other hand, in the group S_∞ of all permutations of $\mathbb{N}$ equipped with the Polish topology obtained by declaring pointwise stabilisers of finite sets to be open, $\mathcal{V} = \big\{\{1\}\big\}$ and the sets bounded in the left-uniformity coincides with the relatively compact sets, while, by Example 2.16, $\mathcal{F} = \mathcal{OB}$ contains all subsets of S_∞.

EXAMPLE 2.18. Consider the free non-abelian group $\mathbb{F}_{\aleph_1}$ on $\aleph_1$ generators $(a_\xi)_{\xi<\aleph_1}$ ($\aleph_1$ is the first uncountable cardinal number) equipped with the discrete topology. Then $\mathcal{K} = \mathcal{F}$ is the class of finite subsets, while $\mathcal{V}$ consists of $\{1\}$.

We claim that any element of $\mathcal{OB}$ is finite and thus $\mathcal{K} = \mathcal{F} = \mathcal{OB}$, despite the fact that $\mathbb{F}_{\aleph_1}$ is not generated over the identity neighbourhood $\{1\}$ by a countable set. Indeed, if $A \subseteq \mathbb{F}_{\aleph_1}$ is infinite, it either contains elements with unbounded word length in $(a_\xi)_{\xi<\aleph_1}$ or it must use an infinite number of generators. In the first case, we let V_n denote the set of $x \in \mathbb{F}_{\aleph_1}$ of word length at most 2^n and see that $A \not\subseteq V_n$ for all n. In the second case, suppose a_{ξ_n} is a sequence of generators all appearing in elements of A. We then find an increasing exhaustive chain $C_1 \subseteq C_2 \subseteq \ldots \subseteq \{a_\xi\}_{\xi<\aleph_1}$ so that $a_{\xi_n} \notin C_n$ and let $V_n = \langle C_n \rangle$. Again, $A \not\subseteq V_n$ for all n. So, in either case, $A \notin \mathcal{OB}$.

EXAMPLE 2.19. For each n, let Γ_n be a copy of the discrete group $\mathbb{Z}$ and let $G = \prod_n \Gamma_n$ be equipped with the product topology. Since the coordinate projections proj_{Γ_n} are all continuous, we see that $A \subseteq G$ is relatively compact if and only if $\mathrm{proj}_{\Gamma_k}(A)$ is finite for all k. Thus, if A is not relatively compact, i.e., $A \notin \mathcal{K}$, fix k so that $\mathrm{proj}_{\Gamma_k}(A)$ is infinite. Then, using $V_n = \{x \in G \mid \mathrm{proj}_{\Gamma_n}(x) \in [-2^n, 2^n]\}$, we see that $A \notin \mathcal{OB}$. In other words, $\mathcal{K} = \mathcal{F} = \mathcal{OB}$ even though G is not locally compact.

If instead Γ_n is an uncountable discrete group for each n, we obtain an example $G = \prod_n \Gamma_n$ that is neither locally compact nor is countably generated over every identity neighbourhood, while nevertheless $\mathcal{K} = \mathcal{F} = \mathcal{OB}$.

4. Metrisability and monogenicity

In what follows, on a topological group, we shall only be considering the left-coarse structure $\mathcal{E}_L$ and the left-uniformity $\mathcal{U}_L$. All concepts will be refering to these.

As is the case with uniform structure, the simplest case to understand is the metrisable coarse spaces.

DEFINITION 2.20. *A coarse space $(X, \mathcal{E})$ is said to be* metrisable *if it is of the form $\mathcal{E}_d$ for some generalised metric $d\colon X \times X \to [0, \infty]$ (thus possibly talking the value ∞).*

Let us note that, since we are not requiring any continuity here, the difference between (generalised) écarts and metrics is not important. Indeed, if d is an écart on X, then $\partial(x,y)=d(x,y)+1$ for all $x\neq y$ defines a metric on X inducing the same coarse structure as d.

In analogy with the characterisation of metrisable uniformities as those that are countably generated, Roe (Theorem 2.55 [**60**]) characterises the metrisable coarse structures $\mathcal{E}$ as those that are *countably generated*, i.e., having a countable cofinal subfamily. Now, if $\mathcal{E}$ is a left-invariant coarse structure on a group G, then, since $\mathcal{E}$ is generated by its G-invariant elements, one may verify that $\mathcal{E}$ is metrisable if and only if the associated ideal $\mathcal{A}_{\mathcal{E}}$ is countably generated, i.e., contains a countable cofinal subfamily $\{A_n\}_n\subseteq\mathcal{A}_{\mathcal{E}}$.

Lemma 2.21. *The following are equivalent for a topological group G.*

1. *The left-coarse structure $\mathcal{E}_L$ is metrisable,*
2. *the ideal $\mathcal{OB}$ is countably generated,*
3. *$\mathcal{E}_L$ is metrised by a left-invariant (possibly discontinuous) metric d on G.*

Proof. Assume that $\mathcal{OB}$ is countably generated, say by a cofinal family $\{A_n\}_n\subseteq\mathcal{OB}$. Set $A'_n=\{1\}\cup A_nA_n^{-1}$ and let $B_0=\{1\}$, $B_{n+1}=A'_{n+1}\cup B_nB_n$. Then $\{B_n\}_n$ is an increasing cofinal sequence in $\mathcal{OB}$ consisting of symmetric sets satisfying $B_nB_n\subseteq B_{n+1}$. It follows that

$$d(x,y)=\min(k\mid x^{-1}y\in B_k)$$

defines a metric on G whose bounded sets are exactly the coarsely bounded sets. □

Definition 2.22. *A topological group G is* locally bounded *if and only it has a coarsely bounded identity neighbourhood.*

Observe that, by Remark 2.15, this happens if and only if

$$\mathcal{U}_L\cap\mathcal{E}_L\neq\emptyset.$$

The importance of this condition will become even clearer from the results below.

Lemma 2.23. *Let G be a topological group and suppose that $\mathcal{E}_L$ is induced by a continuous left-invariant écart d on G. Then G is locally bounded.*

Proof. This is trivial since the continuous left-invariant écart d must be bounded on some identity neighbourhood, which thus must be coarsely bounded in G. □

Before stating the next lemma, let us recall that a topological group G is *Baire* if it satisfies the Baire category theorem, i.e., if the intersection of a countable family of dense open sets is dense in G. Prime examples of Baire groups are of course the locally compact and the completely metrisable groups.

Lemma 2.24. *Let G be a Baire topological group with metrisable left-coarse structure $\mathcal{E}_L$. Then G is locally bounded.*

Proof. Since $\mathcal{E}_L$ is metrisable, the ideal $\mathcal{OB}$ admits a countable cofinal family $\{A_n\}_n$, whence also $\{\overline{A_n}\}_n$ is cofinal in $\mathcal{OB}$. In particular, since $\mathcal{OB}$ contains all singletons, we have $G=\bigcup_n\overline{A_n}$ and hence, as G is Baire, some $\overline{A_n}$ must be non-meagre and thus have non-empty interior W. It follows that the identity neighbourhood $V=WW^{-1}$ is coarsely bounded in G. □

Lemma 2.25. *Suppose G is a topological group countably generated over every identity neighbourhood. Then, for every symmetric open identity neighbourhood V, there is a continuous left-invariant écart d so that a subset $A \subseteq G$ is d-bounded if and only if there are a finite set F and an n so that $A \subseteq (FV)^n$.*

Proof. Fix a symmetric open identity neighbourhood V and choose $x_1, x_2, \ldots \in G$ so that $G = \langle V \cup \{x_1, x_2, \ldots\}\rangle$. We then let

$$V_n = \big(V \cup \{x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}\}\big)^{3^n}$$

and note that the V_n form an increasing exhaustive chain of open symmetric identity neighbourhoods satisfying $V_n^3 \subseteq V_{n+1}$ for all n. We now complement the chain by symmetric open identity neighbourhoods

$$V_0 \supseteq V_{-1} \supseteq V_{-2} \supseteq \ldots$$

so that $V_n^3 \subseteq V_{n+1}$ holds for all $n \in \mathbb{Z}$.

Applying Lemma 2.6, we obtain a continuous left-invariant écart d on G whose balls are each contained in some V_n and so that each V_n has finite d-diameter. It follows that a subset $A \subseteq G$ is d-bounded if and only if $A \subseteq V_n$ for some n. Also, if $F \subseteq G$ is finite, then $F \subseteq V_n$ for some $n \geqslant 1$, whereby $(FV)^k \subseteq V_{n+k}$ must have finite diameter for all $k \geqslant 1$. This shows that a subset $A \subseteq G$ is d-bounded if and only if there are $F \subseteq G$ finite and $k \geqslant 1$ so that $A \subseteq (FV)^k$ as claimed. □

Lemma 2.26. *Let G be a locally bounded topological group, and assume that G is countably generated over every identity neighbourhood. Then $\mathcal{E}_L$ is induced by a continuous left-invariant écart d on G.*

Proof. Fix a symmetric open identity neighbourhood V coarsely bounded in G and let d be a continuous left-invariant écart as in Lemma 2.25. Then a $A \subseteq G$ is d-bounded if and only if A is coarsely bounded in G, whence d induces the left-coarse structure $\mathcal{E}_L$ on G. □

As the combination of being Baire and countably generated over identity neighbourhoods will reappear several times, it will be useful to have a name for this.

Definition 2.27. *A topological group G is* European *if it is Baire and countably generated over every identity neighbourhood.*

Observe that the class of European groups is a proper extension of the class of Polish groups; hence the name. For example, all connected completely metrisable groups, e.g., all Banach spaces, and all locally compact σ-compact groups are European, but may not be Polish. Recall also that the ideals $\mathcal{F}$ and $\mathcal{OB}$ coincide in a European topological group.

Combining the preceding lemmas, we obtain the following characterisation of the metrisability of the coarse structure.

Theorem 2.28. *The following are equivalent for a European topological group G.*

(1) *The left-coarse structure $\mathcal{E}_L$ is metrisable,*
(2) *G is locally bounded,*
(3) *$\mathcal{E}_L$ is induced by a continuous left-invariant écart d on G.*

Theorem 2.28 can be seen as an analogue of well-known facts about locally compact groups. Indeed, S. Kakutani and K. Kodaira [**35**] showed that if G is locally compact, σ-compact, then for any sequence U_n of identity neighbourhoods there is a compact normal subgroup $K \subseteq \bigcap_n U_n$ so that G/K is metrisable.

Moreover, R. Struble [**68**] showed that any metrisable (i.e., second countable) locally compact group admits a compatible left-invariant proper metric, i.e., so that all bounded sets are relatively compact. To construct such a metric, one can simply choose the V_n of Lemma 2.6 to be relatively compact.

Combining these two results, one sees that every locally compact σ-compact group G admits a continuous proper left-invariant écart, thus inducing the group-compact coarse structure $\mathcal{E}_L = \mathcal{E}_{\mathcal{K}}$ on G.

The next concept from [**60**] will help us delineate, e.g., the finitely generated groups in the class of all countable discrete groups.

Definition 2.29. *A coarse structure* $(X, \mathcal{E})$ *is* monogenic *if* $\mathcal{E}$ *is generated by a single entourage* E.

Note that, by replacing the generator $E \in \mathcal{E}$ by $E \cup \Delta$, one sees that $\mathcal{E}$ is monogenic if and only if there is some entourage $E \in \mathcal{E}$ so that $\{E^n\}_n$ is cofinal in $\mathcal{E}$. Also, if $\mathcal{E}_L$ is the left-coarse structure on a group G, this E can be taken of the form E_A for some coarsely bounded set A. Now, recall from earlier that $E_A^n = E_{A^n}$. Using this, we see that the coarse structure $\mathcal{E}_L$ is monogenic if and only if there is some coarsely bounded set A so that $\{A^n\}_n$ is cofinal in $\mathcal{OB}$.

Theorem 2.30. *The following are equivalent for a European topological group* G*.*

1. *The left-coarse structure* $\mathcal{E}_L$ *is monogenic,*
2. G *is* generated by a coarsely bounded set*, i.e., there is some* $A \in \mathcal{OB}$ *algebraically generating* G*,*
3. G *is locally bounded and not the union of a countable chain of proper open subgroups.*

Proof. (2)⇒(1): Suppose that G is generated by a symmetric coarsely bounded subset $A \subseteq G$, i.e., $G = \bigcup_n A^n$. By the Baire category theorem, some A^n must be somewhere dense and thus $B = \overline{A^n}$ is a coarsely bounded set with non-empty interior generating G. To see that $\{B^n\}_n$ is cofinal in $\mathcal{OB}$ and thus that $\mathcal{E}_{\mathcal{OB}}$ is monogenic, observe that if $C \subseteq G$ is coarsely bounded, then, as $\mathrm{int}(B) \neq \emptyset$, there is a finite set $F \subseteq G$ and a k so that $C \subseteq (FB)^k$. As furthermore B generates G, one has $C \subseteq B^m$ for some sufficiently large m.

(1)⇒(3): If $\mathcal{E}_L$ is monogenic, it is countably generated and thus metrisable. Thus, by Theorem 2.28, G is locally bounded. Also, if $H_1 \leqslant H_2 \leqslant \ldots \leqslant G$ is a countable chain of open subgroups exhausting G, then, by definition of the ideal $\mathcal{OB}$, every coarsely bounded set is contained in some H_n. Thus, as G is generated by a coarsely bounded set, it must equal some H_n.

(3)⇒(2): If G is coarsely bounded as witnessed by some identity neighbourhood V, let $\{x_n\}_n$ be a countable set generating G over V. If, moreover, G is not the union of a countable chain of proper open subgroups, it must be generated by some $V \cup \{x_1, \ldots, x_n\}$ and hence be generated by a coarsely bounded set. □

Observe that Theorem 2.30 applies, in particular, to locally compact σ-compact groups where the coarsely bounded and relative compact sets coincide. Thus, among these, the compactly generated are exactly those whose coarse structure is monogenic.

EXAMPLE 2.31. Let X be a Banach space and consider the additive group $(X,+)$ equipped with the weak topology $\mathtt{w}$. Recall that, by the Banach–Steinhaus uniform boundedness principle, a subset $A \subseteq X$ is norm-bounded if and only if it is weakly-bounded, i.e.,

$$\sup_{x\in A} |\phi(x)| < \infty$$

for all $\phi \in X^*$. So, since every functional ϕ defines a $\mathtt{w}$-continuous invariant écart via $d_\phi(x,y) = |\phi(x-y)|$, we see that the norm-bounded sets are exactly the coarsely bounded sets in $(X,+,\mathtt{w})$. It thus follows that the left-coarse structures of the topological groups $(X,+,\|\cdot\|)$ and $(X,+,\mathtt{w})$ coincide. In particular, $(X,+,\mathtt{w})$ is generated by a coarsely bounded set and has metrisable coarse structure, though it fails to be locally bounded except when X is finite-dimensional.

The apparent contradiction with Theorems 2.28 and 2.30 is resolved when we observe that $(X,+,\mathtt{w})$ is not Baire and thus not European when X is infinite-dimensional.

5. Bornologous maps

Let us begin by fixing some notation. If $\phi\colon X \to Y$ is a map, we get a naturally defined map $\phi \times \phi\colon \mathcal{P}(X \times X) \to \mathcal{P}(Y \times Y)$ by

$$\big(\phi \times \phi\big)E = \{\big(\phi(x_1), \phi(x_2)\big) \bigm| (x_1,x_2) \in E\}$$

and, similiarly, a map $(\phi \times \phi)^{-1}\colon \mathcal{P}(Y \times Y) \to \mathcal{P}(X \times X)$ by

$$\big(\phi \times \phi\big)^{-1}F = \{(x_1,x_2) \bigm| \big(\phi(x_1), \phi(x_2)\big) \in F\}.$$

Recall that, if $\phi\colon X \to Y$ is a map between uniform spaces $(X,\mathcal{U})$ and $(Y,\mathcal{V})$, then ϕ is *uniformly continuous* if, for all $F \in \mathcal{V}$, there is $E \in \mathcal{U}$ so that

$$(x_1,x_2) \in E \Rightarrow \big(\phi(x_1), \phi(x_2)\big) \in F,$$

i.e., $E \subseteq (\phi\times\phi)^{-1}F$. Thus, ϕ is uniformly continuous exactly when $(\phi\times\phi)^{-1}\mathcal{V} \subseteq \mathcal{U}$. This motivates the definition of bornologous maps between coarse spaces.

DEFINITION 2.32. *Let $(X,\mathcal{E})$ and $(Y,\mathcal{F})$ be coarse spaces, Z a set and $\phi\colon X \to Y$, $\alpha,\beta\colon Z \to X$ mappings. We say that*

(1) *ϕ is* bornologous *if $(\phi \times \phi)[\mathcal{E}] \subseteq \mathcal{F}$,*
(2) *ϕ is* expanding *if $(\phi \times \phi)^{-1}\mathcal{F} \subseteq \mathcal{E}$,*
(3) *ϕ is* modest *if $\phi[A]$ is $\mathcal{F}$-bounded whenever $A \subseteq X$ is $\mathcal{E}$-bounded,*
(4) *ϕ is* coarsely proper *if $\phi[A]$ is $\mathcal{F}$-unbounded whenever $A \subseteq X$ is $\mathcal{E}$-unbounded,*
(5) *ϕ is a* coarse embedding *if it is both bornologous and expanding,*
(6) *$A \subseteq X$ is* cobounded *in X if there is an entourage $E \in \mathcal{E}$ so that*

$$X = E[A] := \{x \in X \bigm| \exists y \in A\ (x,y) \in E\},$$

(7) *ϕ is* cobounded *if $\phi[X]$ is cobounded in Y,*
(8) *α and β are* close *if there is some $E \in \mathcal{E}$ so that $(\alpha(z),\beta(z)) \in E$ for all $z \in Z$,*
(9) *ϕ is a* coarse equivalence *if it is bornologous and there is a bornologous map $\psi\colon Y \to X$ so that $\psi \circ \phi$ is close to* id_X*, while $\phi \circ \psi$ is close to* id_Y*.*

There are a number of comments that are in order here.
(a) First, a map $\phi\colon G \to H$ between topological groups is coarsely proper if and only if $\phi^{-1}(A)$ is coarsely bounded for every coarsely bounded $A \subseteq H$.
(b) A map ϕ between coarse spaces $(X, \mathcal{E})$ and $(Y, \mathcal{F})$ is bornologous if and only if, for every entourage $E \in \mathcal{E}$, there is an entourage $F \in \mathcal{F}$ so that

$$(x_1, x_2) \in E \Rightarrow (\phi x_1, \phi x_2) \in F.$$

(c) Similarly, ϕ is expanding if and only if, for all $F \in \mathcal{F}$, there is $E \in \mathcal{E}$ so that

$$(x_1, x_2) \notin E \Rightarrow (\phi x_1, \phi x_2) \notin F.$$

(d) In particular, if $\phi\colon (X, d_X) \to (Y, d_Y)$ is a map between spaces with écarts d_X and d_Y, then ϕ is bornologous with respect to the induced coarse structures if and only if, for every $t < \infty$, there is a $\theta(t) < \infty$ so that

$$d_X(x_1, x_2) < t \Rightarrow d_Y(\phi x_1, \phi x_2) < \theta(t)$$

and expanding if and only if, for every $t < \infty$, there is a $\kappa(t) < \infty$ so that

$$d_X(x_1, x_2) > \kappa(t) \Rightarrow d_Y(\phi x_1, \phi x_2) > t.$$

(e) An expanding map is always coarsely proper, while a bornologous map is always modest.
(f) If $\phi\colon X \to Y$ is a coarse equivalence as witnessed by $\psi\colon Y \to X$, then also ψ is a coarse equivalence. Furthermore, if $\eta\colon Y \to Z$ is a coarse equivalence into a coarse space Z, then $\eta \circ \phi$ is a coarse equivalence. So the existence of a coarse equivalence between coarse spaces defines an equivalence relation.
(g) Since $E_{B^{-1}}[A] = AB$, a subset A of a topological group G is cobounded if and only if there is a coarsely bounded set B so that $G = A \cdot B$.

Let us note the following elementary fact.

LEMMA 2.33. *Let $(X, \mathcal{E})$ and $(Y, \mathcal{F})$ be coarse spaces and $\phi\colon X \to Y$, $\psi\colon Y \to X$ mappings so that $\psi \circ \phi$ is close to id_X. It follows that*

1. *if $\psi\colon Y \to X$ is bornologous, then ϕ is expanding,*
2. *if $\psi\colon Y \to X$ is expanding, then ϕ is bornologous and $\phi \circ \psi$ is close to id_Y.*

PROOF. Since $\psi \circ \phi$ is close to id_X, we may fix some $E_1 \in \mathcal{E}$ so that $(\psi\phi(x), x) \in E_1$ for all $x \in X$.

Suppose first that $\psi\colon Y \to X$ is bornologous and let $F \in \mathcal{F}$ be given. As ψ is bornologous, we have $E_2 = (\psi \times \psi)F \in \mathcal{E}$. Thus, if $(x_1, x_2) \in (\phi \times \phi)^{-1}F$, we have $(\psi\phi(x_1), \psi\phi(x_2)) \in E_2$ and so $(x_1, x_2) \in E_1^{-1} \circ E_2 \circ E_1$. I.e., $(\phi \times \phi)^{-1}F \subseteq E_1^{-1} \circ E_2 \circ E_1 \in \mathcal{E}$, showing that ϕ is expanding.

Assume instead that $\psi\colon Y \to X$ is expanding. Given $E_2 \in \mathcal{E}$, we have $F = (\psi \times \psi)^{-1}\big(E_1 \circ E_2 \circ E_1^{-1}\big) \in \mathcal{F}$. Thus, if $(x_1, x_2) \in E_2$, we have $(\psi\phi(x_1), \psi\phi(x_2)) \in E_1 \circ E_2 \circ E_1^{-1}$ and so also $(\phi(x_1), \phi(x_2)) \in F$, showing that $(\phi \times \phi)E_2 \in \mathcal{F}$. In other words, ϕ is bornologous. Moreover, for every $y \in Y$, we have $(\psi\phi\psi(y), \psi y) \in E_1$ and thus $(\phi\psi(y), y) \in (\psi \times \psi)^{-1}E_1 \in \mathcal{F}$, whereby $\phi \circ \psi$ is close to id_Y. $\square$

This in turn leads to the following well-known fact.

LEMMA 2.34. *A map $\phi\colon X \to Y$ between coarse spaces $(X, \mathcal{E})$ and $(Y, \mathcal{F})$ is a coarse equivalence if and only if ϕ is bornologous, expanding and cobounded.*

Proof. Suppose first that ϕ is a coarse equivalence as witnessed by some bornologous $\psi\colon Y\to X$ so that $\psi\circ\phi$ is close to id_X and $\phi\circ\psi$ is close to id_Y. Then ϕ is expanding by Lemma 2.33 (1). Also, as $\phi\circ\psi$ is close to id_Y, we see that $\phi[X]$ must be cobounded in Y.

Conversely, if ϕ is bornologous, expanding and cobounded, pick an entourage $F\in\mathcal F$ so that $\phi[X]$ is F-cobounded and let $\psi\colon Y\to X$ be defined by

$$\psi(y)=x \text{ for some } x\in X \text{ so that } (y,\phi x)\in F.$$

Then by construction we have $\big(y,\phi\psi(y)\big)\in F$ for all $y\in Y$ and thus $\phi\circ\psi$ is close to id_Y. Applying Lemma 2.33 (2) (with the rôles of ϕ and ψ reversed), we see that ψ is bornologous and $\psi\circ\phi$ is close to id_X, i.e., that ϕ is a coarse equivalence. □

Regarding maps between groups, note that, if $\phi\colon G\to H$ is a group homomorphism and $A\subseteq G$ and $B\subseteq H$ are subsets, then

$$(\phi\times\phi)^{-1}E_B=E_{\phi^{-1}(B)}$$

and

$$(\phi\times\phi)E_A\subseteq E_{\phi[A]}.$$

Also, is $\psi\colon G\to H$ is an arbitrary map and $(\psi\times\psi)E_A\subseteq E_B$ for some $A\subseteq G$ and $B\subseteq H$, then

$$(\psi\times\psi)E_{A^n} \;=\; (\psi\times\psi)E_A^n \;\subseteq\; E_B^n \;=\; E_{B^n}$$

for all $n\geqslant 1$.

Lemma 2.35. *Let $\phi\colon G\to H$ be a continuous homomorphism between topological groups. Then ϕ is bornologous. Moreover, the following conditions on ϕ are equivalent.*

(1) *ϕ is a coarse embedding,*
(2) *ϕ is expanding,*
(3) *ϕ is coarsely proper.*

Proof. To see that ϕ is bornologous, let E be a coarse entourage in G. Then there is a coarsely bounded set $A\subseteq G$ so that $E\subseteq E_A$, whence $(\phi\times\phi)E\subseteq(\phi\times\phi)E_A\subseteq E_{\phi[A]}$. It thus suffices to see that $\phi[A]$ is coarsely bounded in H, whereby $E_{\phi[A]}$ is an entourage in H. But this follows from the fact that, if d is a continuous left-invariant écart on H, then $d\big(\phi(\cdot),\phi(\cdot)\big)$ defines a continuous left-invariant écart on G with respect to which A has finite diameter and hence $\phi[A]$ has finite d-diameter. As d was arbitrary, $\phi[A]$ is coarsely bounded in H.

To verify the set of equivalences, since every expanding map is coarsely proper, it suffices to see that ϕ is expanding provided it is coarsely proper. But, if F is an entourage in H, find a coarsely bounded set $B\subseteq H$ so that $F\subseteq E_B$. Then, since ϕ is coarsely proper, $\phi^{-1}(B)$ is coarsely bounded in G, whence $(\phi\times\phi)^{-1}E_B=E_{\phi^{-1}(B)}$ is an entourage in G. I.e., ϕ is expanding. □

Definition 2.36. *Let H be a subgroup of a topological group G. We say that H is* coarsely embedded *in G if the inclusion map $\iota\colon H\to G$ is a coarse embedding, i.e., if the coarse structure on H coincides with the coarse structure on G restricted to H.*

Note that, by Lemma 2.35, the subgroup H is coarsely embedded in G exactly when every subset A of H, which is coarsely bounded in G, is also coarsely bounded in H. In case H is a closed subgroup of a locally compact group, the coarsely bounded sets of H and G are the relatively compact sets, which thus does not depend on wther they are seen as subsets of H or of G. In other words, a closed subgroup of a locally compact group is automatically coarsely embedded. On the other hand, in a Polish group, this is very far from being true. In fact, as we shall see there is a universal Polish group G, i.e., in which every other Polish group can be embedded as a closed subgroup, so that G is coarsely bounded in itself. It thus follows that any closed subgroup H will be coarsely bounded in G, but need not be in itself and thus may not be coarsely embedded. This fact is a source of much of the additional complexity compared with the coarse geometry of locally compact groups.

Proposition 2.37. *Suppose $\phi\colon G \to H$ is a continuous homomorphism between European topological groups so that the image $\phi[G]$ is dense in H. Assume that, for some identity neighbourhood $V \subseteq H$, the preimage $\phi^{-1}(V)$ is coarsely bounded in G. Then ϕ is a coarse equivalence between G and H.*

Proof. Without loss of generality, we may assume that V is open. Let us first see that ϕ is a coarse embedding. So assume that $A \subseteq G$ and that $\phi[A]$ is coarsely bounded in H. Then, as $\phi[G]$ is dense in H, there are a finite set $F \subseteq G$ and a k so that $\phi[A] \subseteq (\phi[F]V)^k$. Now, observe that given $a \in A$, by density of $\phi[G] \cap V$ in the open set V, we can find $g_1, \ldots, g_{k-1} \in \phi^{-1}(V)$ and $f_1, \ldots, f_k \in F$ so that $\phi(a) \in \phi(f_1)\phi(g_1)\cdots\phi(f_{k-1})\phi(g_{k-1})\phi(f_k)V$. But then, since $\phi[G]$ is a subgroup of H, it follows that actually $\phi(a) \in \phi(f_1)\phi(g_1)\cdots\phi(f_{k-1})\phi(g_{k-1})\phi(f_k)\cdot(\phi[G] \cap V)$ and therefore that $\phi(a) \in \big(\phi[F]\cdot(\phi[G]\cap V)\big)^k = \phi\big[(F\phi^{-1}(V))^k\big]$. In other words, $A \subseteq (F\phi^{-1}(V))^k$, showing that A is coarsely bounded in G. So ϕ is coarsely proper and thus a coarse embedding.

Since $\phi^{-1}(V)$ is a coarsely bounded identity neighbourhood, G is locally bounded. We can therefore find a sequence $A_1 \subseteq A_2 \subseteq \ldots \subseteq G$ of coarsely bounded sets cofinal in the ideal $\mathcal{OB}$. We claim that $H = \bigcup_n \overline{\phi[A_n]}$. Indeed, if $h \in H$, choose $g_n \in G$ so that $\phi(g_n) \to h$. Then $\{\phi(g_n)\}_n$ is relatively compact and thus coarsely bounded in H. As ϕ is a coarse embedding, it follows that also $\{g_n\}_n$ is coarsely bounded in G and thus is contained in some set A_k. In other words, $h \in \overline{\phi[A_k]}$. It follows that the ideal $\mathcal{OB}$ on H is countably generated and thus that H is locally bounded. Choosing $U \subseteq H$ to be a coarsely bounded identity neighbourhood, we have $H = \phi[G]U$ and hence $\phi[G]$ is cobounded in H. So ϕ is a coarse equivalence. $\square$

6. Coarsely proper écarts

With the above concepts at hand, we may now return to issues regarding metrisability.

Definition 2.38. *An continuous left-invariant écart d on a topological group G is said to be* coarsely proper *if d induces the left-coarse structure on G, i.e., if $\mathcal{E}_L = \mathcal{E}_d$.*

Thus, by Theorem 2.28, a European group admits a coarsely proper écart if and only if it is locally bounded. On the other hand, the identification of coarsely proper écarts is also of interest.

Lemma 2.39. *The following are equivalent for a continuous left-invariant écart d on a topological group G,*

(1) *d is coarsely proper,*
(2) *a set $A \subseteq G$ is coarsely bounded if and only if it is d-bounded,*
(3) *for every compatible left-invariant écart ∂ on G, the mapping*
$$\mathrm{id}\colon (G,d) \to (G,\partial)$$
is bornologous.

Proof. Since
$$\mathcal{E}_L = \bigcap\{\mathcal{E}_\partial \mid \partial \text{ is a continuous left-invariant écart on } G\},$$
coarse properness of the écart d simply means that $\mathcal{E}_d \subseteq \mathcal{E}_\partial$ for every other continuous left-invariant écart ∂, that is, that the map
$$\mathrm{id}\colon (G,d) \to (G,\partial)$$
is bornologous.

Note that, in general, if d and ∂ are left-invariant écarts on G so that any d-bounded set is ∂-bounded, then $\mathrm{id}\colon (G,d) \to (G,\partial)$ is bornologous. For, if the d-ball of radius R is contained in the ∂-ball of radius S, then
$$d(x,y) = d(1,x^{-1}y) < R \Rightarrow \partial(x,y) = \partial(1,x^{-1}y) < S.$$
Thus, as the coarsely bounded sets are those bounded in every continuous left-invariant écart, this shows that d is coarsely proper if and only if a set $A \subseteq G$ is coarsely bounded exactly when it is d-bounded. □

The following criterion is useful for identifying coarsely proper metrics.

Lemma 2.40. *Suppose that d is a compatible left-invariant metric on a topological group G without proper open subgroups. Then d is coarsely proper if and only if, for all constants Δ and $\delta > 0$, there is a k so that, for any $x \in G$ with $d(x,1) < \Delta$, there are $y_0 = 1, y_1, \ldots, y_{k-1}, y_k = x$ so that $d(y_i, y_{i+1}) < \delta$.*

In particular, every compatible geodesic metric is coarsely proper.

Proof. To see this, suppose first that d is coarsely proper. Then, for any Δ, the open ball $B_d(\Delta) = \{x \in G \mid d(x,1) < \Delta\}$ is coarsely bounded in G and hence, for any $\delta > 0$, there is, as G has no proper open subgroup, some k so that $B_d(\Delta) \subseteq \big(B_d(\delta)\big)^k$. It follows that every $x \in B_d(\Delta)$ can be written as $x = z_1 z_2 \cdots z_k$ with $z_i \in B_d(\delta)$. So, letting $y_i = z_1 \cdots z_i$, we find that the above condition on d is verified.

Assume instead that d satisfies this condition and let $\Delta > 0$ be given. We must show that the ball $B_d(\Delta)$ is coarsely bounded in G. For this, note that, if V is any identity neighbourhood in G, then, since d is a compatible metric, there is $\delta > 0$ so that $B_d(\delta) \subseteq V$. Choosing k as in the assumption on d, we note that $B_d(\Delta) \subseteq \big(B_d(\delta)\big)^k \subseteq V^k$, thus verifying that $B_d(\Delta)$ is coarsely bounded in G. □

Example 2.41. Consider the additive topological group $(X,+)$ of a Banach space $(X, \|\cdot\|)$. Since the norm metric is geodesic on X, by Lemma 2.40, we conclude that this latter is coarsely proper on $(X,+)$.

Example 2.42 (A topology for the coarse structure). Let G be a Hausdorff topological group. Then the ideal $\mathcal{OB}$ of coarsely bounded sets may be used to define a topology $\tau_{\mathcal{OB}}$ on the one-point extension $G \cup \{*\}$. Namely, for $U \subseteq G \cup \{*\}$, we set $U \in \tau_{\mathcal{OB}}$ if

(1) $U \cap G$ is open in G and
(2) $* \in U \Rightarrow G \setminus U \in \mathcal{OB}$.

Since clearly $\emptyset, G \cup \{*\} \in \tau_{\mathcal{OB}}$ and $\tau_{\mathcal{OB}}$ is stable under finite intersections and arbitrary unions, it is a topology on $G \cup \{*\}$. Moreover, when restricted to G, this is simply the usual topology on G and G is open in $G \cup \{*\}$. Also, since $\mathcal{OB}$ is stable under taking topological closures in G, we find that, for $A \subseteq G$,

$$* \in \overline{A}^{\tau_{\mathcal{OB}}} \Leftrightarrow A \notin \mathcal{OB}.$$

We claim that

$$\tau_{\mathcal{OB}} \text{ is Hausdorff } \Leftrightarrow \ G \text{ is locally bounded.}$$

Indeed, if $\tau_{\mathcal{OB}}$ is Hausdorff, then there is an identity neighbourhood U so that $* \notin \overline{U}^{\tau_{\mathcal{OB}}}$, i.e., $U \in \mathcal{OB}$, showing that G is locally bounded. Conversely, suppose G is locally bounded and that U is a coarsely bounded open identity neighbourhood. Then xU and $V = \{*\} \cup G \setminus \overline{xU}$ are open sets separating any point $x \in G$ from $*$. Since G itself is Hausdorff and $\tau_{\mathcal{OB}}$-open, $\tau_{\mathcal{OB}}$ is a Hausdorff topology.

Now, suppose that G is a locally bounded Polish group and fix a coarsely proper continuous left-invariant écart d on G. Setting $U_n = \{x \in G \mid d(x, 1) > n\} \cup \{*\}$, we see that the U_n form a neighbourhood basis at $*$ so that $\overline{U_{n+1}} \subseteq U_n$. Thus, for any open $W \ni *$, there is $U_n \ni *$ so that $\overline{U_n} \subseteq W$. Since also G is regular, this shows that $\tau_{\mathcal{OB}}$ is a regular topology. Being also Hausdorff and second countable, we conclude, by Urysohn's metrisation theorem, that $\tau_{\mathcal{OB}}$ is metrisable. Thus, if ∂ is a compatible metric on $G \cup \{*\}$, we see that, since G is Polish, G and thus also $G \cup \{*\}$ are G_δ in the completion $\overline{G \cup \{*\}}^{\partial}$. It follows that $G \cup \{*\}$ is a Polish space with respect to the topology $\tau_{\mathcal{OB}}$.

EXAMPLE 2.43 (Two notions of divergence). If $\tau_{\mathcal{OB}}$ denotes the topology from Example 2.42, we see that, for $g_n \in G$, we have $g_n \underset{\tau_{\mathcal{OB}}}{\longrightarrow} *$ if and only if the g_n eventually leave every coarsely bounded subset of G. As an alternative to this, we write $g_n \to \infty$ if there is a continuous left-invariant écart d on G so that $d(g_n, 1) \to \infty$. Then, since every coarsely bounded set must have finite d-diameter, we have

$$g_n \to \infty \ \Rightarrow \ g_n \underset{\tau_{\mathcal{OB}}}{\longrightarrow} *.$$

Conversely, if G admits a coarsely proper continuous left-invariant écart ∂, then $g_n \underset{\tau_{\mathcal{OB}}}{\longrightarrow} *$ implies that $\partial(g_n, 1) \to \infty$ and thus also that $g_n \to \infty$. However, as we shall see in Proposition 3.34, this implication fails without the existence of a coarsely proper écart.

Assume that $G \curvearrowright (X, d)$ is a continuous isometric action of a topological group G on a metric space (X, d). Then, for every $x \in X$, the orbit map

$$g \in G \mapsto g \cdot x \in X$$

is uniformly continuous and bornologous. For, if $\epsilon > 0$, then, by continuity, there is an identity neighbourhood $V \ni 1$ so that $d(x, vx) < \epsilon$ for all $v \in V$, whence $d(gx, fx) = d(x, g^{-1}fx) < \epsilon$ whenever $(g, f) \in E_V$, i.e., $g^{-1}f \in V$, verifying uniform continuity. Also, suppose E is a coarse entourage in G, set $\partial(g, f) = d(gx, fx)$ and note that ∂ is a continuous left-invariant écart on G. By Theorem 2.13, we find that $E \in \mathcal{E}_\partial$, that is, $E \subseteq \{(g, f) \in G \times G \mid \partial(g, f) < K\} = \{(g, f) \in$

$G \times G \mid d(gx, fx) < K\}$ for some $K > 0$. In other words, if $(g, f) \in E$, then $d(gx, fx) < K$, showing that $g \mapsto gx$ is bornologous.

We claim that, if the orbit map $g \mapsto gx$ is coarsely proper for some $x \in X$, then $g \mapsto gy$ is a expansive for every $y \in X$. To see this, suppose $x \in X$ is such that $g \mapsto gx$ is coarsely proper, $y \in X$ and $K > 0$. Then, as $g \mapsto gx$ is coarsely proper, the set $\{g \in G \mid d(x, gx) \leqslant 2d(x, y) + K\}$ and hence also the subset $A = \{g \in G \mid d(y, gy) \leqslant K\}$ are coarsely bounded in G. Thus, if $(g, f) \notin E_A$, we have $d(gy, fy) = d(y, g^{-1}fy) > K$. As E_A is a coarse entourage in G, this shows that $g \mapsto gy$ is a expansive.

Thus, the orbit map $g \mapsto gy$ is a uniformly continuous coarse embedding for every $y \in X$ if and only if it is coarsely proper for some $y \in X$.

Similarly, the orbit map $g \mapsto gy$ is cobounded for every $y \in X$ if and only if it is cobounded for some $y \in X$, which again is equivalent to there being an open set $U \subseteq X$ of finite diameter so that $X = G \cdot U$.

Definition 2.44. *An isometric action $G \curvearrowright (X, d)$ of a topological group on a metric space is said to be* coarsely proper*, respectively* cobounded *if*

$$g \in G \mapsto gx \in X$$

is coarsely proper, respectively cobounded, for some $x \in X$.

Setting x to be the identity element 1 in G, one sees that a compatible left-invariant metric d on G is coarsely proper if and only if the left-multiplication action $G \curvearrowright (G, d)$ is coarsely proper.

Conversely, suppose $G \curvearrowright (X, d)$ is a coarsely proper continuous isometric action of a metrisable group G on a metric space (X, d). Let D be a compatible left-invariant metric on G, fix $x \in X$ and define $\partial(g, f) = d(gx, fx) + D(g, f)$. Then ∂ is a continuous left-invariant metric on G. Also, $\partial \geqslant D$, whereby ∂ is compatible with the topology on G. Moreover, since the action is coarsely proper, so is ∂.

To sum up, a metrisable topological group admits a coarsely proper compatible left-invariant metric if and only if it admits a coarsely proper continuous isometric action on a metric space.

7. Quasi-metric spaces and maximal écarts

Definition 2.45. *Let (X, d_X) and (Y, d_Y) be pseudometric spaces. A map $\phi\colon X \to Y$ is said to be a* quasi-isometric embedding *if there are constants K, C so that*

$$\frac{1}{K} \cdot d_X(x_1, x_2) - C \leqslant d_Y(\phi x_1, \phi x_2) \leqslant K \cdot d_X(x_1, x_2) + C.$$

Also, ϕ is a quasi-isometry *if, moreover, $\phi[X]$ is cobounded in Y.*

Similarly, a map $\phi\colon X \to Y$ is Lipschitz for large distances *if there are constants K, C so that*

$$d_Y(\phi x_1, \phi x_2) \leqslant K \cdot d_X(x_1, x_2) + C.$$

In the literature on Banach spaces (e.g., [**8**]), sometimes Lipschitz for large distances means something slightly weaker, namely, that, for all $\alpha > 0$, there is a constant K_α so that $d_Y(\phi x_1, \phi x_2) \leqslant K_\alpha \cdot d_X(x_1, x_2)$, whenever $d_X(x_1, x_2) \geqslant \alpha$. However, in most natural settings, it is equivalent to the above, which is more appropriate since it is a strengthening of bornologous maps.

Two écarts d and ∂ on a set X will be called *quasi-isometric* if the identity map $\mathrm{id}\colon (X,d)\to(X,\partial)$ is a quasi-isometry. Similarly, two spaces (X,d_X) and (Y,d_Y) are *quasi-isometric* if there is a quasi-isometry between them. This is easily seen to define equivalence relations on the set of écarts, respectively, on pseudometric spaces.

DEFINITION 2.46. *A* quasimetric space *is a set X equipped with a quasi-isometric equivalence class $\mathcal{D}$ of écarts d on X.*

Evidently, every quasimetric space $(X,\mathcal{D})$ admits a canonical coarse structure, namely, the coarse structure $\mathcal{E}_d$ induced by some $d\in\mathcal{D}$. As any two d and ∂ in $\mathcal{D}$ are quasi-isometric, we see that $\mathcal{E}_d$ is independent of the choice of $d\in\mathcal{D}$. Quasimetric spaces may therefore be viewed as coarse spaces with an additional structure that allows us to talk about mappings between them as being, e.g., Lipschitz for large distances. Similarly, every pseudometric space (X,d) admits a canonical quasi-isometric structure, namely, the equivalence class of its écart d.

DEFINITION 2.47. *A pseudometric space (X,d) is said to be* large scale geodesic *if there is $K\geqslant 1$ so that, for all $x,y\in X$, there are $z_0=x,z_1,z_2,\ldots,z_n=y$ so that $d(z_i,z_{i+1})\leqslant K$ and*

$$\sum_{i=0}^{n-1} d(z_i,z_{i+1})\leqslant K\cdot d(x,y).$$

For example, if $\mathbb{X}$ is a connected graph, then the shortest path metric ρ on $\mathbb{X}$ makes $(\mathbb{X},\rho)$ large scale geodesic with constant $K=1$.

As is easy to check, large scale geodecity is invariant under quasi-isometries between pseudometric spaces and thus, in particular, is a quasi-isometric invariant of écarts on a space X. Thus, we may define a quasimetric space $(X,\mathcal{D})$ to be *large scale geodesic* if some or, equivalently, all $d\in\mathcal{D}$ are large scale geodesic.

Also, of central importance is the following well-known fact generalising the classical Corson-Klee Lemma [**20**] (for a proof see, e.g., Theorem 1.4.13 [**55**]).

LEMMA 2.48. *Let $\phi\colon X\to Y$ be a bornologous map between quasi-metric spaces $(X,\mathcal{D}_X)$ and $(Y,\mathcal{D}_Y)$ and assume that $(X,\mathcal{D}_X)$ is large scale geodesic. Then ϕ is Lipschitz for large distances.*

J. Roe (Proposition 2.57 [**60**]) showed that a connected coarse space $(X,\mathcal{E})$ is monogenic if and only if it is coarsely equivalent to a geodesic metric space and the same proof also works for large scale geodesic quasimetric spaces in place of geodesic metric spaces. Indeed, suppose first that $(X,\mathcal{E})$ is monogenic, i.e., that $\mathcal{E}$ is generated by a symmetric entourage E. Then $\{E^n\}_n$ is cofinal in $\mathcal{E}$ and, as $\mathcal{E}$ is connected, (X,E) is a connected graph and the shortest path distance ρ will induce the coarse structure $\mathcal{E}$.

Conversely, if $\phi\colon Y\to X$ is a coarse equivalence from a large scale geodesic quasimetric space $(Y,\mathcal{D})$, pick $d\in\mathcal{D}$, $K>0$ and $E\in\mathcal{E}$ so that d is large scale geodesic with constant K, E is symmetric and contains the diagonal Δ and $X=E\big[\phi[Y]\big]$. Set $F=\{(y_1,y_2)\in Y\times Y \mid d(y_1,y_2)\leqslant K\}$. We claim that $E\circ(\phi\times\phi)F\circ E$ generates $\mathcal{E}$. To see this, suppose $E'\in\mathcal{E}$ and let $F'=(\phi\times\phi)^{-1}\big(E\circ E'\circ E\big)$. As ϕ is a coarse equivalence, F' is a coarse entourage in Y, say $d(y_1,y_2)\leqslant m\cdot K$ for all $(y_1,y_2)\in F'$. As d is large scale geodesic, $F'\subseteq F^{K^2}$, so

$$(\phi\times\phi)F'\subseteq(\phi\times\phi)\big(F^{K^2}\big)=\big((\phi\times\phi)F\big)^{K^2}.$$

Now assume that $(x_1,x_2)\in E'$ and find $y_1,y_2\in Y$ so that $(x_i,\phi y_i)\in E$, i.e., $(y_1,y_2)\in F'$ and thus $(\phi y_1,\phi y_2)\in \big((\phi\times\phi)F\big)^{K^2}$. It follows that $(x_1,x_2)\in E\circ\big((\phi\times\phi)F\big)^{K^2}\circ E$, that is,

$$E'\subseteq E\circ\big((\phi\times\phi)F\big)^{K^2}\circ E\subseteq\big(E\circ(\phi\times\phi)F\circ E\big)^{K^2}$$

as required.

We may define an ordering on the space of écarts on a set X by letting $\partial\ll d$ if there are constants K and C so that $\partial\leqslant K\cdot d+C$, that is, if the identity map $\mathrm{id}\colon (X,d)\to(X,\partial)$ is Lipschitz for large distances and thus, a fortiori, bornologous. With these considerations, the following definition is natural.

Definition 2.49. *A continuous left-invariant écart d on a topological group G is said to be* maximal *if, for every other continuous left-invariant écart ∂, there are constants K,C so that $\partial\leqslant K\cdot d+C$.*

Note that, if d is a maximal écart on a metrisable group G, then, for every compatible left-invariant metric D on G, $d+D$ defines a maximal compatible metric on G.

We remark also that, unless G is discrete, this is really the strongest notion of maximality possible for d. Indeed, if G is non-discrete and thus d takes arbitrarily small values, then

$$\mathrm{id}\colon (G,d)\to(G,\sqrt{d})$$

fails to be Lipschitz for small distances, while $\sqrt{d}$ is a continuous left-invariant écart on G.

Note that, by Lemma 2.39, maximal écarts are automatically coarsely proper and the ordering $\ll$ provides a finer graduation within the class of coarsely proper écarts. Moreover, any two maximal écarts are clearly quasi-isometric. This latter observation shows the unambiguity of the following definition.

Definition 2.50. *Let G be a topological group admitting a maximal écart. The* quasimetric structure *on G is the quasi-isometric equivalence class of its maximal écarts.*

Let us now see how maximal metrics may be constructed. Recall first that, if Σ is a symmetric generating set for a topological group G, then we can define an associated *word metric* $\rho_\Sigma\colon G\to\mathbb{N}$ by

$$\rho_\Sigma(g,h)=\min\big(k\geqslant 0\;\big|\;\exists s_1,\dots,s_k\in\Sigma\;\; g=hs_1\cdots s_k\big).$$

Thus, ρ_Σ is a left-invariant metric on G, but, since it only takes values in $\mathbb{N}$, it will never be continuous unless of course G is discrete. However, in certain cases, this may be remedied.

Lemma 2.51. *Suppose d is a compatible left-invariant metric on a topological group G and $V\ni 1$ is a symmetric open identity neighbourhood generating G and having finite d-diameter. Define ∂ by*

$$\partial(f,h)=\inf\Big(\sum_{i=1}^n d(g_i,1)\;\Big|\;g_i\in V\;\&\;f=hg_1\cdots g_n\Big).$$

Then ∂ is a compatible left-invariant metric, quasi-isometric to the word metric ρ_V.

PROOF. Note first that ∂ is a left-invariant écart. Since V is open and d is continuous, also ∂ is continuous. Moreover, as $\partial \geqslant d$, it is a metric generating the topology on G.

To see that ∂ is quasi-isometric to ρ_V, note first that

$$\partial(f,h) \leqslant \mathrm{diam}_d(V)\cdot \rho_V(f,h).$$

For the other direction, pick some $\epsilon > 0$ so that V contains the identity neighbourhood $\{g \in G \mid d(g,1) < 2\epsilon\}$. Now, fix $f,h \in G$ and find a shortest sequence $g_1,\dots,g_n \in V$ so that $f = hg_1\cdots g_n$ and $\sum_{i=1}^n d(g_i,1) \leqslant \partial(f,h)+1$. Note that, for all i, we have $g_ig_{i+1} \notin V$, since otherwise we could coalesce g_i and g_{i+1} into a single term g_ig_{i+1} to get a shorter sequence where $d(g_ig_{i+1},1) \leqslant d(g_i,1) + d(g_{i+1},1)$. It thus follows that either $d(g_i,1) \geqslant \epsilon$ or $d(g_{i+1},1) \geqslant \epsilon$, whereby there are at least $\frac{n-1}{2}$ terms g_i so that $d(g_i,1) > \epsilon$. In particular,

$$\frac{n-1}{2}\cdot\epsilon < \sum_{i=1}^n d(g_i,1) \leqslant \partial(f,h)+1$$

and so, as $\rho_V(f,h) \leqslant n$, we have

$$\frac{\epsilon}{2}\cdot\rho_V(f,h) - \big(1+\frac{\epsilon}{2}\big) \leqslant \partial(f,h) \leqslant \mathrm{diam}_d(V)\cdot\rho_V(f,h)$$

showing that ∂ and ρ_V are quasi-isometric. □

PROPOSITION 2.52. *The following conditions are equivalent for a continuous left-invariant écart d on a topological group G,*

(1) *d is maximal,*
(2) *d is coarsely proper and (G,d) is large scale geodesic,*
(3) *d is quasi-isometric to the word metric ρ_A given by a coarsely bounded symmetric generating set $A \subseteq G$.*

PROOF. (2)⇒(1): Assume that d is coarsely proper and (G,d) is large scale geodesic with constant $K \geqslant 1$. Suppose ∂ is another continuous left-invariant écart on G. Since d is coarsely proper, the identity map from (G,d) to (G,∂) is bornologous. By Lemma 2.48, it follows that it is also Lipschitz for large distances, showing the maximality of d.

(1)⇒(3): Suppose d is maximal. We claim that G is generated by some closed ball $B_k = \{g \in G \mid d(g,1) \leqslant k\}$. Note that, if this fails, then G is the increasing union of the chain of proper open subgroups $V_n = \langle B_n\rangle$, $n \geqslant 1$. However, it is now easy, using Lemma 2.6, to construct an écart ∂ from the V_n contradicting the maximality of d. First, complementing with symmetric open sets $V_0 \supseteq V_{-1} \supseteq V_{-2} \supseteq \ldots \ni 1$ so that $V_{-n}^3 \subseteq V_{-n+1}$, and letting ∂ denote the continuous left-invariant écart obtained via Lemma 2.6 from $(V_n)_{n\in\mathbb{Z}}$, we see that, for all $g \in B_n \setminus V_{n-1} \subseteq V_n \setminus V_{n-1}$, we have

$$\partial(g,1) \geqslant 2^{n-1} \geqslant n \geqslant d(g,1).$$

Since $B_n \setminus V_{n-1} \neq \emptyset$ for infinitely many $n \geqslant 1$, this contradicts the maximality of d. We therefore conclude that $G = V_k = \langle B_k\rangle$ for some $k \geqslant 1$.

Let now ∂ denote the écart obtained from $V = B_k$ and d via Lemma 2.51. Then $d \leqslant \partial$ and, since d is maximal, we have $\partial \leqslant K\cdot d + C$ for some constants K, C, showing that d, ∂ and hence also the word metric ρ_{B_ϵ} are all quasi-isometric. As d is maximal, the generating set B_k is coarsely bounded in G.

(3)⇒(2): We note that the word metric ρ_A is simply the shortest path metric on the Cayley graph of G with respect to the symmetric generating set A, i.e., the graph whose vertex set is G and whose edges are $\{g, gs\}$, for $g \in G$ and $s \in A$. Thus, (G, ρ_A) is large scale geodesic and, since d is quasi-isometric to ρ_A, so is (G, d). It follows that every d-bounded set is ρ_A-bounded and hence included in some power A^n. It follows that d-bounded sets are coarsely bounded, showing that d is coarsely proper. □

THEOREM 2.53. *The following are equivalent for a European topological group G.*

(1) *G admits a continuous left-invariant maximal écart d ,*
(2) *G is generated by a coarsely bounded set,*
(3) *G is locally bounded and not the union of a countable chain of proper open subgroups,*
(4) *the coarse structure is monogenic.*

PROOF. The equivalence of (2), (3) and (4) has already been established in Theorem 2.30. Also, if these equivalent conditions hold, then G admits a continuous left-invariant coarsely proper écart d so that some open ball $V = \{x \in G \mid d(1, x) < k\}$ generates G. The écart ∂ defined from V and d by Lemma 2.51 is then maximal.

Conversely, if d is a maximal écart on G, then, by Proposition 2.52, G is generated by a coarsely bounded set. □

EXAMPLE 2.54. By Lemma 2.40 and Theorem 2.53, we see that every compatible left-invariant geodesic metric is maximal.

Observe that, by Theorem 2.30, if G is a European topological group generated by a coarsely bounded set, then G is also generated by an open coarsely bounded set. But, in a Polish group, we can also show that, if A and B are analytic coarsely bounded generating sets, then $A \subseteq B^n$ and $B \subseteq A^n$ for some sufficiently large n and thus the word metrics ρ_A and ρ_B are quasi-isometric. We recall here that a set in a Polish space is *analytic* if it is the continuous image of some other Polish space. E.g., Borel sets are analytic.

To verify the statement above, observe that $G = \bigcup_{n \geqslant 1} A^n$, so, as the A^n are also analytic and thus have the Baire property, we have that by Baire's Theorem some A^n must be somewhere comeagre. From Pettis' lemma [**57**] it follows that A^{2n} has non-empty interior, whereby $B \subseteq (A^{2n})^m = A^{2nm}$ for some m.

As the next example shows, this may fail if A and B are no longer assumed to be analytic. Nevertheless, there is a large number of Polish groups in which it is true, for example, in every group G having ample generics (cf. Lemma 6.15 [**40**]).

EXAMPLE 2.55. *$\mathbb{R}$ is generated by a coarsely bounded set D so that*

$$k \cdot D = \underbrace{D + \ldots + D}_{k \text{ times}}$$

has empty interior for all k. In particular, the word metric ρ_D is not quasi-isometric to the euclidean metric.

PROOF. Let us first show that there is a symmetric set $A \subseteq]-1, 1[$ generating $(\mathbb{Q}, +)$ and rationals $0 < r_k < 1$ so that $r_k \notin \underbrace{A + \ldots + A}_{k \text{ times}}$ for all $k \geqslant 1$.

The generating set will be of the form $A = \{0, \frac{1}{n_1!}, \frac{1}{n_2!}, \ldots, \}^{\pm}$, where $n_1 < n_2 < n_3 < \ldots$ is an inductively defined sequence of natural numbers. For this, observe that, if $n_1, \ldots n_k$ have been defined and we set $B = \{0, \frac{1}{n_1!}, \frac{1}{n_2!}, \ldots, \frac{1}{n_k!}\}^{\pm}$, then $\underbrace{B + \ldots + B}_{k \text{ times}}$ will be finite. So pick a rational number $0 < r_k < 1$ at some positive distance 2ϵ from $\underbrace{B + \ldots + B}_{k \text{ times}}$. Assume now that $n_{k+1} < n_{k+2} < \ldots$ are chosen so that $\frac{k}{n_{k+1}!} < \epsilon$. Then, letting $C = \{0, \frac{1}{n_{k+1}!}, \frac{1}{n_{k+2}!}, \ldots\}^{\pm}$, we find that

$$\begin{aligned} \underbrace{A + \ldots + A}_{k \text{ times}} &\subseteq \underbrace{B + \ldots + B}_{k \text{ times}} + \underbrace{C + \ldots + C}_{k \text{ times}} \\ &\subseteq \underbrace{B + \ldots + B}_{k \text{ times}} +]-\epsilon, \epsilon[, \end{aligned}$$

whence $r_k \notin \underbrace{A + \ldots + A}_{k \text{ times}}$. This shows how to inductively construct the points r_k, the sequence $n_1, n_2, \ldots$ and hence also the symmetric generating set $A = \{0, \frac{1}{n_1!}, \frac{1}{n_2!}, \ldots, \}^{\pm}$.

Pick now a Hamel basis $\mathcal{B} \subseteq]0, 1[$ for $\mathbb{R}$, i.e., a basis for $\mathbb{R}$ as a $\mathbb{Q}$-vector space and let $D = \bigcup_{s \in A} s \cdot \mathcal{B}$. Then D is a relatively compact symmetric generating set for $(\mathbb{R}, +)$. We claim that $\underbrace{D + \ldots + D}_{k \text{ times}}$ has empty interior for all k. Since D is symmetric, it suffices to show that it does not contain the interval $]0, 1[$. So note that, if $x \in \mathcal{B}$, then $r_k \cdot x$ is the unique representation of this number as a $\mathbb{Q}$-linear combination over the basis $\mathcal{B}$. Since $r_k \notin \underbrace{A + \ldots + A}_{k \text{ times}}$, this means that

$$r_k \cdot x \notin \underbrace{D + \ldots + D}_{k \text{ times}},$$

which concludes the example. □

Thus far, we have been able to, on the one hand, characterise the maximal écarts and, on the other, characterise the groups admitting these. However, oftentimes it will be useful to have other criteria that guarantee existence. In the context of finitely generated groups, the main such criterium is the Milnor–Schwarz Lemma [**52, 69**] of which we will have a close analogue.

THEOREM 2.56 (First Milnor–Schwarz Lemma). *Suppose $G \curvearrowright (X, d)$ is a coarsely proper, cobounded, continuous isometric action of a topological group on a connected metric space. Then G admits a maximal écart.*

PROOF. Since the action is cobounded, there is an open set $U \subseteq X$ of finite diameter so that $G \cdot U = X$. We let

$$V = \{g \in G \mid g \cdot U \cap U \neq \emptyset\}$$

and observe that V is an open identity neighbourhood in G. Since the action is coarsely proper, so is the orbit map $g \mapsto gx$ for any $x \in U$. But, as $\mathrm{diam}_d(V \cdot x) \leqslant 3 \cdot \mathrm{diam}_d(U) < \infty$, this shows that V is coarsely bounded in G.

To see that G admits a maximal écart, by Proposition 2.52, it now suffices to verify that G is generated by V. For this, observe that, if $g, f \in G$, then

$$\begin{aligned}\Big(g\langle V\rangle \cdot U\Big) \cap \Big(f\langle V\rangle \cdot U\Big) \neq \emptyset &\Rightarrow \Big(\langle V\rangle f^{-1}g\langle V\rangle \cdot U\Big) \cap U \neq \emptyset \\ &\Rightarrow \Big(\langle V\rangle f^{-1}g\langle V\rangle\Big) \cap V \neq \emptyset \\ &\Rightarrow f^{-1}g \in \langle V\rangle \\ &\Rightarrow g\langle V\rangle = f\langle V\rangle.\end{aligned}$$

Thus, distinct left cosets $g\langle V\rangle$ and $f\langle V\rangle$ give rise to disjoint open subsets $g\langle V\rangle \cdot U$ and $f\langle V\rangle \cdot U$ of X. However, $X = \bigcup_{g\in G} g\langle V\rangle \cdot U$ and X is connected, which implies that there can only be a single left coset of $\langle V\rangle$, i.e., $G = \langle V\rangle$. □

Theorem 2.57 (Second Milnor–Schwarz Lemma). *Suppose $G \curvearrowright (X,d)$ is a coarsely proper, cobounded, continuous isometric action of a topological group on a large scale geodesic metric space. Then G admits a maximal écart.*

Moreover, for every $x \in X$, the map

$$g \in G \mapsto gx \in X$$

is a quasi-isometry between G and (X,d).

Proof. Let $x \in X$ be given and set $\partial(g,f) = d(gx, fx)$, which defines a continuous left-invariant écart on G. Moreover, since the action is coarsely proper, so is ∂. Also, $g \mapsto gx$ is a cobounded isometric embedding of (G,∂) into (X,d), i.e., a quasi-isometry of (G,∂) with (X,d). Thus, as (X,d) is large scale geodesic, so is (G,∂), whence ∂ is maximal by Proposition 2.52. □

Example 2.58 (Power growth of elements). One of the main imports of the existence of a canonical quasimetric structure on a group is that it allows for a formulation of the growth rate of elements and discrete subgroups. On the other hand, for a non-trivial definition of the growth rate of the group itself, one must require something more, namely, *bounded geometry.* We return to this in Chapter 5.

So assume G is a topological group admitting a maximal écart d. Then the *power growth* of an element $g \in G$ is given by the function $\mathfrak{p}_d(n) = d(g^n, 1)$. Observe that, if ∂ is a different maximal écart, then the associated power growth is equivalent to that of d, in the sense that

$$\frac{1}{K}\mathfrak{p}_d(n) - K \leqslant \mathfrak{p}_\partial(n) \leqslant K \cdot \mathfrak{p}_d(n) + K$$

for some K and all n. In particular, up to this notion of equivalence, power growth is a conjugacy invariant of group elements.

CHAPTER 3

Structure theory

1. The Roelcke uniformity

A topological tool that will turn out to be fundamental in the study of coarse structure of Polish groups is the so-called Roelcke uniformity.

Definition 3.1. *The* Roelcke uniformity *on a topological group G is the meet $\mathcal{U}_L \wedge \mathcal{U}_R$ of the left and right uniformities on G. That is, it is the uniformity generated by the basic entourages*

$$E_V = \{(x, y) \in G \times G \mid y \in VxV\},$$

where V ranges over identity neighbourhoods in G.

When G is metrisable, one may pick a left-invariant compatible metric d on G, in which case, the metric

$$\begin{aligned} d_\wedge(x, y) &= \inf_{z \in G} d(x, z) + d(z^{-1}, y^{-1}) \\ &= \inf_{y = vxw} d(v, 1) + d(w, 1) \end{aligned}$$

is a compatible metric for the Roelcke uniformity.

Now, a subset A of G is said to be *Roelcke precompact* if it is relatively compact in the completion of G with respect to the Roelcke uniformity. This is equivalent to demanding that A is totally bounded or that, for every identity neighbourhood V there is a finite set $F \subseteq G$ so that $A \subseteq VFV$. Thus, a Roelcke precompact set is automatically coarsely bounded. Observe that, if $A \subseteq VFV$, then $\mathrm{cl}\, A \subseteq V^2FV^2$. Therefore, the family of Roelcke precompact sets is a subideal of the coarsely bounded sets, stable under taking topological closures, i.e., under $A \mapsto \mathrm{cl}\, A$.

Definition 3.2. *A topological group G is* Roelcke precompact *if if it precompact in the Roelcke uniformity and* locally Roelcke precompact *if it has a Roelcke precompact identity neighbourhood.*

Within the class of Polish groups, there is a useful characterisation of these as the automorphism groups of ω-categorical metric structures [**6, 63**]. More precisely, a Polish group G is Roelcke precompact if it is isomorphic to an *approximately oligomorphic* closed subgroup H of the isometry group $\mathrm{Isom}(X, d)$ of a separable complete metric space (X, d), meaning that, for every $n \geqslant 1$ and $\epsilon > 0$, there is a finite set $A \subseteq X^n$ so that $H \cdot A$ is ϵ-dense in X^n.

The class of Roelcke precompact Polish groups includes many familiar isometry groups of highly homogeneous metric structures, e.g., the infinite symmetric group S_∞, the unitary group $U(\mathcal{H})$ of separable Hilbert space with the strong operator topology, the group $\mathrm{Aut}([0, 1], \lambda)$ of measure-preserving automorphisms of the unit interval with the weak topology and the homeomorphism group of the unit interval $\mathrm{Homeo}([0, 1])$ [**61**] with the compact-open topology.

Obviously, every locally Roelcke precompact group is locally bounded and thus provides an important source of examples.

EXAMPLE 3.3 (Metrically homogeneous graphs). Every connected graph Γ is naturally a metric space when equipped with the shortest path metric ρ and automorphisms of Γ are then exactly the isometries of the metric space. The graph Γ is the said to be *metrically homogeneous* if any isometry $\phi\colon A \to B$ between two finite subsets of Γ extends to a surjective isometry $\tilde{\phi}\colon \Gamma \to \Gamma$. We observe that this is stronger than requiring Γ to be *vertex transitive*, i.e., that the automorphism group acts transitively on the set of vertices. A classification program of metrically homogeneous countable graphs is currently underway, see, e.g., the book by G. Cherlin [**15**].

PROPOSITION 3.4. *Let Γ be a countable metrically homogenous connected graph. Then the automorphism group $\mathrm{Aut}(\Gamma)$ is locally Roelcke precompact and, for any root $t \in \Gamma$, the mapping $g \in \mathrm{Aut}(\Gamma) \mapsto g(t) \in \Gamma$ is a quasi-isometry.*

PROOF. As Γ is a countable discrete object, the automorphism group $\mathrm{Aut}(\Gamma)$ is equipped with the permutation group topology which is obtained by declaring pointwise stabilisers $V_A = \{g \in \mathrm{Aut}(\Gamma) \mid g(a) = a \text{ for } a \in A\}$ of finite sets of vertices A to be open. This makes $\mathrm{Aut}(\Gamma)$ into a Polish group.

We claim that, for any root $a_0 \in \Gamma$ and any $k \geqslant 0$, the set

$$U = \{g \in \mathrm{Aut}(\Gamma) \mid \rho(g(a_0), a_0) \leqslant k\}$$

is Roelcke precompact. To see this, let V be an identity neighbourhood in $\mathrm{Aut}(\Gamma)$ and find vertices $a_1, \ldots, a_n \in \Gamma$ so that $V_{\{a_1,\ldots,a_n\}} \subseteq V$. Let $r = \mathrm{diam}_\rho(\{a_0, a_1, \ldots, a_n\})$ and note that there are only finitely many types of metric spaces of size $\leqslant 2n+2$ with integral distances at most $2r+k$. That means that there is a finite set $F \subseteq \mathrm{Aut}(\Gamma)$ so that, for any $g \in U$, there is $f \in F$ for which

$$\rho(a_i, g(a_j)) = \rho(a_i, f(a_j))$$

for all i, j. Observe that for such g, f the isometry

$$\phi\colon \{a_0, a_1, \ldots, a_n, g(a_1), \ldots, g(a_n)\} \to \{a_0, a_1, \ldots, a_n, f(a_1), \ldots, f(a_n)\}$$

defined by $\phi(a_i) = a_i$ and $\phi(g(a_i)) = f(a_i)$ extends to an element $h \in V_{\{a_1,\ldots,a_n\}} \subseteq \mathrm{Aut}(\Gamma)$. But $f^{-1}hg(a_i) = a_i$ for all i and so $f^{-1}hg \in V_{\{a_1,\ldots,a_n\}}$, i.e.,

$$g \in h^{-1}fV_{\{a_1,\ldots,a_n\}} \subseteq V_{\{a_1,\ldots,a_n\}}FV_{\{a_1,\ldots,a_n\}}.$$

So $U \subseteq V_{\{a_1,\ldots,a_n\}}FV_{\{a_1,\ldots,a_n\}}$, showing that U is Roelcke precompact.

Now, observe that every such U is an identity neighbourhood, so $\mathrm{Aut}(\Gamma)$ is locally Roelcke precompact. Also, as Roelcke precompact sets are coarsely bounded, this shows that the transitive isometric action of $\mathrm{Aut}(\Gamma)$ on the geodesic metric space (Γ, ρ) is coarsely proper. So, by the second Milnor–Schwarz Lemma, Theorem 2.57, the orbit mapping

$$g \in \mathrm{Aut}(\Gamma) \mapsto g(a_0) \in \Gamma$$

is a quasi-isometry. □

Basic examples of metrically homogeneous graphs include the n-regular trees T_n for $n = 2, 3, \ldots, \infty$ and the integral Urysohn metric space $\mathbb{Z}\mathbb{U}$, i.e., the Fraïssé limit of the class of all finite metric spaces with integral distances (where $\mathbb{Z}\mathbb{U}$ is given the edge relation of having distance 1). It thus follows that, for $t \in T_n$ and $x \in \mathbb{Z}\mathbb{U}$, the mappings

$$g \in \mathrm{Aut}(T_n) \mapsto g(t) \in T_n$$

and

$$g \in \mathrm{Isom}(\mathbb{Z}\mathbb{U}) \mapsto g(x) \in \mathbb{Z}\mathbb{U}$$

are quasi-isometries.

EXAMPLE 3.5 (Finite asymptotic dimension). A metric space (X, d) is said to have *finite asymptotic dimension at most k* if, for every R, there are families $\mathcal{U}_0, \ldots, \mathcal{U}_k$ of subsets of X so that, for all i,

(1) $\sup_{U \in \mathcal{U}_i} \mathrm{diam}_d(U) < \infty$,
(2) $\mathrm{dist}(U, U') > R$ for distinct $U, U' \in \mathcal{U}_i$,
(3) $X = \bigcup \mathcal{U}_0 \cup \ldots \cup \bigcup \mathcal{U}_k$.

For example, if T_n denotes the n-regular tree, $n = 2, 3, \ldots, \infty$, then T_n has finite asymptotic dimension 1 (see, e.g., Proposition 2.3.1 [**55**]). Now, finite asymptotic dimension k is a coarse invariant of metric spaces, so we conclude that $\mathrm{Aut}(T_\infty)$ is a non-locally compact Polish group of finite asymptotic dimension 1.

Finite asymptotic dimension may also be defined for arbitrary coarse spaces [**60**], but the Polish groups with finite asymptotic dimension that we are aware of are all locally bounded and thus have metrisable coarse structure. Moreover, the results of Section 5 indicate that this may not be by accident.

PROBLEM 3.6. *Let G be a Polish group of finite asymptotic dimension. Is G necessarily locally bounded?*

EXAMPLE 3.7 (The Urysohn space). The Urysohn space is a separable complete metric space $\mathbb{U}$ satisfying the following extension property. For every finite metric space X and subspace $Y \subseteq X$, every isometric embedding $f : Y \to \mathbb{U}$ extends to an isometric embedding $\tilde{f} : X \to \mathbb{U}$. By a result of P. Urysohn [**74**], these properties completely determine $\mathbb{U}$ up to isometry. Moreover, by a back-and-forth argument, one sees that $\mathbb{U}$ is homogeneous in the sense that every isometry between finite subsets of $\mathbb{U}$ extends to a surjective isometry from $\mathbb{U}$ onto itself. We let $\mathrm{Isom}(\mathbb{U})$ be the group of all isometries of $\mathbb{U}$ equipped with the topology of pointwise convergence on $\mathbb{U}$, that is, $g_i \to g$ in $\mathrm{Isom}(\mathbb{U})$ if and only if $d(g_i x, gx) \to 0$ for all $x \in \mathbb{U}$. With this topology, $\mathrm{Isom}(\mathbb{U})$ is Polish.

PROPOSITION 3.8. $\mathrm{Isom}(\mathbb{U})$ *is locally Roelcke precompact and, for any $x \in \mathbb{U}$, the map*

$$g \in \mathrm{Isom}(\mathbb{U}) \mapsto gx \in \mathbb{U}$$

is a quasi-isometry between $\mathrm{Isom}(\mathbb{U})$ *and* $\mathbb{U}$.

PROOF. Fix $x \in \mathbb{U}$ and $\alpha < \infty$. We claim that the set

$$U = \{g \in \mathrm{Isom}(\mathbb{U}) \mid d(g(x), x) < \alpha\}$$

is Roelcke precompact in $\mathrm{Isom}(\mathbb{U})$.

To see this, suppose V is any identity neighbourhood. By shrinking V, we may suppose that

$$V = \big\{g \in \mathrm{Isom}(\mathbb{U}) \bigm| d(g(y), y) < \epsilon,\ \forall y \in A\big\},$$

where $A \subseteq \mathbb{U}$ is finite, $x \in A$ and $\epsilon > 0$.

Choose now finitely many $f_1, \ldots, f_n \in U$ so that, for every other $g \in U$, there is $k \leqslant n$ with

$$\big|d\big(y, g(y)\big) - d\big(y, f_k(y)\big)\big| < \epsilon$$

for all $y \in A$. That this is possible follows from simple compactness considerations, see, e.g., Section 5 [**62**]. We claim that $U \subseteq V\{f_1, \ldots, f_n\}V$, showing that $\mathrm{Isom}(\mathbb{U})$ is locally Roelcke precompact.

To see this, suppose $g \in U$ and pick f_k is as above. By the extension property of $\mathbb{U}$, there is some isometry h pointwise fixing A so that

$$d(g(y), hf_k(y)) < \epsilon$$

for all $y \in A$. In particular, $h \in V$ and $f_k^{-1}h^{-1}g \in V$, whence $g \in hf_kV \subseteq Vf_kV$.

By the claim, we see that the orbit map $g \in \mathrm{Isom}(\mathbb{U}) \mapsto g(x) \in \mathbb{U}$ is coarsely proper, whence the tautological action of $\mathrm{Isom}(\mathbb{U})$ on $\mathbb{U}$ is coarsely proper. As the action is also transitive and $\mathbb{U}$ is geodesic, we conclude, by the Milnor–Schwarz Lemma, Theorem 2.57, that the orbit map is a quasi-isometry between $\mathrm{Isom}(\mathbb{U})$ and $\mathbb{U}$. $\square$

In the above two examples, one may observe that the orbit mappings are a bit more than only coarsely proper, namely that the inverse image of a set of bounded diameter is actually Roelcke precompact. This is not entirely by chance, since a J. Zielinski [**83**] has shown that, in a locally Roelcke precompact group, the Roelcke precompact sets coincide with the coarsely bounded sets. Thus, in locally Roelcke precompact groups, we have an especially tight connection between the coarse geometry and the Roelcke uniformity.

Some other very interesting examples of locally Roelcke precompact groups are $\mathrm{Aut}_{\mathbb{Z}}(\mathbb{Q})$ and $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$, the groups of order-preserving bijections of $\mathbb{Q}$, respectively, order-preserving homeomorphisms of $\mathbb{R}$ commuting with integral shifts. We return to these in Chapter 5.

2. Examples of Polish groups

In the listing of the geometry of various groups, the first to be mentioned are the geometrically trivial examples, i.e., those quasi-isometric to a single point space.

DEFINITION 3.9. *A topological group G is said to be* coarsely bounded[1] *if it is coarsely bounded in itself, i.e., if every continuous left-invariant écart on G is bounded.*

Whereas coarsely bounded topological groups may not be very small in a topological sense, they may be viewed as those that are "geometrically compact" and indeed contain the compact groups as a subclass. Note also that a European group G is coarsely bounded if and only if, for every identity neighbourhood V, there is a finite set F and a k so that $G = (FV)^k$.

Any compact group is Roelcke precompact and any Roelcke precompact group is coarsely bounded, but there are many sources of coarsely bounded groups beyond Roelcke precompactness.

[1]We note that these are exactly the groups that have *property (OB)* in the language of [**62**].

EXAMPLE 3.10 (Homeomorphism groups of spheres). As shown by M. Culler and the author in [**63**], homeomorphism groups of compact manifolds of dimension $\geqslant 2$ are never locally Roelcke precompact, since Dehn twists of different orders can be shown to be well separated in the Roelcke uniformity[2]. Similarly, the homeomorphism group $\mathrm{Homeo}([0,1]^{\mathbb{N}})$ of the Hilbert cube fails to be Roelcke precompact. However, as shown in [**62**], for all dimensions $n \geqslant 1$, the homeomorphism group $\mathrm{Homeo}(\mathbb{S}^n)$ of the n-sphere is coarsely bounded and the same holds for $\mathrm{Homeo}([0,1]^{\mathbb{N}})$.

Beyond the geometrically trivial groups, we can identify the coarse or quasimetric geometric geometry of many other groups. First of all, the left-coarse structure $\mathcal{E}_L$ of a countable discrete group is that given by any left-invariant proper metric, i.e., whose balls are finite. In the case of a finitely generated group Γ, we see by Proposition 2.52 that the word metric ρ_S induced by a finite symmetric generating set $S \subseteq \Gamma$ is maximal, whence the quasi-isometry type of Γ is the usual one. The same holds true for a compactly generated locally compact group, i.e., its quasi-isometry type is given by the word metric of a compact generating set.

EXAMPLE 3.11 (The additive group of a Banach space). Consider again the additive group $(X,+)$ of a Banach space $(X, \|\cdot\|)$. Since the norm metric on X is geodesic, it follows from Example 2.54 that it is maximal. In other words, the quasi-isometry type of the topological group $(X,+)$ is none other than the quasi-isometry type of $(X, \|\cdot\|)$ itself.

EXAMPLE 3.12 (Groups of affine isometries). Assume $(X, \|\cdot\|)$ is a Banach space and let $\mathrm{Aff}(X)$ be the group of all *affine* surjective isometries $f\colon X \to X$, which, by the Mazur–Ulam Theorem, coincides with the group of all surjective isometries of X. Thus, every element $f\colon X \to X$ splits into a linear isometry $\pi(f)\colon X \to X$ and a vector $b(f) \in X$ so that $f(y) = \pi(f)(y) + b(f)$ for all $y \in X$.

We equip $\mathrm{Aff}(X)$ with the topology of pointwise convergence on X, i.e., $g_i \to g$ if and only if $\|g_i(x) - g(x)\| \to 0$ for all $x \in X$, and let $\mathrm{Isom}(X)$ be the closed subgroup consisting of linear isometries. So the induced topology on $\mathrm{Isom}(X)$ is simply the strong operator topology. Also, $(X,+)$ may be identified with the closed group of translations in $\mathrm{Aff}(X)$. Then $\mathrm{Aff}(X)$ may be written as a topological semidirect product

$$\mathrm{Aff}(X) = \mathrm{Isom}(X) \ltimes (X,+)$$

for the natural action of $\mathrm{Isom}(X)$ on X and so is Polish provided X is separable.

PROPOSITION 3.13. *Let $(X, \|\cdot\|)$ be a Banach space. Then the transitive isometric action $\mathrm{Aff}(X) \curvearrowright X$ is coarsely proper if and only if $\mathrm{Isom}(X)$ is a coarsely bounded group. In this case, the cocycle*

$$b\colon \mathrm{Aff}(X) \to X$$

is a quasi-isometry between $\mathrm{Aff}(X)$ and $(X, \|\cdot\|)$.

PROOF. Observe first that the cocycle $b\colon \mathrm{Aff}(X) \to X$ is simply the orbit map

$$f \in \mathrm{Aff}(X) \mapsto f(0) \in X.$$

So the action of $\mathrm{Aff}(X)$ on X is coarsely proper if and only if b is coarsely proper.

[2]The result in [**63**] only states that the groups are not Roelcke precompact, but the proof implicitly treats local Roelcke precompactness too.

Now, $b(f) = 0$ for all $f \in \mathrm{Isom}(X)$. So, if b is coarsely proper, then $\mathrm{Isom}(X)$ must be coarsely bounded in $\mathrm{Aff}(X)$. However, $\mathrm{Isom}(X)$ is the quotient of $\mathrm{Aff}(X)$ by the normal subgroup $(X, +)$, whence $\mathrm{Isom}(X)$ is the continuous homomorphic image of a coarsely bounded set and thus must be coarsely bounded in itself.

Conversely, assume that $\mathrm{Isom}(X)$ is a coarsely bounded group. We must show that, for any α, the set $U = \{f \in \mathrm{Aff}(X) \mid \|b(f)\| \leqslant \alpha\}$ is coarsely bounded. But, for any $f \in \mathrm{Aff}(X)$, $f = \tau_{b(f)} \circ \pi(f)$, where $\tau_{b(f)} \in \mathrm{Aff}(X)$ denotes the translation by $b(f)$. Therefore $U \subseteq B_\alpha \cdot \mathrm{Isom}(X)$, where B_α is the ball of radius α in X, which is coarsely bounded in $(X, +)$ and thus in $\mathrm{Aff}(X)$.

That b is a quasi-isometry follows immediately from the second Milnor–Schwarz Lemma, Theorem 2.57. □

Since, as noted above, the unitary group $U(\mathcal{H})$ of separable infinite-dimensional Hilbert space with the strong operator topology is Roelcke precompact and thus coarsely bounded, we see that the group of affine isometries of $\mathcal{H}$ is quasi-isometric to $\mathcal{H}$ itself.

Also, S. Banach described the linear isometry groups of ℓ^p, $1 < p < \infty$, $p \neq 2$, as consisting entirely of sign changes and permutations of the basis elements. Thus, the isometry group is the semidirect product $S_\infty \ltimes \{-1, 1\}^{\mathbb{N}}$ of the Roelcke precompact group S_∞ and a compact group and hence is Roelcke precompact itself. Therefore, the affine isometry group $\mathrm{Aff}(\ell^p)$ is quasi-isometric to ℓ^p.

By results due to C. W. Henson [**5**], the L^p-lattice $L^p([0, 1], \lambda)$, with λ being Lebesgue measure and $1 < p < \infty$, is ω-categorical in the sense of model theory for metric structures. This also implies that the Banach space reduct $L^p([0, 1], \lambda)$ is ω-categorical and hence that the action by its isometry group on the unit ball is approximately oligomorphic. By Theorem 5.2 [**62**], it follows that the isometry group $\mathrm{Isom}(L^p)$ is coarsely bounded and thus, as before, that the affine isometry group $\mathrm{Aff}(L^p)$ is quasi-isometric to L^p.

Now, by results of W. B. Johnson, J. Lindenstrauss and G. Schechtman [**33**] (see also Theorem 10.21 [**8**]), any Banach space quasi-isometric to ℓ^p for $1 < p < \infty$ is, in fact, linearly isomorphic to ℓ^p. Also, for $1 < p < q < \infty$, the spaces L^p and L^q are not coarsely equivalent since they then would be quasi-isometric (being geodesic spaces) and, by taking ultrapowers, would be Lipschitz equivalent, contradicting Corollary 7.8 [**8**].

Thus, it follows that all of $\mathrm{Aff}(\ell^p)$ and $\mathrm{Aff}(L^p)$ for $1 < p < \infty$, $p \neq 2$, have distinct quasi-isometry types and, in particular, cannot be isomorphic as topological groups.

Example 3.14. Consider again the infinite symmetric group S_∞ of all permutations of $\mathbb{N}$ and let $F \leqslant S_\infty$ be the normal subgroup of finitely supported permutations. Viewing F as a countable discrete group, we may define a continous action by automorphisms

$$S_\infty \curvearrowright F$$

simply by setting $\alpha.f = \alpha f \alpha^{-1}$ for $f \in F$ and $\alpha \in S_\infty$. Let $S_\infty \ltimes F$ be the corresponding topological semidirect product, which is a Polish group. Thus, elements of $S_\infty \ltimes F$ may be represented uniquely as products $f \cdot \alpha$, where $f \in F$ and $\alpha \in S_\infty$. Moreover, $\alpha \cdot f = f^\alpha \cdot \alpha$, where $f^\alpha \in F$ is the conjugation of f by α.

Letting (nm) denote the transposition switching n and m and noting that S_∞ is coarsely bounded, we see that $B = \{(12) \cdot \alpha \mid \alpha \in S_\infty\}$ is coarsely bounded in

$S_\infty \ltimes F$. Noting then that

$$(12)\cdot(12)\alpha\cdot(12)\alpha^{-1}\beta = \big(\alpha(1)\ \alpha(2)\big)\beta,$$

we see that B^3 contains the set A of all products $(nm)\beta$ for $n \neq m$ and $\beta \in S_\infty$. Moreover, since every finitely supported permutation may be written as a product of transpositions, we find that the coarsely bounded set A generates $S_\infty \ltimes F$.

We claim that $S_\infty \ltimes F$ is quasi-isometric to F with the metric

$$d(f,g) = \min(k \mid g^{-1}f \text{ can be written as a product of } k \text{ transpositions}).$$

Indeed, since $S_\infty \ltimes F$ may be written as the product FS_∞, we find that F is cobounded in $S_\infty \ltimes F$. Moreover, for $f \in F$ and $\alpha_i \in S_\infty$,

$$f_1\alpha_1 f_2\alpha_2\cdots f_n\alpha_n = f_1 f_2^{\alpha_1} f_3^{\alpha_1\alpha_2}\cdots f_n^{\alpha_1\alpha_2\cdots\alpha_{n-1}}\alpha_1\alpha_2\cdots\alpha_n.$$

So, if the f_i are all transpositions, then $f_1\alpha_1 f_2\alpha_2\cdots f_n\alpha_n = g\beta$, where $g \in F$ is a product of n transpositions and $\beta \in S_\infty$. It follows that, up to an additive error of 1, we have that the A-word length of a product $g\beta$, $g \in F$ and $\beta \in S_\infty$, equals

$$\min(k \mid g \text{ can be written as a product of } k \text{ transpositions}).$$

Now, for $f, g \in F$ and $\alpha, \beta \in S_\infty$,

$$\rho_A(f\alpha, g\beta) = \rho_A(\beta^{-1}g^{-1}f\alpha, 1) = \rho_A\big((g^{-1}f)^{\beta^{-1}}\beta^{-1}\alpha, 1).$$

So since the minimal number of transpositions with product $(g^{-1}f)^{\beta^{-1}}$ equals that for $g^{-1}f$, we find that

$$\big|\rho_A(f\alpha, g\beta) - d(f,g)\big| \leqslant 1.$$

In other words, $S_\infty \ltimes F$ is quasi-isometric to (F, d).

EXAMPLE 3.15 (The fragmentation norm on homeomorphism groups). Let M be a closed manifold and $\mathrm{Homeo}_0(M)$ the identity component of its homeomorphism group $\mathrm{Homeo}(M)$ with the compact-open topology. That is, $\mathrm{Homeo}_0(M)$ is the group of isotopically trivial homeomorphisms of M. In [**46**], it is shown that $\mathrm{Homeo}_0(M)$ is a Polish group generated by a coarsely bounded set and thus has a well-defined quasi-isometry type. Moreover, in contradistinction to the case of spheres, it is also shown that this quasi-isometry type is highly non-trivial once the fundamental group $\pi_1(M)$ has an element of infinite order.

The fact that $\mathrm{Homeo}_0(M)$ is generated by a coarsely bounded set relies on the Fragmentation Lemma of R. D. Edwards and R. C. Kirby [**22**], which states that, if $\mathcal{U} = \{U_1, \ldots, U_n\}$ is an open cover of M, there is an identity neighbourhood V in $\mathrm{Homeo}_0(M)$ so that every $g \in V$ can be factored into $g = h_1\cdots h_n$ with $\mathrm{supp}(h_i) \subseteq U_i$. It follows that we may define a fragmentation norm on $\mathrm{Homeo}_0(M)$ by letting

$$\|g\|_{\mathcal{U}} = \min(k \mid g = h_1\cdots h_k \ \&\ \forall i\ \exists j\ \mathrm{supp}(h_i) \subseteq U_j).$$

As is observed in [**46**], provided the cover $\mathcal{U}$ is sufficiently fine, the identity neighbourhood V is coarsely bounded and thus the maximal metric on $\mathrm{Homeo}_0(M)$ is quasi-isometric to the metric induced by the fragmentation norm $\|\cdot\|_{\mathcal{U}}$.

In previous work, E. Militon [**50**] was able to take this even further for the case of compact surfaces by identifying the fragmentation norm $\|\cdot\|_{\mathcal{U}}$ with a metric of maximal displacement on the universal cover of M.

EXAMPLE 3.16 (Diffeomorphism groups). In [**16**], M. Cohen considers the diffeomorphism groups $\mathrm{Diff}^k_+(M)$ for $1 \leqslant k \leqslant \infty$ of the one dimensional manifolds $M = \mathbb{S}^1$ and $M = [0,1]$. In particular, he shows that a subset $A \subseteq \mathrm{Diff}^k_+(M)$ is coarsely bounded if and only if

$$\sup_{f\in A}\sup_{x\in M}\big|\log f'(x)\big| < \infty \quad \text{and} \quad \sup_{f\in A}\sup_{x\in M}\big|f^{(i)}(x)\big| < \infty$$

for all integers $2 \leqslant i \leqslant k$. If follows that, for $1 \leqslant k < \infty$, the group $\mathrm{Diff}^k_+(M)$ is generated by a coarsely bounded set and, in fact, is quasi-isometric to the Banach space $C([0,1])$.

EXAMPLE 3.17. P. J. Cameron and A. M. Vershik [**13**] have shown that there is an invariant metric d on the group $\mathbb{Z}$ for which the metric space $(\mathbb{Z}, d)$ is isometric to the rational Urysohn metric space $\mathbb{QU}$. Since d is two-sided invariant, the topology τ it induces on $\mathbb{Z}$ is necessarily a group topology, i.e., the group operations are continuous. Thus, $(\mathbb{Z},\tau)$ is a metrisable topological group and we claim that $(\mathbb{Z},\tau)$ has a well-defined quasi-isometry type, namely, the Urysohn metric space $\mathbb{U}$ or, equivalently, $\mathbb{QU}$.

To see this, we first verify that d is coarsely proper on $(\mathbb{Z},\tau)$. For this, note that, since $(\mathbb{Z},\tau)$ is isometric to $\mathbb{QU}$, we have that, for all $n,m \in \mathbb{Z}$ and $\epsilon > 0$, if $r = \lceil \frac{d(n,m)}{\epsilon} \rceil$, then there are $k_0 = n, k_1, k_2, \ldots, k_r = m \in \mathbb{Z}$ so that $d(k_{i-1}, k_i) \leqslant \epsilon$. Thus, as r is a function only of ϵ and of the distance $d(n,m)$, we see that d satisfies the criteria in Example 2.40 and hence is coarsely proper on $(\mathbb{Z},\tau)$. Also, as $\mathbb{QU}$ is large scale geodesic, so is $(\mathbb{Z}, d)$. It follows that the shift action of the topological group $(\mathbb{Z},\tau)$ on $(\mathbb{Z}, d)$ is a coarsely proper transitive action on a large scale geodesic space. So, by the Milnor–Schwarz Lemma, Theorem 2.57, the identity map is a quasi-isometry between the topological group $(\mathbb{Z},\tau)$ and the metric space $(\mathbb{Z}, d)$. As the latter is quasi-isometric to $\mathbb{QU}$, so is $(\mathbb{Z},\tau)$.

By taking the completion of $(\mathbb{Z},\tau)$, this also provides us with monothetic Polish groups quasi-isometric to the Urysohn space $\mathbb{U}$.

3. Rigidity of categories

In our study we have considered topological groups in four different categories, namely, as uniform and coarse spaces and then in the restrictive subcategory of metrisable coarse spaces and finally quasimetric spaces. Each of these come with appropriate notions of morphisms, i.e., uniformly continuous and bornologous maps, while the latter also allow for the finer concept of Lipschitz for large distance maps.

We have already seen some relation between the uniform and coarse structures in that a European group has a metrisable coarse structure if and only if it is locally bounded or, equivalently, if $\mathcal{E}_L \cap \mathcal{U}_L \neq \emptyset$. Not surprisingly, also at the level of morphisms there is a connection.

PROPOSITION 3.18. *Suppose $\phi\colon G \to (X,d)$ is a uniformly continuous map from a topological group G to a metric space (X,d) and assume that G has no proper open subgroups. Then ϕ is bornologous.*

PROOF. Since ϕ is uniformly continuous, there is an identity neighbourhood $V \subseteq G$ so that $d(\phi x, \phi y) < 1$, whenever $x^{-1}y \in V$. So, if $x^{-1}y \in V^n$, write

$y = xv_1 \cdots v_n$, for $v_i \in V$, and note that

$$\begin{aligned} d(\phi x, \phi y) &\leqslant d\big(\phi x, \phi(xv_1)\big) + d\big(\phi(xv_1), \phi(xv_1v_2)\big) + \ldots + d\big(\phi(xv_1 \cdots v_{n-1}), \phi y\big) \\ &< n. \end{aligned}$$

On the other hand, since G has no proper open subgroups, then, if E is a coarse entourage on G, there is an n so that $x^{-1}y \in V^n$ for all $(x, y) \in E$. It follows that $d(\phi x, \phi y) < n$ for all $(x, y) \in E$, showing that ϕ is bornologous. □

PROPOSITION 3.19. *Suppose $\phi\colon G \to H$ is a uniformly continuous map from a topological group G without proper open subgroups to a topological group H. Then ϕ is bornologous.*

PROOF. To see that ϕ is bornologous, let $E \subseteq G \times G$ be a coarse entourage. In order to verify that $(\phi \times \phi)E$ is a coarse entourage on H, it suffices to show that, for every identity neighbourhood W in H, there is an n so that $(\phi \times \phi)E \subseteq E_{W^n}$. So let W be given and choose by uniform continuity some identity neighbourhood V so that $(\phi \times \phi)E_V \subseteq E_W$. Now, as G has no proper open subgroup, there is n so that $E \subseteq E_{V^n}$, whence

$$(\phi \times \phi)E \;\subseteq\; (\phi \times \phi)E_{V^n} \;=\; (\phi \times \phi)E_V^n \;\subseteq\; E_W^n \;=\; E_{W^n}$$

as required. □

PROPOSITION 3.20. *Suppose $\phi\colon G \to H$ is a bornologous cobounded map between European topological groups. Then, if G is locally bounded, so is H. Similarly, if G is generated by a coarsely bounded set, so is H.*

PROOF. As ϕ is cobounded, there is a coarsely bounded set $B \subseteq H$ so that $H = \phi[G] \cdot B$.

Assume first that G is locally bounded. Then G admits a countable covering by open sets U_n coarsely bounded in G, whence $H = \phi[G] \cdot B = \bigcup_n \overline{\phi[U_n] \cdot B}$. Now, as ϕ is bornologous, the $\phi[U_n]$ are coarsely bounded, whence the same is true of the $\overline{\phi[U_n] \cdot B}$. By the Baire Category Theorem, it follows that some $\overline{\phi[U_n] \cdot B}$ has non-empty interior, showing that H is locally bounded.

Assume now that G is generated by a symmetric coarsely bounded set $A \subseteq G$ with $1 \in A$ and hence that the E_{A^n} are cofinal in the coarse structure on G. Since ϕ is bornologous, there is a coarsely bounded set $C \subseteq H$ so that $(\phi \times \phi)E_A \subseteq E_C$ and thus that $(\phi \times \phi)E_{A^n} \subseteq E_{C^n}$ for all n. As $\{1\} \times A^n \subseteq E_{A^n}$, we have that $\{\phi(1)\} \times \phi[A^n] \subseteq E_{C^n}$, whereby $\phi[A^n] \subseteq \phi(1)C^n$. Thus, since $G = \bigcup_n A^n$, we find that

$$H = \phi[G] \cdot B = \bigcup_n \phi[A^n] \cdot B = \bigcup_n \phi(1)C^n B,$$

showing that the coarsely bounded set $\{\phi(1)\} \cup C \cup B$ generates H. □

A complication in the above proposition is that even a dense subgroup of a Polish group need not be cobounded. For example, the countable group $\bigoplus_{n\in\mathbb{N}} \mathbb{Z}$ is a dense subgroup of the Polish group $\prod_{n\in\mathbb{N}} \mathbb{Z}$, but fails to be cobounded. For any coarsely bounded set $B \subseteq \prod_{n\in\mathbb{N}} \mathbb{Z}$ is contained in some product $F_1 \times F_2 \times \ldots$ of finite sets (see Proposition 3.30), whence $\big(\bigoplus_{n\in\mathbb{N}} \mathbb{Z}\big) \cdot B$ is a proper subset of $\prod_{n\in\mathbb{N}} \mathbb{Z}$.

PROPOSITION 3.21. *Let $\phi\colon G \to H$ be a bornologous map between European groups generated by coarsely bounded sets. Then ϕ is Lipschitz for large distances.*

Proof. By Theorem 2.53 and Proposition 2.52, European groups generated by coarsely bounded sets have a well-defined quasimetric structure in which they are large scale geodesic. The proposition therefore follows from an application of Lemma 2.48. □

Corollary 3.22. *Among European groups, the properties of being locally bounded and of being generated by a coarsely bounded set are both coarse invariants, i.e., are preserved under coarse equivalence.*

Moreover, every coarse equivalence between European groups generated by coarsely bounded sets is automatically a quasi-isometry.

It is natural to wonder whether the coarse properties of a map can be matched by topological properties. Clearly, a bornologous map cannot in general be approximated by a continuous bornologous map, e.g., the integral part map $\lfloor\cdot\rfloor\colon \mathbb{R}\to\mathbb{Z}$ is a coarse equivalence, while every continuous map $\mathbb{R}\to\mathbb{Z}$ is constant. Nevertheless, measurability can be attained.

Recall first that a subset A of a topological space X is said to have the *Baire property* in X if there is an open set $V\subseteq X$ so that $A\triangle V$ is meagre, i.e., $A\triangle V$ is the union of countable many nowhere dense sets. Similarly, A is a *C-set* if it belongs to the smallest σ-algebra in $\mathcal{P}(X)$ closed under the Souslin operation $\mathcal{A}$ (see Section 29.D [**38**]). Every Borel set is a C-set and, by a theorem of O. M. Nikodým (see (29.14) [**38**]), every C-set has the Baire property.

Moreover, a map $\phi\colon X\to Y$ between topological spaces is said to be *Baire measurable* if the inverse image $\phi^{-1}(U)$ of every open set $U\subseteq Y$ has the property of Baire in X. Similarly for C and Borel measurable. By the above, we have the following implications among functions between topological spaces.

$$\text{Borel measurable} \;\Rightarrow\; C\text{-measurable} \;\Rightarrow\; \text{Baire measurable}.$$

However, in contradistinction to Baire measurable functions, the composition of two C-measurable functions is again C-measurable, which makes it an appropriate class of functions to work with.

In the next lemma, by $\mathcal{G}\phi$, we denote the graph of a function ϕ.

Lemma 3.23. *Suppose $\phi,\psi\colon H\to G$ are maps between metrisable topological groups so that $\mathcal{G}\psi\subseteq\overline{\mathcal{G}\phi}$. Then the following holds.*

(1) *If ϕ is modest, so is ψ.*
(2) *If ϕ is bornologous, then ψ is close to ϕ and thus is bornologous too.*

Proof. Note first that $\mathcal{G}\psi\subseteq\overline{\mathcal{G}\phi}$ is equivalent to the condition

$$\psi(x)\in\bigcup_{\substack{U\ni x\\ \text{open}}}\overline{\phi[U]}\qquad\text{for all } x\in H.$$

For (1), suppose for a contradiction that $A\subseteq H$ is coarsely bounded and that $\psi[A]$ fails to be coarsely bounded in G. Then there is a continuous left-invariant écart d on G and a sequence $x_n\in A$ so that $d(\psi(x_n),\psi(x_1))\underset{n\to\infty}{\longrightarrow}\infty$. By the condition above, we can then find $z_n\in H$ with $z_n\underset{n\to\infty}{\longrightarrow}1$ so that also

$$d\big(\psi(x_n),\phi(x_nz_n)\big)\underset{n\to\infty}{\longrightarrow}0,$$

whereby

$$d\big(\phi(x_nz_n),\phi(x_1z_1)\big)\underset{n\to\infty}{\longrightarrow}\infty.$$

However, $\overline{\{z_n\}_n}$ is compact and thus coarsely bounded, whereby $\{x_nz_n\}_n \subseteq A \cdot \overline{\{z_n\}_n}$ are coarsely bounded in H. However, then $\{\phi(x_nz_n)\}_n$ will be coarsely bounded in G, contradicting the above.

For (2), assume instead for a contradiction that ψ is not close to ϕ. Then there is a continuous left-invariant écart d on G and a sequence $x_n \in H$ so that $d(\psi(x_n), \phi(x_n)) \underset{n\to\infty}{\longrightarrow} \infty$. As before, we find $z_n \in H$ with $z_n \underset{n\to\infty}{\longrightarrow} 1$ so that

$$d\big(\psi(x_n), \phi(x_nz_n)\big) \underset{n\to\infty}{\longrightarrow} 0,$$

whence

$$d\big(\phi(x_nz_n), \phi(x_n)\big) \underset{n\to\infty}{\longrightarrow} \infty.$$

However, as $x_n^{-1}x_nz_n = z_n$ belongs to the compact and thus coarsely bounded set $\overline{\{z_n\}_n}$, this contradicts that ϕ is bornologous. $\square$

Proposition 3.24. *Every bornologous map $\phi\colon H \to G$ between Polish groups is close to a C-measurable bornologous map ψ with $\mathcal{G}\psi \subseteq \overline{\mathcal{G}\phi}$.*

Similarly, if instead $\phi\colon H \to G$ is modest, then there is a C-measurable modest map ψ with $\mathcal{G}\psi \subseteq \overline{\mathcal{G}\phi}$.

Proof. Observe that $\overline{\mathcal{G}\phi}$ is a closed subset of the Polish space $H \times G$ all of whose vertical sections

$$\big(\overline{\mathcal{G}\phi}\big)_x = \{g \in G \mid (x, g) \in \overline{\mathcal{G}\phi}\}$$

are non-empty. So by the Jankov–von Neumann selection theorem (see (18.1) [**38**]) there is a C-measurable selector for $\overline{\mathcal{G}\phi}$, i.e., a map $H \xrightarrow{\psi} G$ with $\mathcal{G}\psi \subseteq \overline{\mathcal{G}\phi}$. Then properties of ψ now follow from Lemma 3.23. $\square$

The assumption that $\mathcal{G}\psi \subseteq \overline{\mathcal{G}\phi}$ is can be useful in different contexts. For example if ϕ is a section for a continuous epimorphism $G \xrightarrow{\pi} H$, then the graph $\mathcal{G}\pi$ is closed and so $\mathcal{G}\psi \subseteq \overline{\mathcal{G}\phi} \subseteq \big(\mathcal{G}\pi\big)^{-1}$. It follows that also ψ is a section for π.

4. Comparison of left and right-coarse structures

Whereas hitherto we have only studied the left-invariant coarse structure $\mathcal{E}_L$ generated by the ideal $\mathcal{OB}$ of coarsely bounded sets in a topological group G, one may equally well consider the right-coarse structure $\mathcal{E}_R$ generated by the entourages

$$F_A = \{(x, y) \in G \times G \mid xy^{-1} \in A\},$$

where A varies over coarsely bounded subsets of G. Of course, since the inversion map $\texttt{inv}\colon x \mapsto x^{-1}$ is seen to be a coarse equivalence between $(G, \mathcal{E}_L)$ and $(G, \mathcal{E}_R)$, the coarse spaces are very much alike and we are instead interested in when they outright coincide.

Lemma 3.25. *Let G be a topological group and $\mathcal{E}_L$ and $\mathcal{E}_R$ be its left and right-coarse structures. Then the following conditions are equivalent.*

1. *The left and right-coarse structures coincide, $\mathcal{E}_L = \mathcal{E}_R$,*
2. *the inversion map $\texttt{inv}\colon (G, \mathcal{E}_L) \to (G, \mathcal{E}_L)$ is bornologous,*
3. *if A is coarsely bounded is G, then so is $A^G = \{gag^{-1} \mid a \in A \,\&\, g \in G\}$.*

Proof. The equivalence between (1) and (2) follows easily from the fact that the inversion map is a coarse equivalence between $(G,\mathcal{E}_L)$ and $(G,\mathcal{E}_R)$.

(2)⇒(3): Assume that $\mathtt{inv}\colon (G,\mathcal{E}_L)\to(G,\mathcal{E}_L)$ is bornologous and that $A\subseteq G$ is coarsely bounded in G. Replacing A by $A\cup A^{-1}$, we may suppose that A is symmetric. Then $(\mathtt{inv}\times\mathtt{inv})E_A\in\mathcal{E}_L$ and hence in contained in some basic entourage E_B, with B coarsely bounded in G. In other words, $x^{-1}y\in A$ implies $xy^{-1}\in B$ for all $x,y\in G$. In particular, if $a\in A$ and $g\in G$, then $g^{-1}\cdot ga^{-1}=a^{-1}\in A$, whence $g\cdot ag^{-1}\in B$, showing that $A^G\subseteq B$ and thus that A^G is coarsely bounded in G.

(3)⇒(2): Assume that (3) holds and that E_A is a basic coarse entourage with $A\subseteq G$ symmetric. Then

$$(\mathtt{inv}\times\mathtt{inv})E_A=\{(x^{-1},y^{-1})\mid x^{-1}y\in A\}=\{(x,y)\mid xy^{-1}\in A\}.$$

But clearly, if $xy^{-1}\in A$, then $yx^{-1}\in A^{-1}=A$ and thus $x^{-1}y=x^{-1}\cdot yx^{-1}\cdot x\in A^G$, showing that $(\mathtt{inv}\times\mathtt{inv})E_A\subseteq E_{A^G}$. As A^G is coarsely bounded in G, this shows that $\mathtt{inv}\colon (G,\mathcal{E}_L)\to(G,\mathcal{E}_L)$ is bornologous. □

Observe that condition (3) may equivalently be stated as $\mathcal{OB}$ having a cofinal basis consisting of conjugacy invariant sets.

Example 3.26. If G is a countable discrete group, the coarsely bounded sets in G are simply the finite sets, so $\mathcal{E}_L=\mathcal{E}_R$ if and only if every conjugacy class is finite, i.e., if G is an FCC group (for *finite conjugacy classes*).

Observe that, if d_L is a coarsely proper continuous left-invariant écart G, i.e., inducing the left-invariant coarse structure $\mathcal{E}_L$, then the écart d_R given by $d_R(g,h)=d_L(g^{-1},h^{-1})$ is right-invariant. Moreover, since $d_L(g,1)=d_L(1,g^{-1})=d_R(g,1)$, we see that sets of finite d_R-diameter are coarsely bounded in G and hence that d_R induces the right-coarse structure $\mathcal{E}_R$. Thus, by Lemma 3.25, the identity mapping

$$\mathrm{id}\colon (G,d_L)\to(G,d_R)$$

is bornologous if and only if A^G is coarsely bounded in G for every coarsely bounded set $A\subseteq G$.

Example 3.27. It is worth pointing out that the equivalent properties of Lemma 3.25 are not coarse invariants of topological groups, that is, are not preserved under coarse equivalence. Indeed, let D_∞ denote the infinite dihedral group, i.e., the group of all isometries of $\mathbb{Z}$ with the euclidean metric. Then every element g of D_∞ can be written uniquely as $g=\tau_n$ or $g=\tau_n\cdot\rho$, where τ_n is a translation of amplitude n and ρ is the reflection around 0. It follows that the group $\mathbb{Z}$ of translations is an index 2 subgroup of D_∞ and hence is quasi-isometric to D_∞. On the other hand, as $\tau_m\rho\tau_{-m}=\tau_{2m}\rho$, we see that ρ has an infinite conjugacy class in D_∞. So $\mathbb{Z}$ and D_∞ are quasi-isometric, but only $\mathbb{Z}$ is FCC.

Similar to the considerations above is the question of when the left and right uniformities $\mathcal{U}_L$ and $\mathcal{U}_R$ coincide. As above one sees that this is equivalent to the inversion map on G being left-uniformly continuous, which again is equivalent to the being a neighbourhood basis at the identity consisting of conjugacy invariant sets. Groups with this property are called SIN for *small invariant neighbourhoods*. Moreover, by a result of V. Klee [**41**], metrisable SIN groups are exactly those admitting a compatible bi-invariant metric.

PROPOSITION 3.28. *The following are equivalent for a Polish group G,*

(1) *G admits a coarsely proper bi-invariant compatible metric d,*
(2) *G is SIN, locally bounded and every conjugacy class is coarsely bounded.*

PROOF. The implication (1)⇒(2) is trivial since d-balls are conjugacy invariant.

(2)⇒(1): Assume that (2) holds and let U be a coarsely bounded identity neighbourhood. Since G is SIN, there is a symmetric open conjugacy invariant identity neighbourhood $V \subseteq U$. Suppose that $A \subseteq G$ is a coarsely bounded set and find a finite $F \subseteq G$ and $k \geqslant 1$ so that $A \subseteq (FV)^k$, whereby also $A^G \subseteq (F^G V^G)^k = (F^G V)^k$. As, F^G is a union of finitely many conjugacy classes, it is coarsely bounded in G, showing that also $(F^G V)^k$ and thus A^G are coarsely bounded in G. In other words, if A is coarsely bounded, so is A^G. Since G is separable, we may therefore find symmetric open and conjugacy invariant sets $V = V_0 \subseteq V_1 \subseteq V_2 \subseteq \ldots$ so that

$$V_n^3 \subseteq V_{n+1}$$

for all $n \geqslant 0$ and with $G = \bigcup_k V_k$.

Using that G is SIN, there are a neighbourhood basis at the identity $V_0 \supseteq V_{-1} \supseteq V_{-2} \supseteq \ldots \ni 1$ consisting of conjugacy invariant, coarsely bounded, symmetric open sets so that now $V_n^3 \subseteq V_{n+1}$ for all $n \in \mathbb{Z}$. It follows that the metric defined from $(V_n)_{n\in\mathbb{Z}}$ via Lemma 2.6 is bi-invariant, coarsely proper and compatible with the topology on G. □

If G is a locally bounded Polish group whose left and right-coarse structures coincide, but which fails to be SIN, one may still hope that this could be reflected in the continuous écarts on the group.

PROBLEM 3.29. *Suppose G is a locally bounded Polish group whose left and right-coarse structures coincide, i.e., $\mathcal{E}_L = \mathcal{E}_R$. Does G admits a continuous bi-invariant coarsely proper écart?*

5. Coarse geometry of product groups

Note that, if $H = \prod_{i\in I} H_i$ is a product of topological groups H_i equipped with the Tychonoff topology, then a basis for the uniformity on H is given by entourages

$$E = \{\big((x_i),(y_i)\big) \in H \;\big|\; x_i^{-1}y_i \in V_i,\ \forall i \in J\},$$

where J is a *finite* subset of I and $V_i \subseteq H_i$ are identity neighbourhoods. This means that the entourages depend only on a finite set of coordinates, which, as we shall see, is opposed to the coarse structure.

LEMMA 3.30. *Let $H = \prod_{i\in I} H_i$ be a product of topological groups H_i. Then a subset $A \subseteq H$ is coarsely bounded in H if and only if there are coarsely bounded sets $A_i \subseteq H_i$ with $A \subseteq \prod_{i\in I} A_i$.*

PROOF. Suppose that $A \subseteq H$ and set $A_i = \mathrm{proj}_i(A)$. Then, if some A_j is not coarsely bounded in H_j, we may find a continuous left-invariant écart d on H_j with $\mathrm{diam}_d(A_j) = \infty$. So let ∂ be defined on H by

$$\partial\big((x_i),(y_i)\big) = d(x_j, y_j)$$

and observe that ∂ is a continuous left-invariant écart with respect to which A has infinite diameter, i.e., A is not coarsely bounded in H.

Conversely, if each A_i is coarsely bounded in H_i and d is a continuous left-invariant écart on H, then, by continuity of d, there is a finite set $J \subseteq I$ so that

$$\mathrm{diam}_d\Big(\prod_{i\in J}\{1\}\times\prod_{i\notin J}H_i\Big)<1.$$

Moreover, since each A_j is coarsely bounded in H_j, also $A_j\times\prod_{i\neq j}\{1\}$ is coarsely bounded in H and hence has finite d-diameter. Writing $J=\{j_1,\ldots,j_n\}$, this implies that the finite product with respect to the group multiplication in H,

$$\prod_{i\in J}A_i\times\prod_{i\notin J}H_i=\Big(A_{j_1}\times\prod_{i\neq j_1}\{1\}\Big)\cdots\Big(A_{j_n}\times\prod_{i\neq j_n}\{1\}\Big)\cdot\Big(\prod_{i\in J}\{1\}\times\prod_{i\notin J}H_i\Big),$$

has finite d-diameter. As $A\subseteq\prod_{i\in J}A_i\times\prod_{i\notin J}H_i$, this shows that A has finite diameter with respect to any continuous left-invariant écart on H, i.e., that A is coarsely bounded in H. □

By Lemma 3.30, product sets $\prod_{i\in I}A_i$ with A_i coarsely bounded in H_i are thus cofinal in $\mathcal{OB}$, which means that the entourages

$$E_{\prod_{i\in I}A_i}=\{\big((x_i),(y_i)\big)\mid x_i^{-1}y_i\in A_i,\ \forall i\in I\}=\prod_{i\in I}E_{A_i}$$

are also cofinal in the coarse structure on H. Therefore, a map $\phi\colon X\to H$ from a coarse space $(X,\mathcal{E})$ is bornologous if and only if each $\mathrm{proj}_i\circ\phi\colon X\to H_i$ is bornologous.

The above considerations also motivate the following definition.

Definition 3.31. *Let $\big\{(X_i,\mathcal{E}_i)\big\}_{i\in I}$ be a family of coarse spaces. Then the coarse structure on the product $\prod_{i\in I}X_i$ is that generated by entourages of the form*

$$\prod_{i\in I}E_i$$

with $E_i\in\mathcal{E}_i$.

Thus, with this definition, the coarse structure on a product group $\prod_{i\in I}H_i$ is simply the product of the coarse structures on the H_i.

We now return to the Urysohn space $\mathbb{U}$ from Example 3.7. Whereas the Urysohn space is universal for all separable metric spaces, i.e., embeds any separable metric space, its isometry group $\mathrm{Isom}(\mathbb{U})$ is universal for all Polish groups as shown by V. V. Uspenskiĭ [**75**]. We show that similar results are valid in the coarse category.

Theorem 3.32. *Let G be a locally bounded Polish group. Then G admits a simultaneously coarse and isomorphic embedding $\phi\colon G\to\mathrm{Isom}(\mathbb{U})$. Moreover, if G is actually generated by a coarsely bounded set, ϕ can be made simultaneously a quasi-isometric and isomorphic embedding.*

Proof. If G is locally bounded, we let ∂ be a coarsely proper compatible left-invariant metric on G, while if G is furthermore generated by a coarsely bounded set, we choose ∂ to be maximal.

We now reprise a construction due to M. Katětov [**37**] and used by Uspenskiĭ [**75**] to prove universality of $\mathrm{Isom}(\mathbb{U})$. The construction associates to every separable metric space X an isometric embedding $\iota\colon X\to\mathbb{U}$ which is functorial in the sense that there is a corresponding isomorphic embedding $\theta\colon\mathrm{Isom}(X)\to\mathrm{Isom}(\mathbb{U})$ so that

$$\theta(g)\big(\iota x\big)=\iota\big(g(x)\big)$$

for all $x \in X$ and $g \in \mathrm{Isom}(X)$.

Taking $X = (G, \partial)$ and embedding G into $\mathrm{Isom}(G, \partial)$ via the left-regular representation $g \mapsto \lambda_g$, we obtain an isometric embedding

$$\iota \colon (G, d) \to \mathbb{U}$$

and an isomorphic embedding $\phi \colon G \to \mathrm{Isom}(\mathbb{U})$, $\phi(g) = \theta(\lambda_g)$, so that

$$\phi(g)\big(\iota f\big) = \iota\big(\lambda_g(f)\big) = \iota(gf)$$

for all $g, f \in G$. Thus, if $x = \iota 1$ is the image of the identity element in G, we have

$$d\big(\phi(g)x, \phi(f)x\big) = d\big(\iota(g1), \iota(f1)\big) = d(\iota g, \iota f) = \partial(g, f),$$

i.e., the orbit map $g \in G \mapsto \phi(g)x \in \mathbb{U}$ is an isometric embedding of (G, ∂) into $\mathbb{U}$. As, on the other hand, the orbit map $a \in \mathrm{Isom}(\mathbb{U}) \mapsto ax \in \mathbb{U}$ is a quasi-isometry between $\mathrm{Isom}(\mathbb{U})$ and $\mathbb{U}$ by Proposition 3.8, we find that $g \mapsto \phi(g)$ is a quasi-isometric embedding of (G, ∂) into $\mathrm{Isom}(\mathbb{U})$. □

THEOREM 3.33. *Let G be Polish group. Then G admits a simultaneously coarse and isomorphic embedding into $\prod_{n \in \mathbb{N}} \mathrm{Isom}(\mathbb{U})$.*

PROOF. Let $\{U_n\}_{n \in \mathbb{N}}$ be a neighbourhood basis at the identity consisting of open symmetric sets. Then, by Lemma 2.25, there are continuous left-invariant écarts d_n so that a subset $A \subseteq G$ is d_n-bounded if and only if there is a finite set $D \subseteq G$ and an m with $A \subseteq (DU_n)^m$. Fixing a compatible left-invariant metric $\partial \leqslant 1$ on G and replacing d_n with $d_n + \partial$, we may suppose that the d_n are compatible left-invariant metrics.

By the proof of Theorem 3.32, there are isomorphic embeddings

$$\phi_n \colon G \to \mathrm{Isom}(\mathbb{U})$$

which are quasi-isometric embeddings with respect to the metric d_n on G. So let $\phi \colon G \to \prod_{n \in \mathbb{N}} \mathrm{Isom}(\mathbb{U})$ be the product $\phi = \bigotimes_{n \in \mathbb{N}} \phi_n$. Evidently, ϕ is an isomorphic embedding and hence also bornologous. So it remains to verify that it is expanding. Now, if E is a coarse entourage on $\prod_{n \in \mathbb{N}} \mathrm{Isom}(\mathbb{U})$, by the discussion above, there are coarse entourages E_n on $\mathrm{Isom}(\mathbb{U})$ so that $E \subseteq \prod_{n \in \mathbb{N}} E_n$. Since each ϕ_n is a quasi-isometric embedding with respect to the metric d_n, there are constants K_n so that $d_n(x, y) < K_n$ whenever $\big(\phi_n(x), \phi_n(y)\big) \in E_n$, whereby

$$\big(\phi(x), \phi(y)\big) \in E \;\;\Rightarrow\;\; \forall n \;\; d_n(x, y) < K_n.$$

To see that $F = \{(x, y) \mid \forall n \;\; d_n(x, y) < K_n\}$ is a coarse entourage on G, suppose that d is a continuous left-invariant écart on G. Then, by continuity, some U_n has finite d-diameter and since the ball $B_{d_n}(K_n) = \{x \in G \mid d_n(x, 1) < K_n\}$ is contained in some set $(DU_n)^m$, for $D \subseteq G$ finite and $m \geqslant 1$, also $B_{d_n}(K_n)$ has finite d-diameter. Thus, for some C, we have $d(x, y) < C$, whenever $d_n(x, y) < K_n$, showing that $F \in \mathcal{E}_d$ and hence that F is a coarse entourage on G. In other words, ϕ is expanding and therefore a coarse embedding. □

For the next proposition, recall that, as in Examples 2.42 and 2.43, for elements h_m in a topological group G, we write $g_n \xrightarrow[\tau_{\mathcal{OB}}]{} *$ if the g_n eventually leave every coarsely bounded set and $g_n \to \infty$ if there is a continuous left-invariant écart d so that $d(g_n, 1) \to \infty$.

PROPOSITION 3.34. *The following are equivalent for a Polish group G.*

(1) *G is locally bounded,*
(2) *for all sequences (g_n), we have $g_n \underset{\tau_{\mathcal{OB}}}{\longrightarrow} *$ if and only if $g_n \to \infty$.*

Proof. The implication from (1) to (2) has already been established in Example 2.43. So suppose instead that G fails to be locally bounded. As in the proof of Theorem 3.33, let $\{U_n\}_{n\in\mathbb{N}}$ be a neighbourhood basis at the identity consisting of open symmetric sets and find compatible left-invariant metrics d_n so that a subset $A \subseteq G$ is d_n-bounded if and only if there is a finite set $D \subseteq G$ and an m with $A \subseteq (DU_n)^m$. Set also $\partial_n = d_1 + \ldots + d_n$. Then a subset A of G is coarsely bounded if and only if $\mathrm{diam}_{\partial_n}(A) < \infty$ for all n. Moreover, if d is any compatible left-invariant metric on G, there is an n so that $d \leqslant \rho \circ \partial_n$ for some monotone function $\rho\colon \mathbb{R}_+ \to \mathbb{R}_+$.

Note that, since G fails to be locally bounded, no ball $B_{\partial_n}(1_G, 1)$ is coarsely bounded. Thus, by passing to a subsequence, we may suppose that $\mathrm{diam}_{\partial_1}(G) = \infty$ and that

$$\mathrm{diam}_{\partial_{n+1}}\big(B_{\partial_n}(1_G, 1)\big) = \infty$$

for all n. So pick $f_n \in G$ with $\partial_1(f_n, 1) \geqslant n$ and find $h_{n,k} \in B_{\partial_n}(1_G, 1)$ so that $\partial_{n+1}(h_{n,k}, 1) \geqslant k$. We let $g_{n,k} = f_n h_{n,k}$ and consider the sequence $(g_{n,k})_{(n,k)\in\mathbb{N}^2}$ enumerated in ordertype $\mathbb{N}$.

Observe first that $(g_{n,k})_{(n,k)\in\mathbb{N}^2} \not\to \infty$. Indeed, if d is any compatible left-invariant metric on G, pick n and a monotone function $\rho\colon \mathbb{R}_+ \to \mathbb{R}_+$ so that $d \leqslant \rho \circ \partial_n$. Then, for all k,

$$\begin{aligned} d(1, g_{n,k}) &\leqslant \rho\big(\partial_n(1, g_{n,k})\big) \\ &\leqslant \rho\big(\partial_n(1, f_n) + \partial_n(f_n, f_n h_{n,k})\big) \\ &= \rho\big(\partial_n(1, f_n) + \partial_n(1, h_{n,k})\big) \\ &\leqslant \rho\big(\partial_n(1, f_n) + 1\big), \end{aligned}$$

showing that the infinite subsequence $(g_{n,k})_{k\in\mathbb{N}}$ is d-bounded.

On the other hand, no coarsely bounded set A contains more than finitely many terms of $(g_{n,k})_{(n,k)\in\mathbb{N}^2}$. For if $g_{n,k} = f_n h_{n,k} \in A$, then

$$\begin{aligned} n &\leqslant \partial_1(f_n, 1) \\ &\leqslant \partial_1(f_n, f_n h_{n,k}) + \partial_1(f_n h_{n,k}, 1) \\ &\leqslant \partial_1(1, h_{n,k}) + \partial_1(g_{n,k}, 1) \\ &\leqslant \partial_n(1, h_{n,k}) + \mathrm{diam}_{\partial_1}(A) \\ &\leqslant 1 + \mathrm{diam}_{\partial_1}(A), \end{aligned}$$

while

$$\begin{aligned} k &\leqslant \partial_{n+1}(h_{n,k}, 1) \\ &\leqslant \partial_{n+1}(h_{n,k}, f_n^{-1}) + \partial_{n+1}(f_n^{-1}, 1) \\ &\leqslant \partial_{n+1}(f_n h_{n,k}, 1) + \partial_{n+1}(f_n^{-1}, 1) \\ &\leqslant \mathrm{diam}_{\partial_{n+1}}(A) + \partial_{n+1}(f_n^{-1}, 1). \end{aligned}$$

Thus, on the one hand, n is bounded independently of k, while k is bounded as a function of n, whence only finitely many terms $g_{n,k}$ belong to A. $\square$

Our discussion here also provides a framework for investigating a problem concerning coarse embeddability of groups. Suppose H is a closed subgroup of a Polish group G. Then H is coarsely embedded in G if the coarse structure on H, when H is viewed as a topological group in its own right, coincides with the coarse structure on G restricted to H, i.e., if a subset $A \subseteq H$ is coarsely bounded in H exactly when it is coarsely bounded in G. In particular, the latter reformulation shows that being coarsely embedded is independent of whether we talk of the left or the right coarse structure on H and G (as long as we make the same choice for H and G).

Assume now that, in addition to being coarsely embedded, H is locally bounded and thus admits a coarsely proper metric. If G is also locally bounded, then the restriction $d|_H$ to H of a coarsely proper metric d on G will also be coarsely proper on H, but one may wonder whether this holds more generally.

Definition 3.35. *Let H be a closed subgroup of a Polish group G. We say that H is* well-embedded *in G if there is a compatible left-invariant metric d on G so that the restriction $d|_H$ is coarsely proper on H.*

So are coarsely embedded, locally bounded subgroups automatically well-embedded? As seen in the next example, the answer is in general no, even considering countable discrete groups H.

Example 3.36 (Coarsely, but not well-embedded subgroups). Consider the free abelian group $\mathbb{A}_X \cong \bigoplus_n \mathbb{Z}$ on a denumerable set of generators X. Let also $w_0\colon X \to \mathbb{N}$ be a function so that every fibre $w_0^{-1}(n)$ is infinite and choose, for every $n \geqslant 1$, some function $w_n\colon X \to \mathbb{N}$ agreeing with w_0 on $X \setminus w_0^{-1}(n)$, while being injective on $w_0^{-1}(n)$. For every $n \geqslant 0$, let $d_n\colon \mathbb{A}_X \to \mathbb{N}_0$ be the invariant metric with weight w_n, i.e., for distinct $g, f \in \mathbb{A}_X$,

$$d_n(g,f) = \min\big(w_n(x_1) + \ldots + w_n(x_k) \;\big|\; g = f x_1^{\pm} \cdots x_k^{\pm} \;\&\; x_i \in X\big).$$

We claim that, if $A \subseteq \mathbb{A}_X$ is infinite, then A is d_n-unbounded for some $n \geqslant 0$. Indeed, if elements of A include generators from infinitely many distinct fibres $w_0^{-1}(n)$, then A is already d_0-unbounded. Similarly, if the elements of A include infinitely many distinct generators from some single fibre $w_0^{-1}(n)$, then A is d_n-unbounded. And finally, the last remaining option is that only finitely many generators appear in elements of A, whereby the word length must be unbounded on A and so A is d_0-unbounded.

On the other hand, for every k, the fibre $A = w_0^{-1}(k+1)$ is an infinite set bounded in each of the metrics $d_0, d_1, \ldots, d_k$.

Now, as in the proof of Theorem 3.32, the metric group $(\mathbb{A}_X, d_n)$ admits a quasi-isometric isomorphic embedding into $\mathrm{Isom}(\mathbb{U})$. In this way, the countable discrete diagonal subgroup $\Delta = \{(f,f,f,\ldots) \mid f \in \mathbb{A}_X\}$ is coarsely embedded in the infinite product $\prod_{n=0}^{\infty} \mathrm{Isom}(\mathbb{U})$, while the projections $\pi_k\colon \Delta \to \prod_{n=0}^{k} \mathrm{Isom}(\mathbb{U})$ all fail to be coarsely proper. Now, assume d is a compatible left-invariant metric on $\prod_{n=0}^{\infty} \mathrm{Isom}(\mathbb{U})$. Then d is bounded on some tail $\{\mathrm{id}\}^k \times \prod_{n=k}^{\infty} \mathrm{Isom}(\mathbb{U})$, whereby the restriction $d|_\Delta$ fails to be coarsely proper.

In other words, $\Delta \cong \bigoplus_n \mathbb{Z}$ is a coarsely embedded countable discrete subgroup of $\prod_{n=0}^{\infty} \mathrm{Isom}(\mathbb{U})$, but fails to be well-embedded.

As it turns out, we have positive results when considering coarsely embedded subgroups of products $\prod_n G_n$ of a restricted class of groups.

DEFINITION 3.37. *A Polish group G is said to have* bounded geometry *if there is a coarsely bounded set $A \subseteq G$ covering every coarsely bounded set $B \subseteq G$ by finitely many left-translates, i.e., $B \subseteq FA$ for some finite $F \subseteq G$.*

Evidently, every locally compact second countable group has bounded geometry, but also more complex groups such as the group $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ of homeomorphisms of $\mathbb{R}$ commuting with integral translations have bounded geometry.

PROPOSITION 3.38. *Suppose G_n is a sequence of Polish groups with bounded geometry. Then every coarsely embedded, closed, locally bounded subgroup of $\prod_n G_n$ is well-embedded.*

PROOF. Assume that H is a coarsely embedded, closed, locally bounded subgroup of $\prod_n G_n$. For each n, fix a coarsely bounded symmetric subset $A_n \subseteq G_n$ covering coarsely bounded sets by finitely many left translates. We claim that, for some k, the intersection

$$C_k = H \cap (A_1^2 \times \ldots \times A_k^2 \times G_{k+1} \times G_{k+2} \times \ldots)$$

is coarsely bounded in H. If not, let ∂ be a coarsely proper compatible left-invariant metric on H, and, for every k, choose some $x_k \in C_k$ with $\partial(x_k, 1) > k$. Then $\{x_n\}_n$ is clearly not coarsely bounded in H. So, as H is coarsely embedded in $\prod_n G_n$, $\{x_n\}_n$ is not coarsely bounded in G, whence there is a compatible left-invariant metric d on G with $\mathrm{diam}_d(\{x_n\}_n) = \infty$. By continuity of d, we find that some $\{1\} \times \ldots \times \{1\} \times G_{k+1} \times G_{k+2} \times \ldots$ has finite d-diameter. Thus, by restricting d to the subproduct, $G_1 \times \ldots \times G_k$, we find that $\pi_k[\{x_n\}_n]$ fails to be coarsely bounded in $G_1 \times \ldots \times G_k$ despite the cofinite subset $\pi_k[\{x_{k+1}, x_{k+2}, \ldots\}]$ lying in the coarsely bounded subset $A_1^2 \times \ldots \times A_k^2$.

So choose k so that C_k is coarsely bounded in H. To see that $\pi_k \colon H \to \prod_{i=1}^k G_i$ is a coarse embedding, let $B \subseteq H$ fail to be coarsely bounded in H. Then B cannot be covered by finitely many left translates of C_k and we can therefore choose an infinite subset $D \subseteq B$ so that $y \notin zC_k$ and hence $\pi_k(z)^{-1}\pi_k(y) \notin A_1^2 \times \ldots \times A_k^2$ for all $y \neq z$ in D. On the other hand, if $\pi_k[B]$ is coarsely bounded in $G_1 \times \ldots \times G_k$, then by choice of the A_n there is a finite set $F \subseteq G_1 \times \ldots \times G_k$ so that $\pi_k[B] \subseteq F \cdot (A_1 \times \ldots \times A_k)$. In particular, there must be distinct $y, z \in D$ and some $f \in F$ so that $\pi_k(y), \pi_k(z) \in f \cdot (A_1 \times \ldots \times A_k)$ and thus $\pi_k(z)^{-1}\pi_k(y) \in A_1^2 \times \ldots \times A_k^2$, which is absurd. So $\pi_k[B]$ fails to be coarsely bounded in $G_1 \times \ldots G_k$, whereby π_k is coarsely proper and hence a coarse embedding.

To finish the proof, we now use that fact that every Polish group of bounded geometry is automatically locally bounded (see Lemma 5.5). It thus follows that $G_1 \times \ldots \times G_k$ is locally bounded and therefore admits a coarsely proper compatible metric ∂. Letting ρ be a compatible left-invariant metric on $G_{k+1} \times G_{k+2} \times \ldots$, we obtain a compatible metric $\partial + \rho$ on the product group

$$\big(G_1 \times \ldots \times G_k\big) \times \big(G_{k+1} \times G_{k+2} \times \ldots\big),$$

which is coarsely proper on H. So H is well-embedded. □

When the G_n are not just of bounded geometry, but locally compact, an even stronger statement is true.

PROPOSITION 3.39. *Suppose G_n is a sequence of locally compact Polish groups. Then every closed, locally bounded subgroup of $\prod_n G_n$ is well-embedded.*

PROOF. Observe that, by Lemma 3.30, the coarsely bounded sets in $\prod_n G_n$ are exactly the relatively compact sets. Thus, if H is a closed subgroup of $\prod_n G_n$ and $A \subseteq H$ is coarsely bounded in $\prod_n G_n$, then $\overline{A}$ is compact and thus A must be coarsely bounded in H too. So H is coarsely embedded. If, moreover, H is locally bounded, i.e., locally compact, it follows from Proposition 3.38, that H is also well-embedded. $\square$

6. Distorted elements and subgroups

Suppose G is a Polish group with a maximal metric d_G. Recall that a closed subgroup H of G is coarsely embedded if the coarse structure of H agrees with that induced from G. However, if H itself admits a maximal metric d_H, one may also directly compare the metrics d_G and d_H.

DEFINITION 3.40. *Suppose G is a Polish group with a maximal metric d_G. Assume also that H is a subgroup of G that is Polish in a finer group topology and admits a maximal metric d_H. We say that H is* undistorted *in G if the inclusion map*

$$(H, d_H) \to (G, d_G)$$

is a quasi-isometric embedding, i.e., when $d_H \ll d_G$.

A case of special interest is when H is a finitely generated subgroup of G. Then, for H to be undistorted in G, it must, in particular, be discrete in G. While finite subgroups are always undistorted in G, it will be convenient to consider an element $g \in G$ to be distorted even if it has finite order.

DEFINITION 3.41. *Let G be a Polish group with a maximal metric d. An element $g \in G$ is said to be* distorted *or to be a* distortion element *if*

$$\lim_{n\to\infty} \frac{d(g^n, 1)}{n} = 0.$$

Let us observe that, since $d(g^{n+m}, 1) \leqslant d(g^n, 1) + d(g^m, 1)$ for all n, m, the limit $\lim_{n\to\infty} \frac{d(g^n,1)}{n}$ exists for all elements of G. Indeed, let $\epsilon > 0$ be given and pick n so that

$$\frac{d(g^n, 1)}{n} < \liminf_k \frac{d(g^k, 1)}{k} + \epsilon.$$

Then, for all $m > \frac{n\cdot d(g,1)}{\epsilon}$, write $m = pn + j$ with $0 \leqslant j < n$ and $p \geqslant 0$, whereby

$$\frac{d(g^m, 1)}{m} \leqslant \frac{p \cdot d(g^n, 1) + j \cdot d(g, 1)}{m} < \frac{d(g^n, 1)}{n} + \frac{n \cdot d(g, 1)}{m} < \liminf_k \frac{d(g^k, 1)}{k} + 2\epsilon.$$

Thus $\limsup_m \frac{d(g^m,1)}{m} < \liminf_k \frac{d(g^k,1)}{k} + 2\epsilon$ and hence $\lim_{n\to\infty} \frac{d(g^n,1)}{n}$ exists.

Using this calculation, we see that the set of distortion elements

$$\{g \in G \mid \lim_n \frac{d(g^n, 1)}{n} = 0\} = \bigcap_k \bigcup_n \{g \in G \mid \frac{d(g^n, 1)}{n} < \frac{1}{k}\}$$

is G_δ in G.

Also, while it is easy to see that the set of distortion elements is invariant under conjugacy, a bit more can be said. For this, we let g^G denote the conjugacy class of g and $\overline{g^G}$ its closure.

LEMMA 3.42. *Let G be a Polish group with a maximal metric d and let $g \in G$. If $\overline{g^G}$ contains a distortion element, then also g is distorted. In particular, if g is undistorted, then* $\inf_{h\in G} d(hgh^{-1}, 1) > 0$.

PROOF. As we are dealing with powers of individual elements, it will be useful to work with the length function ℓ associated to d, namely, $\ell(h) = d(h, 1)$. Observe then that $\ell(h^{-1}) = \ell(h)$ and $\ell(h_1h_2) \leqslant \ell(h_1) + \ell(h_2)$ for all $h, h_i \in G$.

Suppose $f \in \overline{g^G}$ is distorted. Fix $\epsilon > 0$ and pick n so that $\frac{\ell(f^n)}{n} < \epsilon$. Pick then $h \in G$ so that $d(hg^nh^{-1}, f) = d\big((hgh^{-1})^n, f^n\big) < \epsilon$. Then, for $k \geqslant 1$, we have

$$\begin{aligned}\ell(g^{kn}) &\leqslant \ell(hg^{kn}h^{-1}) + 2\ell(h)\\ &\leqslant \ell\big((hg^nh^{-1})^k\big) + 2\ell(h)\\ &\leqslant k\cdot \ell(hg^nh^{-1}) + 2\ell(h)\\ &\leqslant k\cdot\big(\ell(f^n) + d(hg^nh^{-1}, f^n)\big) + 2\ell(h)\\ &\leqslant kn\epsilon + k\epsilon + 2\ell(h)\end{aligned}$$

and so

$$\lim_{k\to\infty}\frac{\ell(g^k)}{k} = \lim_{k\to\infty}\frac{\ell(g^{kn})}{kn} \leqslant \liminf_{k\to\infty}\Big(\epsilon + \frac{\epsilon}{n} + \frac{2\ell(h)}{kn}\Big) \leqslant 2\epsilon.$$

As $\epsilon > 0$ is arbitrary, g is distorted in G.

As 1 is distorted in G, we have as an immediate consequence that, if $g \in G$ is undistorted, then $\inf_{h\in G} d(hgh^{-1}, 1) > 0$. □

From Lemma 3.42, we see that

$$C = \{g \in G \mid \inf_{h\in G} d(hgh^{-1}, 1) = 0\} = \{g \in G \mid 1 \in \overline{g^G}\},$$

is a conjugacy invariant G_δ subset of the distortion elements. Observe also that, similarly to Lemma 3.42, if $\overline{g^G} \cap C \neq \emptyset$, then $g \in C$ too. Indeed, if $f \in \overline{g^G} \cap C$, then $f^G \subseteq \overline{g^G}$ and so $1 \in \overline{f^G} \subseteq \overline{g^G}$, i.e., $g \in C$.

EXAMPLE 3.43 (Distortion elements in affine isometry groups). In the following, let X be a separable reflexive real Banach space and A a surjective isometry of X. Then, by the Mazur–Ulam Theorem, A is affine and hence can be written as $A(x) = T(x) + b$ for some linear isometry $T \in \mathrm{Isom}(X)$ and a vector $b \in X$. Let also

$$X = X^T \oplus X_T$$

be the Yosida decomposition [**80**] associated with the linear isometry T. That is, $X^T = \ker(T - \mathrm{Id})$ and $X_T = \overline{\mathrm{rg}}(T - \mathrm{Id})$ are closed linear subspaces decomposing X and

$$\frac{1}{n}\big(T^{n-1} + T^{n-2} + \ldots + \mathrm{Id}\big)(x) \underset{n\to\infty}{\longrightarrow} P(x)$$

for all $x \in X$, where P is the projection onto X^T with $\ker P = X_T$.

Observe that, for $x \in X$, we have

$$\frac{1}{n}A^nx = \frac{1}{n}\big(T^nx + T^{n-1}b + T^{n-2}b + \ldots + Tb + b\big) \underset{n\to\infty}{\longrightarrow} 0 + Pb = Pb$$

and

$$\begin{aligned}\|A^nx - x\| &\leqslant \|A^nx - A^{n-1}x\| + \|A^{n-1}x - A^{n-2}x\| + \ldots + \|Ax - x\|\\ &= \|Ax - x\| + \|Ax - x\| + \ldots + \|Ax - x\|\\ &= n\|Ax - x\|.\end{aligned}$$

So, for any $x \in X$,

$$\|Pb\| = \lim_{n\to\infty} \frac{1}{n}\|A^n x\| = \lim_{n\to\infty} \frac{1}{n}\|A^n x - x\| \leqslant \|Ax - x\|.$$

On the other hand, $Pb - b \in \ker P = \overline{\mathrm{rg}}(T - \mathrm{Id})$ and thus

$$Pb - b = \lim_{n\to\infty} Tx_n - x_n$$

for some sequence $x_n \in X$. Then

$$\lim_{n\to\infty} \|Ax_n - x_n\| = \lim_{n\to\infty} \|Tx_n + b - x_n\| = \|Pb - b + b\| = \|Pb\|.$$

All in all, this shows that

$$\|Pb\| = \inf_{x\in X} \|Ax - x\| = \lim_{n\to\infty} \frac{1}{n}\|A^n y\|$$

for all $y \in X$.

Proposition 3.44. *Let X be a separable reflexive Banach space so that the group* $\mathrm{Isom}(X)$ *of linear isometries is coarsely bounded in the strong operator topology. Let A be an affine isometry with linear part T and translation vector b.*

(1) *Then A is distorted in the group* $\mathrm{Aff}(X)$ *of affine isometries if and only if,*

$$\inf_{x\in X} \|Ax - x\| = 0,$$

which happens if and only if $Pb = \lim_n \frac{1}{n}(T^{n-1} + \ldots + T + \mathrm{Id})b = 0.$

(2) *Moreover,* Id *is a limit of conjugates of A in* $\mathrm{Aff}(X)$ *if and only if both* Id *is a limit of conjugates of T in* $\mathrm{Isom}(X)$ *and* $\inf_{x\in X}\|Ax - x\| = 0$.

Proof. Since $\mathrm{Isom}(X)$ is coarsely bounded, by Proposition 3.13, the orbit map $A \mapsto A(0)$ is a quasi-isometry between $\mathrm{Aff}(X)$ and X. Therefore, A is distorted in $\mathrm{Aff}(X)$ if and only if

$$\lim_{n\to\infty} \frac{1}{n}\|A^n(0)\| = \lim_{n\to\infty} \frac{1}{n}\|A^n(0) - \mathrm{Id}(0)\| = 0.$$

However, as explained above, for all $y \in X$,

$$\|Pb\| = \inf_{x\in X} \|Ax - x\| = \lim_{n\to\infty} \frac{1}{n}\|A^n y\|$$

and hence (1) follows by taking $y = 0$.

For (2), suppose B_n are affine isometries of X with linear part R_n and translation vector c_n. Then, for all $x \in X$,

$$\begin{aligned} B_nAB_n^{-1}(x) &= B_nA(R_n^{-1}x - R_n^{-1}c_n) \\ &= B_n(TR_n^{-1}x - TR_n^{-1}c_n + b) \\ &= R_nTR_n^{-1}x - R_nTR_n^{-1}c_n + R_nb + c_n \end{aligned}$$

and so $\lim_n B_nAB_n^{-1} = \mathrm{Id}$ in $\mathrm{Aff}(X)$ if and only if $\lim_n R_nTR_n^{-1} = \mathrm{Id}$ in $\mathrm{Isom}(X)$ and

$$\lim_n \big\|b - (T - \mathrm{Id})R_n^{-1}c_n\big\| = \lim_n \big\| - R_nTR_n^{-1}c_n + R_nb + c_n\big\| = 0.$$

In particular, if $\lim_n B_nAB_n^{-1} = \mathrm{Id}$ in $\mathrm{Aff}(X)$, then Id is a limit of conjugates of T in $\mathrm{Isom}(X)$ and $b \in \overline{\mathrm{rg}}(T - \mathrm{Id}) = \ker P$, which by (1) happens if A is distorted in $\mathrm{Aff}(X)$.

Conversely, if A is distorted in $\mathrm{Aff}(X)$ and $\lim_n R_n T R_n^{-1} = \mathrm{Id}$ in $\mathrm{Isom}(X)$ for some sequence R_n in $\mathrm{Isom}(X)$, then $b \in \overline{\mathrm{rg}}(T - \mathrm{Id})$ and hence $b = \lim_n (T - \mathrm{Id}) b_n$ for some sequence of $b_n \in X$. Setting B_n to be the affine isometry with linear part R_n and translation vector $c_n = R_n b_n$, we see that $\lim_n B_n A B_n^{-1} = \mathrm{Id}$ in $\mathrm{Aff}(X)$, which finishes the proof of (2). $\square$

EXAMPLE 3.45 (Distortion elements in homeomorphism groups of surfaces). Suppose S is a compact surface different from the disc, the closed annulus and the Möbius strip. Assume also that S has non-empty boundary ∂S. Let $\tilde{S} \xrightarrow{p} S$ be the universal cover and equip $\tilde{S}$ with a compatible proper metric d that is invariant under the group $\pi_1(S)$ of deck-transformations. Fix also a fundamental domain D for $\tilde{S}$, i.e., a compact connected subset mapping onto S via the covering map and so that $\mathrm{int}(D) \cap \mathrm{int}(\gamma(D)) = \emptyset$ for any non-trivial deck-transformation γ.

Suppose f is an isotopically trivial homeomorphism of S. Then if $\tilde{f}_1$ and $\tilde{f}_2$ are any two lifts of f to $\tilde{S}$ and $n \geqslant 1$, both $\tilde{f}_1^n$ and $\tilde{f}_2^n$ are lifts of f^n and therefore differ by a deck-transformation γ, i.e., $\tilde{f}_1^n = \gamma \tilde{f}_2^n$. As d is invariant under deck-transformations, it thus follows that

$$\mathrm{diam}_d(\tilde{f}_1^n[D]) = \mathrm{diam}_d(\gamma \tilde{f}_2^n[D]) = \mathrm{diam}_d(\tilde{f}_2^n[D]).$$

Therefore,

$$\lim_n \frac{\mathrm{diam}_d(\tilde{f}^n[D])}{n} = 0$$

holds for *some* lift $\tilde{f}$ of f if and only if it holds for *all* lifts $\tilde{f}$ of f. A homeomorphism f with this property is said to be *non-spreading* in E. Militon's paper [**51**].

Theorem 1.11 of [**51**] states that, if f is a non-spreading homeomorphism of S, then the identity homeomorphism id_S is a limit of conjugates of f, that is

$$\mathrm{id}_S \in \overline{f^{\mathrm{Homeo}_0(S)}}.$$

Now, by Theorem 1 of [**46**], the group $\mathrm{Homeo}_0(S)$ of isotopically trivial homeomorphisms of S admits a maximal metric. Coupling this with Militon's theorem, we obtain the following.

PROPOSITION 3.46. *The following are equivalent for an isotopically trivial homeomorphism f of S.*

(1) *f distorted in* $\mathrm{Homeo}_0(S)$,
(2) id_S *is a limit of conjugates of f,*
(3) *f is non-spreading.*

PROOF. The implication from (3) to (2) is just the above cited result of Militon, while the implication from (2) to (1) follows immediately from Lemma 3.42.

For the implication of (1) to (3), let $\mathcal{U} = \{U_1, \dots, U_n\}$ be an open covering of S so that each $p^{-1}(U_i)$ is a disjoint union of d-bounded sets $W_{i,j}$ that are all mapped homeomorphically onto U_i via p. Note that every homeomorphism h with $\mathrm{supp}(h) \subseteq U_i$ admits a lift $\tilde{h}$ that leaves each $W_{i,j}$ invariant and hence satisfies

$$\sup_{x \in \tilde{S}} d(\tilde{h}(x), x) \leqslant K,$$

where K is an upper bound for the diameters of the sets $W_{i,j}$.

Now, suppose $f \in \mathrm{Homeo}_0(S)$ can be written as a product $f = h_1 \cdots h_m$ of homeomorphisms h_i each supported in some $U_{j(i)}$. Then we can choose lifts $\tilde{h}_i$ as above and find that

$$\begin{aligned}\sup_{x\in\tilde{S}} d(\tilde{h}_1 \cdots \tilde{h}_m(x), x) &\leqslant \sup_{y\in\tilde{S}} d(\tilde{h}_1(y), y) + \sup_{x\in\tilde{S}} d(\tilde{h}_2 \cdots \tilde{h}_k(x), x)\\ &\leqslant K + \sup_{x\in\tilde{S}} d(\tilde{h}_2 \cdots \tilde{h}_m(x), x)\\ &\leqslant \ldots\\ &\leqslant m \cdot K.\end{aligned}$$

In particular, for $\tilde{f} = \tilde{h}_1 \cdots \tilde{h}_m$, we have

$$\mathrm{diam}_d(\tilde{f}[D]) \leqslant 2K \cdot m + \mathrm{diam}_d(D).$$

As discussed in Example 3.15, it follows from the Fragmentation Lemma of R. D. Edwards and R. C. Kirby [**22**] that there is an identity neighbourhood V in $\mathrm{Homeo}_0(S)$ so that every $g \in V$ can be factored into $g = h_1 \cdots h_n$ with $\mathrm{supp}(h_i) \subseteq U_i$. Thus, from the above it follows that

$$f \in V^m \;\Rightarrow\; \mathrm{diam}_d(\tilde{f}[D]) \leqslant 2Kn \cdot m + \mathrm{diam}_d(D).$$

As $\mathrm{Homeo}_0(S)$ is connected, it is generated by V and we let ρ_V designate the word metric induced by generating set V. If also ∂ denotes the maximal metric on $\mathrm{Homeo}_0(S)$, we have $\rho_V \leqslant C \cdot \partial + C$ for some C. Now, suppose f is distorted, whence $\lim_k \frac{\partial(f^k,1)}{k} = 0$. Then, if $\tilde{f}$ is a lift of f, we have

$$\limsup_k \frac{\mathrm{diam}_d(\tilde{f}^k[D])}{k} \leqslant 2Kn \cdot \limsup_k \frac{\rho_V(f^k, \mathrm{id})}{k} = 2KCn \cdot \lim_k \frac{\partial(f^k, 1)}{k} = 0,$$

i.e., f is non-spreading. $\square$

CHAPTER 4

Sections, cocycles and group extensions

We will now consider how coarse structure is preserved in short exact sequences

$$1 \to K \xrightarrow{\iota} G \xrightarrow{\pi} H \to 1$$

of topological groups, where K is a closed normal subgroup of G and ι the inclusion map. This situation is typical and it will be useful to have a general terminology and set of tools to describe and compute the coarse structure of G from those of K and H.

Analogously to the algebraic study of group extensions, a central question is whether the extension splits coarsely. This will require a bornologous section $H \xrightarrow{\phi} G$ for the quotient map $G \xrightarrow{\pi} H$ and, for this reason, we shall explore the tight relationship between sections ϕ and their associated cocycles $\omega_\phi \colon H \times H \to G$ given by

$$\omega_\phi(x, y) = \phi(xy)^{-1}\phi(x)\phi(y).$$

We have three levels of regularity of ϕ along with useful reformulations in terms of the cocycle ω_ϕ.

(1) ϕ is modest, roughly corresponding to when $\omega_\phi[B \times B]$ is coarsely bounded for all coarsely bounded $B \subseteq H$,
(2) ϕ is bornologous, roughly corresponding to when $\omega_\phi[H \times B]$ is coarsely bounded for all coarsely bounded $B \subseteq H$,
(3) ϕ is a quasimorphism, roughly corresponding to when $\omega_\phi[H \times H]$ is coarsely bounded.

As second issue, which only appears in the case of Polish or general topological groups and not in the more specific context of locally compact groups, is when a closed subgroup K is coarsely embedded in G. Similarly, the question of when G is locally bounded and whether this can be deduced from the local boundedness of K and H is, of course, not relevant to locally compact groups, but will turn out to be crucial in our study.

1. Quasimorphisms and bounded cocycles

A quasimorphism from a group H into $\mathbb{R}$ is a map $\phi \colon H \to \mathbb{R}$ so that

$$|\phi(x) + \phi(y) - \phi(xy)| < K$$

for some constant K and all $x, y \in H$. Quasimorphisms appear naturally in topology and in questions concerning bounded cohomology of groups, but also admit the following generalisation of broader interest.

Definition 4.1. *A map $\phi \colon H \to G$ from a group H to a topological group G is a* quasimorphism *if the two maps $(x, y) \mapsto \phi(xy)$ and $(x, y) \mapsto \phi(x)\phi(y)$ are close with respect to the coarse structure on G.*

Given a map $\phi\colon H\to G$, we define a map $\omega_\phi\colon H\times H\to G$ measuring the failure of ϕ to be a homomorphism by

$$\omega_\phi(x,y)=\phi(xy)^{-1}\phi(x)\phi(y).$$

Then we see that ϕ is a quasimorphism if and only if the *defect* of ϕ,

$$\Delta=\omega_\phi[H\times H]\cup\omega_\phi[H\times H]^{-1}=\{\phi(xy)^{-1}\phi(x)\phi(y)\mid x,y\in H\}^{\pm},$$

is coarsely bounded in G. Note that then $\phi(1)=\phi(1\cdot 1)^{-1}\phi(1)\phi(1)\in\Delta$, $\phi(xy)\in\phi(x)\phi(y)\Delta$ and $\phi(x^{-1}y)=\phi(x)^{-1}\phi(x)\phi(x^{-1}y)\in\phi(x)^{-1}\phi(xx^{-1}y)\Delta=\phi(x)^{-1}\phi(y)\Delta$ for all $x,y\in H$. Therefore, if $x_i\in H$ and $\epsilon_i=\pm1$, we have

$$\begin{aligned}\phi(x_1^{\epsilon_1}x_2^{\epsilon_2}\cdots x_n^{\epsilon_n})&\in\phi(x_1)^{\epsilon_1}\phi(x_2^{\epsilon_2}\cdots x_n^{\epsilon_n})\Delta\\&\subseteq\ldots\\&\subseteq\phi(x_1)^{\epsilon_1}\phi(x_2)^{\epsilon_2}\cdots\phi(x_n^{\epsilon_n})\Delta^{n-1}\\&\subseteq\phi(x_1)^{\epsilon_1}\phi(x_2)^{\epsilon_2}\cdots\phi(x_n)^{\epsilon_n}\phi(1)\Delta^{n}\\&\subseteq\phi(x_1)^{\epsilon_1}\phi(x_2)^{\epsilon_2}\cdots\phi(x_n)^{\epsilon_n}\Delta^{n+1}.\end{aligned}$$

In particular, for all $x,y,g\in H$,

$$\phi(g)^{\pm}\cdot\phi(xy)^{-1}\phi(x)\phi(y)\cdot\phi(g)^{\mp}\in\phi(g^{\pm}(xy)^{-1}xyg^{\mp})\Delta^6=\phi(1)\Delta^6\subseteq\Delta^7,$$

i.e., $\phi(g)\Delta\phi(g)^{-1}\cup\phi(g)^{-1}\Delta\phi(g)\subseteq\Delta^7$, showing that $\phi[H]$ normalises the subgroup $F=\langle\Delta\rangle$ generated by Δ. In particular, from ϕ we obtain a canonical homomorphism

$$\tilde\phi\colon H\to N_G(F)/F,$$

where $N_G(F)$ is the normaliser of F in G.

Note also that not every quasimorphism between Polish groups is bornologous. For example, a linear operator between two Banach spaces is certainly a quasimorphism, but it is bornologous if and only if it is continuous, i.e., a bounded linear operator.

Proposition 4.2. *Let $\phi\colon H\to G$ be a Baire measurable quasimorphism between European topological groups. Then ϕ is bornologous.*

Proof. Let Δ be the defect of ϕ and observe that, for $x^{-1}y\in A\subseteq H$, we have $\phi(x)^{-1}\phi(y)\in\phi(x^{-1}y)\Delta\subseteq\phi[A]\Delta$, so

$$(\phi\times\phi)E_A\subseteq E_{\phi[A]\Delta}.$$

To see that ϕ is bornologous, it thus suffices to show that $\phi[A]$ is coarsely bounded for any coarsely bounded $A\subseteq H$.

So fix a coarsely bounded A and a symmetric open identity neighbourhood V in G. As G is European, we may find a chain $1\in F_1\subseteq F_2\subseteq\ldots\subseteq G$ of finite symmetric sets whose union generates G over V. Thus, $F_1V\subseteq\big(F_2V\big)^2\subseteq\big(F_3V\big)^3\subseteq\ldots$ is an exhaustive chain of open subsets of G and $H=\bigcup_n\phi^{-1}\big((F_nV)^n\big)$ is a covering of H by countably many sets with the Baire property. Since H is a Baire group, it follows that some $\phi^{-1}\big((F_nV)^n\big)$ must be somewhere comeagre in H and hence, by a lemma of B. J. Pettis [**57**], that $\big[\phi^{-1}\big((F_nV)^n\big)\big]^2$ has non-empty interior in H. As A is coarsely bounded, there is a finite set $D\subseteq H$ and an m so that

$$A\subseteq\Big(D\cdot\big[\phi^{-1}\big((F_nV)^n\big)\big]^2\Big)^m$$

and thus

$$\phi[A] \subseteq \Big(\phi[D] \cdot \big((F_nV)^n\big)^2\Big)^m \Delta^{3m-1}.$$

As $\phi[D]$ and F_n are finite and $\Delta \subseteq H$ coarsely bounded, it follows that there is a finite set $F \subseteq G$ and a k so that $\phi[A] \subseteq (FV)^k$. Since V was arbitrary, we conclude that $\phi[A]$ is coarsely bounded and hence that ϕ is bornologous. $\square$

Suppose G is a topological group so that whenever $W_1 \subseteq W_2 \subseteq \ldots \subseteq G$ is a countable exhaustive chain of symmetric subsets, then some W_n^k, $n, k \geqslant 1$, has non-empty interior. Then the above proof shows that every quasimorphism from G into a European topological group is bornologous. Such G include for example Polish groups with ample generics or homeomorphism groups of compact manifolds.

As we are primarily interested in weaker conditions on ϕ, it will be useful to establish the following equivalence similar to Proposition 4.2. We recall that a map $\phi\colon X \to Y$ between coarse spaces is modest if the image of every coarsely bounded set in X is coarsely bounded in Y.

Proposition 4.3. *Suppose $\phi\colon H \to G$ is a Baire measurable map from a locally bounded European group H to a European group G. Then ϕ is modest if and only if $\omega_\phi[B \times B]$ is coarsely bounded for every coarsely bounded B.*

Proof. Suppose first that ω_ϕ is modest, i.e., that $\omega_\phi[B \times B]$ is coarsely bounded for every coarsely bounded set B and fix a coarsely bounded set $A \subseteq H$. To see that ϕ is modest, we must show that also $\phi[A]$ is coarsely bounded.

To show this, let V be an arbitrary symmetric open identity neighbourhood in G. Then, as in the proof of Proposition 4.2, there is a finite set F and some n so that $\phi^{-1}\big((VFV)^n\big)$ is comeagre in some coarsely bounded open set $W \subseteq H$, whence by Pettis' lemma [**57**] we have $W^2 = \big[\phi^{-1}\big((VFV)^n\big) \cap W\big]^2$. As A is coarsely bounded, find a finite set $E \ni 1$ and an m so that $A \subseteq (EW^2)^m$. Set also $B = (E \cup W^2)^{\leqslant 2m}$, which is coarsely bounded.

Observe now that, for $e_i \in E$ and $u_i \in W^2$,

$$\begin{aligned}
&\phi(e_1u_1e_2u_2\cdots e_mu_m)\\
&\qquad = \phi(e_1)\phi(u_1e_2u_2\cdots e_mu_m)\omega_\phi(e_1, u_1e_2u_2\cdots e_mu_m)^{-1}\\
&\qquad \in \phi[E]\phi(u_1e_2u_2\cdots e_mu_m)\omega_\phi[B \times B]^{-1}\\
&\qquad \subseteq \phi[E]\phi(u_1)\phi(e_2u_2\cdots e_mu_m)\omega_\phi(u_1, e_2u_2\cdots e_mu_m)^{-1}\omega_\phi[B \times B]^{-1}\\
&\qquad \subseteq \phi[E]\phi[W^2]\phi(e_2u_2\cdots e_mu_m)\omega_\phi[B \times B]^{-1}\omega_\phi[B \times B]^{-1}\\
&\qquad \subseteq \ldots\\
&\qquad \subseteq \big(\phi[E]\phi[W^2]\big)^m\big(\omega_\phi[B \times B]^{-1}\big)^{2m-1}.
\end{aligned}$$

I.e., $\phi[A] \subseteq \big(\phi[E]\phi[W^2]\big)^m\big(\omega_\phi[B \times B]^{-1}\big)^{2m-1}$. But any element $h \in W^2$ can be written as $h = xy$ for some $x, y \in \phi^{-1}\big((VFV)^n\big) \cap W$ and thus

$$\phi(h) = \phi(x)\phi(y)\omega_\phi(x, y)^{-1} \in (VFV)^{2n}\omega_\phi[W \times W]^{-1},$$

that is, $\phi[W^2] \subseteq (VFV)^{2n}\omega_\phi[W \times W]^{-1}$.

Thus, finally,

$$\phi[A] \subseteq \big(\phi[E]\ (VFV)^{2n}\omega_\phi[W \times W]^{-1}\big)^m\big(\omega_\phi[B \times B]^{-1}\big)^{2m-1},$$

where $\omega_\phi[W\times W]$ and $\omega_\phi[B\times B]$ are coarsely bounded and $\phi[E]$ and F finite, showing that also $\phi[A]$ is coarsely bounded.

The converse implication that ω is modest whenever ϕ is follows directly from the inclusion $\omega[B\times B]\subseteq \phi[B^2]^{-1}\phi[B]^2$. □

To understand instead when a map ϕ is bornologous, we need only the following simple observation.

Lemma 4.4. *The following are equivalent for a map $\phi\colon H\to G$ between topological groups.*

1. *ϕ is bornologous,*
2. *ϕ is modest and $\omega_\phi[H\times B]$ is coarsely bounded for every coarsely bounded subset $B\subseteq H$.*

Proof. Assume first that ϕ is bornologous and let $B\subseteq H$ be coarsely bounded. As ϕ is bornologous, fix a coarsely bounded set $C\subseteq G$ so that, $\phi(x)^{-1}\phi(y)\in C$ whenever $x^{-1}y\in B^{-1}$. Then, for all $x\in H$ and $b\in B$, we have

$$\omega_\phi(x,b)=\phi(xb)^{-1}\phi(x)\phi(b)\in C\cdot\phi[B].$$

As ϕ is bornologous and thus modest, $C\cdot\phi[B]$ is coarsely bounded and so also $\omega_\phi[H\times B]$ is coarsely bounded.

Conversely, if (2) holds and $B\subseteq H$ is coarsely bounded, note that, for $x\in H$ and $b\in B$, we have

$$\phi(x)^{-1}\phi(xb)=\phi(xb\cdot b^{-1})^{-1}\phi(xb)\phi(b^{-1})\phi(b^{-1})^{-1}\in\omega_\phi[H\times B^{-1}]\cdot\phi[B^{-1}]^{-1},$$

showing that $\phi(x)^{-1}\phi(y)$ belongs to the coarsely bounded set $\omega_\phi[H\times B^{-1}]\cdot\phi[B^{-1}]^{-1}$ whenever $x^{-1}y\in B$. So ϕ is bornologous. □

The following is now an immediate consequence of Proposition 4.3 and Lemma 4.4.

Proposition 4.5. *Suppose $\phi\colon H\to G$ is a Baire measurable map from a locally bounded European group H to a European group G. Then ϕ is bornologous if and only if $\omega_\phi[H\times B]$ is coarsely bounded for every coarsely bounded B.*

While Proposition 4.5 does not immediately seem to simplify matters, its utility lies in the situation when ϕ is a section for a quotient map $\pi\colon G\to H$. For in this case the image of ω_ϕ is contained in the kernel $K=\ker\pi$, whose coarse structure we may know independently of that of G. For example, if, for every coarsely bounded B, $\omega_\phi[H\times B]$ is coarsely bounded in K and thus also in G, then we may conclude that $\phi\colon H\to G$ is bornologous.

2. Local boundedness of extensions

Before directly addressing the coarse structure of extensions, we will consider the specific issue of when a Polish group extension is locally bounded. That is, suppose K is a closed normal subgroup of a Polish group G. Under what assumptions on K and G/K is G locally bounded?

Observe first that, as the quotient map $\pi\colon G\to G/K$ is open, the image of a coarsely bounded identity neighbourhood in G will be a coarsely bounded identity neighbourhood in G/K. So G/K is locally bounded whenever G is. Conversely, one would like to establish local boundedness of G exclusively from knowledge of K

and the quotient G/K. Though the following problem is unlikely to have a positive answer, several positive instances will be established.

PROBLEM 4.6. *Suppose that K is a closed subgroup of a Polish group G and that both K and G/K are locally bounded. Does it follow that also G is locally bounded?*

We begin by characterising coarsely bounded identity neighbourhoods in G.

LEMMA 4.7. *Suppose K is a closed normal subgroup of a Polish group G and let $\pi\colon G \to G/K$ denote the quotient map. Then the following are equivalent for an identity neighbourhood V in G.*

(1) *V is coarsely bounded,*
(2) *$\pi[V]$ coarsely bounded in G/K and $(FV)^m \cap K$ is coarsely bounded in G for all finite $F \subseteq G$ and m.*

PROOF. The implication from (1) to (2) is trivial, so assume instead that (2) holds. To see that V is coarsely bounded in G, assume that $W \subseteq V$ is a symmetric identity neighbourhood in G. Then, as π is an open mapping, $\pi[W]$ is an identity neighbourhood in G/K. Since $\pi[V]$ is coarsely bounded in G/K, there are $F \subseteq G/K$ finite and m so that $\pi[V] \subseteq (F\pi[W])^m$. Choose a finite set $E \subseteq G$ so that $\pi[E] = F$ and note that then $\pi[V] \subseteq \pi\big[(EW)^m\big]$, i.e.,

$$V \subseteq (EW)^m \cdot \ker \pi = (EW)^m \cdot K.$$

Thus, every $v \in V$ may be written as $v = xk$, for some $x \in (EW)^m$ and $k \in K$, that is, $k = x^{-1}v \in (WE^{-1})^m V \cap K \subseteq (VE^{-1})^m V \cap K$. However, since $(VE^{-1})^m V \cap K$ is coarsely bounded in G, there are a finite set $D \subseteq G$ and an n so that $(VE^{-1})^m V \cap K \subseteq (DW)^n$. It follows that

$$V \subseteq (EW)^m \cdot (DW)^n,$$

verifying that V is coarsely bounded in G. □

We now establish an exact equivalence for local boundedness of G.

PROPOSITION 4.8. *Suppose K is a closed normal subgroup of a Polish group G. Then the following conditions are equivalent.*

(1) *G is locally bounded,*
(2) *G/K is locally bounded and there is a compatible left-invariant metric d on G so that $d|_K$ metrises the coarse structure on K induced from G.*

PROOF. Suppose first that G is locally bounded and fix a coarsely proper metric on G. Then d metrises the coarse structure on G and thus $d|_K$ also metrises the coarse structure on K induced from G. Similarly, as noted above, G/K will be locally bounded.

Conversely, suppose that (2) holds and let d be the given metric. As G/K is locally bounded, fix a coarsely proper metric ∂ on G/K. We claim that the metric

$$D(g, f) = d(g, f) + \partial(\pi(g), \pi(f)),$$

is coarsely proper on G, where $\pi\colon G \to G/K$ is the quotient map $\pi(g) = gK$.

Indeed, if not, there is a D-bounded sequence (g_n) in G, which is unbounded in some other compatible left-invariant metric $\rho \geqslant d$. Note now that the formula

$$\rho_H(gK, fK) = \inf_{k \in K} \rho(g, fk)$$

defines a continuous left-invariant metric on the quotient group G/K. As (g_n) is D-bounded, $\big(\pi(g_n)\big)$ is ∂-bounded and thus coarsely bounded in G/K. In particular, $\big(\pi(g_n)\big)$ is ρ_H-bounded, whereby

$$\sup_n d(g_n, k_n) \leqslant \sup_n \rho(g_n, k_n) < \infty$$

for some sequence (k_n) in K. But then (k_n) is ρ-unbounded and thus also coarsely unbounded in G. As $d|_K$ metrises the coarse structure on K induced from G, (k_n) must be d-unbounded, whence (g_n) must be d and D-unbounded, which is absurd. □

Recall that a closed subgroup K of a Polish group G is coarsely embedded if the coarse structure on K coincides with that induced from G and that K is well-embedded if there is a compatible left-invariant metric d on G so that $d|_K$ is coarsely proper on K.

Corollary 4.9. *Suppose K is a coarsely embedded closed normal subgroup of a Polish group G. Then G is locally bounded if and only if G/K is locally bounded and K is well-embedded.*

Proof. Observe that, if K is coarsely embedded in G and d is a compatible left-invariant metric on G, then $d|_K$ is coarsely proper on K if and only if it metrises the coarse structure on K induced from G. □

Thus, if one can construct a coarsely, but not well-embedded closed normal subgroup K of a Polish group G so that the quotient G/K is locally bounded, then this will provide a counter-example to Problem 4.6.

When G is generated by closed locally bounded subgroups, local boundedness of G easily follows.

Lemma 4.10. *Suppose G is a Polish group generated by closed locally bounded subgroups K and F. Then G is locally bounded.*

Proof. Since K and F are locally bounded, they admit coverings $U_1 \subseteq U_2 \subseteq \ldots \subseteq K$ and $V_1 \subseteq V_2 \subseteq \ldots \subseteq F$ by coarsely bounded open identity neighbourhoods. It follows that the $F_n = \overline{(U_n V_n)^n}$ are coarsely bounded closed subsets covering G and hence, by the Baire category theorem, some F_n must have non-empty interior. Thus $F_n^{-1}F_n$ witnesses local boundedness of G. □

Let Γ be a denumerable discrete group and consider the semidirect product $G = \mathbb{Z} \ltimes \Gamma^{\mathbb{Z}}$, which is easily seen to be locally bounded, but nevertheless has an open subgroup, namely $\Gamma^{\mathbb{Z}}$, which is not locally bounded. On the other hand, there are groups whose local boundedness is truly locally caused. For this, suppose that U is a subset of a group G. We define a product $x_1x_2\cdots x_n$ of elements $x_i \in U$ to be U-admissible if there is a way to distribute parentheses so that the product may be evaluated in the local group $(U,\cdot)$. Formally, the set of *U-admissible* products is the smallest set of products so that

(1) if $x \in U$, the single factor product x is U-admissible,
(2) if $x_1\cdots x_n$ and $y_1\cdots y_m$ are U-admissible and $x_1\cdots x_n \cdot y_1\cdots y_m \in U$, then also $x_1\cdots x_n \cdot y_1\cdots y_m$ is U-admissible.

Strictly speaking, we should be talking about the U-admissibility of the sequence $(x_1,\ldots,x_n)$ rather than of the explicit product $x_1\cdots x_n$, but this suggestive misuse of language should not cause any confusion.

For example, while all terms and the value of the product $3 \cdot 3^{-1} \cdot 3$ belong to $\{3, 3^{-1}\}$, the product $3 \cdot 3^{-1} \cdot 3$ is not $\{3, 3^{-1}\}$-admissible since for both ways of placing the parentheses, $(3 \cdot 3^{-1}) \cdot 3$ and $3 \cdot (3^{-1} \cdot 3)$, one will get a product $(3 \cdot 3^{-1})$ or $(3^{-1} \cdot 3)$ whose value is not in $\{3, 3^{-1}\}$.

Definition 4.11. *A topological group G is* ultralocally bounded *if every identity neighbourhood U contains a further identity neighbourhood V so that, whenever W is an identity neighbourhood, there is a finite set F and a $k \geqslant 1$ for which every element $v \in V$ can be written as a U-admissible product $v = x_1 \cdots x_k$ with terms $x_i \in F \cup W$.*

In other words, a group is ultralocally bounded if it is locally bounded and this can be witnessed entirely within arbitrary small identity neighbourhoods. Let us also point out the following series of implications.

$$\begin{aligned} &\text{locally compact} \;\Rightarrow\; \text{locally Roelcke precompact} \\ &\Rightarrow\; \text{ultralocally bounded} \;\Rightarrow\; \text{locally bounded.} \end{aligned}$$

The only new fact here is that locally Roelcke precompact groups are ultralocally bounded. To see this, suppose U is an identity neighbourhood in a locally Roelcke precompact group G. Pick a Roelcke precompact symmetric identity neighbourhood V so that $V^5 \subseteq U$ and assume a further identity neighbourhood $W \subseteq V$ is given. By Roelcke precompactness, there is a finite set $F \subseteq G$ so that $V \subseteq WFW$. Thus, any element $v \in V$ can be written as a product $v = w_1 f w_2$ of elements $f \in F$ and $w_i \in W \subseteq V$, whence also $f = w_1^{-1} v w_2^{-1} \in V^3$. In other words, setting $F' = F \cap V^3$, we still have $V \subseteq WF'W$. Now, since $V^5 \subseteq U$, $W \subseteq V$ and $F' \subseteq V^3$, any product of the type $w_1 f w_2$ for $w_i \in W$ and $f \in F'$ is U-admissible, so this verifies ultralocal boundedness of G.

Similarly, suppose that d is a compatible left-invariant geodesic metric on a topological group G. Then G is ultralocally bounded. For, if U is any identity neighbourhood, pick $\alpha > 0$ so that the open ball $V = B_d(\alpha)$ is contained in U. Suppose now that $W = B_d(\beta)$ is a further identity neighbourhood and that $v \in V$. Then we can write $v = w_1 \cdots w_n$ for some $w_i \in W$ and $n \leqslant \lceil \frac{\alpha}{\beta} \rceil$ minimal so that $n\beta > d(v, 1)$. In particular, $d(w_1 \cdots w_i, 1) \leqslant d(w_1, 1) + \ldots + d(w_i, 1) < i \cdot \beta \leqslant d(v, 1) \leqslant \alpha$ for all $i < n$ and so distributing parentheses to the left shows that $w_1 \cdots w_n$ is a U-admissible product. Evidently, the assumption that d is geodesic may be weakened considerably.

As an easy application of the following proposition, we have that any extension of an ultralocally bounded topological group by a discrete group is locally bounded.

Proposition 4.12. *Suppose Γ is a discrete normal subgroup of a topological group G. Then G is ultralocally bounded if and only if G/Γ is. Similarly, G is locally Roelcke precompact if and only if G/Γ is.*

Proof. Since Γ is discrete, pick a symmetric open identity neighbourhood $U \subseteq G$ so that $\Gamma \cap U^3 = \{1\}$, whence the restriction $\pi \colon U \to G/\Gamma$ is injective. As π is also an open map, it follows that $\pi[U]$ is an open identity neighbourhood in G/Γ and that the inverse $\omega \colon \pi[U] \to U$ to $\pi \colon U \to \pi[U]$ is continuous.

We claim that $\pi \colon U \to \pi[U]$ is a homeomorphic isomorphism of the local groups $(U, \cdot)$ and $(\pi[U], \cdot)$ with inverse $\omega \colon \pi[U] \to U$. Specifically, the following two properties hold.

(1) If $u_1, u_2 \in U$ satisfy $u_1u_2 \in U$, then also $\pi(u_1)\pi(u_2) \in \pi[U]$ and
$$\pi(u_1)\pi(u_2) = \pi(u_1u_2).$$
(2) If $w_1, w_2 \in \pi[U]$ satisfy $w_1w_2 \in \pi[U]$, then also $\omega(w_1)\omega(w_2) \in U$ and
$$\omega(w_1)\omega(w_2) = \omega(w_1w_2).$$

Property (1) is immediate from the fact that π is a group homomorphism. On the other hand, to verify (2), suppose that $w_1, w_2 \in \pi[U]$ satisfy $w_1w_2 \in \pi[U]$. Then, as
$$\pi\big(\omega(w_1)\omega(w_2)\big) = \pi\big(\omega(w_1)\big)\pi\big(\omega(w_2)\big) = w_1w_2 = \pi\big(\omega(w_1w_2)\big),$$
there is $\gamma \in \Gamma$ so that $\omega(w_1)\omega(w_2) = \omega(w_1w_2)\gamma$ and so $\gamma \in \Gamma \cap U^3 = \{1\}$, i.e., $\omega(w_1)\omega(w_2) = \omega(w_1w_2) \in U$.

Now, ultralocal boundedness of G and G/Γ is exclusively dependent on the isomorphic topological local groups $(U,\cdot)$ and $(\pi[U],\cdot)$ and therefore G is ultralocally bounded if and only if G/Γ is. Indeed, it suffices to notice that π maps U-admissible products to $\pi[U]$-admissible products and, conversely, ω maps $\pi[U]$-admissible products to U-admissible products.

Similarly, as shown above, a topological group is locally Roelcke precompact exactly when every identity neighbourhood U contains a further identity neighbourhood V with the following property: For any identity neighbourhood W, there is a finite set F so that any element $v \in V$ can be written as a U-admissible product $v = w_1fw_2$ with $w_i \in W$, $f \in F$. So again this preserved under isomorphism of the local groups given by identity neighbourhoods. □

Now, as shown by M. Culler and the author in [**63**], if M is a compact manifold of dimension $\geqslant 2$, the group $\mathrm{Homeo}_0(M)$ of isotopically trivial homeomorhisms is not locally Roelcke precompact. Nevertheless, by the results of R. D. Edwards and R. C. Kirby [**22**], $\mathrm{Homeo}_0(M)$ is locally contractible and, by [**46**], also locally bounded. While this latter fact is established using the results of Edwards and Kirby, their results appear not to decide the following problem.

PROBLEM 4.13. *Suppose M is a compact manifold. Is* $\mathrm{Homeo}_0(M)$ *ultralocally bounded?*

Apart from the connection between local boundedness and metrisability of the coarse structure, one reason for our interest in local boundedness of group extensions is that it allows us to construct well-behaved sections for the quotient map. While, by a result of J. Dixmier [**21**], every continuous epimorphism between Polish groups admits a Borel section, modesty seems to require additional assumptions.

PROPOSITION 4.14. *Suppose $\pi\colon G \to H$ is a continuous epimorphism between Polish groups and assume that G is locally bounded. Then there is Borel measurable modest section for π.*

PROOF. Set $K = \ker\pi$ and fix an increasing exhaustive sequence of coarsely bounded open sets $V_1 \subseteq V_2 \subseteq \ldots \subseteq G$ so that $V_n^2 \subseteq V_{n+1}$. For each n, consider the map $x \in V_n \mapsto V_n \cap xK \in F(V_n)$ into the Effros–Borel space of closed subsets of the Polish space V_n. This will be Borel measurable, since for every open subset $U \subseteq V_n$ we have
$$(V_n \cap xK) \cap U \neq \emptyset \;\Leftrightarrow\; x \in UK.$$

By the selection theorem of K. Kuratowski and C. Ryll-Nardzewski [**45**], there is a Borel selector $s\colon F(V_n)\setminus\{\emptyset\}\to V_n$, and so $\sigma(x)=s(V_n\cap xK)$ defines a Borel map $\sigma\colon V_n\to V_n$.

Observe now that, by the open mapping theorem, the images $U_n=\pi[V_n]$ are open in H and define $\phi_n\colon U_n\to V_n$ by

$$\phi_n(y)=x \quad\Leftrightarrow\quad \sigma(x)=x \;\&\; \pi(x)=y.$$

As ϕ_n has Borel graph, it is Borel measurable and is clearly a section for $\pi\colon V_n\to U_n$.

In order to obtain a Borel measurable global section for $\pi\colon G\to H$, it now suffices to set

$$\phi(y)=\phi_n(y) \text{ where } y\in U_n\setminus U_{n-1}.$$

To see that ϕ is modest, note that the U_n are an increasing exhaustive sequence of open subsets of H satisfying $U_n^2\subseteq U_{n+1}$. Thus, if A is coarsely bounded in H, it will be contained in some U_n, whereby $\phi[A]\subseteq\phi[U_n]\subseteq V_n$ and thus $\phi[A]$ is coarsely bounded in G. □

Let us also note that Proposition 4.14 admits a partial converse.

Corollary 4.15. *Suppose $\pi\colon G\to H$ is a continuous epimorphism between Polish groups and assume that both H and $K=\ker\pi$ are locally bounded. Then the following are equivalent,*

(1) *G is locally bounded,*
(2) *there is modest section for π,*
(3) *there is Borel measurable modest section for π.*

Proof. By Proposition 4.14, if G is locally bounded, then there is a modest Borel measurable section for π. This shows (1)⇒(3). Also, (3)⇒(2) is trivial. Finally, for (2)⇒(1), suppose ϕ is a modest section for π. Note that, as K and H are locally bounded, there are increasing exhaustive sequences $A_1\subseteq A_2\subseteq\ldots\subseteq K$ and $B_1\subseteq B_2\subseteq\ldots\subseteq H$ of coarsely bounded sets. Since also A_n and $\phi[B_n]$ are coarsely bounded in G, it follows that $A_n\cdot\phi[B_n]$ is an increasing sequence of coarsely bounded sets covering G, whence G is locally bounded. □

The utility of this corollary is nevertheless somewhat restricted by the fact that to verify that a certain section ϕ is modest, one must already have some understanding of the coarse structure of G.

3. Refinements of topologies

We shall now encounter a somewhat surprising phenomenon concerning sections of quotient maps. Indeed, suppose $\pi\colon G\to H$ is a continuous epimorphism between topological groups and suppose $H'\leqslant H$ is a subgroup equipped with some finer group topology. Then there is a canonical group topology on the group of lifts $G'=\pi^{-1}(H')$, namely, the one in which open sets have the form

$$V\cap\pi^{-1}(U),$$

for V open in G and U open in H'.

We observe that, if G is a Polish group and H' is Polish in its finer group topology, then also G' is Polish in this canonical group topology. To see this, observe first that G' is separable. Secondly, let d_G and $d_{H'}$ be compatible, complete, but not necessarily left-invariant metrics on G and H' respectively. Then we obtain a compatible metric on G' by setting

$$d_{G'}(g,f)=d_G(g,f)+d_{H'}\big(\pi(g),\pi(f)\big)$$

for $g,f\in G'$. It thus suffices to note that $d_{G'}$ is complete. So suppose (g_n) is $d_{G'}$-Cauchy. Then (g_n) is d_G-Cauchy and $\big(\pi(g_n)\big)$ is $d_{H'}$-Cauchy, whence $g=\lim_{d_G} g_n$ and $h=\lim_{d_{H'}}\pi(g_n)$ exist. As the topology on H' refines that of H and $\pi\colon G\to H$ is continuous, we find that also $h=\lim\pi(g_n)=\pi(g)$ in H. It follows that $g=\lim_{d_{G'}} g_n\in\pi^{-1}(H')=G'$, showing that $d_{G'}$ is complete on G'.

Example 4.16. Let M be a compact differentiable manifold with universal cover $\tilde M\xrightarrow{p}M$. Assume that $H=\mathrm{Homeo}(M)$ and that $G\leqslant\mathrm{Homeo}(\tilde M)$ is the set of lifts of homeomorphisms of M to homeomorphisms of $\tilde M$, i.e., homeomorphisms g of $\tilde M$ so that $p\big(g(x)\big)=p\big(g(y)\big)$ whenever $p(x)=p(y)$. As every homeomorphism of M admits a lift to $\tilde M$, this defines a continuous epimorphism $\pi\colon G\to H$ by

$$\pi(g)\big(p(x)\big)=p\big(g(x)\big),$$

where G and H are equipped with the Polish topologies of uniform convergence on $\tilde M$ and M respectively. If now $H'=\mathrm{Diff}^k(M)$, then H' becomes a Polish group in a finer group topology and thus $G'=\pi^{-1}(H')$ is Polish in the lifted group topology described above.

Observe that, if $\phi\colon H\to G$ is a section for the quotient map $\pi\colon G\to H$, then ϕ maps H' into G' and thus remains a section for the restricted quotient map $\pi\colon G'\to H'$. The next result shows that, in common situations, if $\phi\colon H\to G$ is bornologous, then so is $\phi\colon H'\to G'$ despite the changes of topology and thus also of coarse structure.

Proposition 4.17. *Suppose $G\xrightarrow{\pi}H$ is a continuous epimorphism between locally bounded Polish groups so that $K=\ker\pi$ is coarsely embedded in G. Assume also that $H'\leqslant H$ is a subgroup equipped with a finer Polish group topology and let $G'=\pi^{-1}(H')$ be the group of lifts.*

Then if $\phi\colon H\to G$ is a bornologous section for π, also $\phi\colon H'\to G'$ is a bornologous section for the restriction $\pi\colon G'\to H'$.

Proof. Since the G' topology refines that of G, we see that K is closed in G'. Therefore, as there can be no strictly finer Polish group topology on K, the G and G' topologies must coincide on K and hence the inclusion map $K\to G'$ is a continuous homomorphism. Also, if a subset of K is coarsely bounded in G', it is also coarsely bounded in G and thus also in K, as K is coarsely embedded in G. This shows that K is a coarsely embedded closed subgroup of G'.

We first show that a section ϕ is modest as a map from H' to G' provided it is modest from H to G. So assume that $B\subseteq H'$ is coarsely bounded in H'. We must show that $\phi[B]$ is coarsely bounded in G'.

So pick an identity neighbourhood $V\subseteq G'$, which is coarsely bounded as a subset of G, and note, as $G'\xrightarrow{\pi}H'$ is an open map, that $\pi[V]$ is also an identity neighbourhood in H'. So find a finite set $E\subseteq G'$ and an n so that $B\subseteq\big(\pi[E]\pi[V]\big)^n=\pi\big[(EV)^n\big]$. Then, for each $x\in B$, we have $\phi(x)k\in(EV)^n$ for some $k\in K$, in fact, $k\in K\cap\phi[B]^{-1}(EV)^n$. As the inclusion map $H'\to H$ is a continuous homomorphism, B is also coarsely bounded in H, and, since $\phi\colon H\to G$ is modest, we see that $\phi[B]^{-1}(EV)^n$ is coarsely bounded G. Therefore, because K

is coarsely embedded in G, we find that the intersection $A = K \cap \phi[B]^{-1}(EV)^n$ is coarsely bounded in K and hence also in G'. It thus follows that $A^{-1} \subseteq (FV)^m$ for some finite set $E \subseteq F \subseteq G'$ and $m \geqslant 1$, whence

$$\phi[B] \subseteq (EV)^n A^{-1} \subseteq (FV)^{n+m}.$$

Given that V is an arbitrarily small identity neighbourhood in G', this implies that $\phi[B]$ is coarsely bounded in G' and hence that $\phi\colon H' \to G'$ is a modest mapping.

Now, suppose instead that $\phi\colon H \to G$ is bornologous. Fix also a coarsely bounded subset B of H'. Then B is coarsely bounded in H, as the inclusion $H' \to H$ is continuous. Furthermore, by Lemma 4.4, the subset $\omega_\phi[H \times B]$ of K is coarsely bounded in G and thus also in K, since K is coarsely embedded in G. Therefore, also $\omega_\phi[H' \times B]$ is coarsely bounded in K and hence also in G'. Since $\phi\colon H' \to G'$ is modest, by Lemma 4.4, we conclude that $\phi\colon H' \to G'$ is bornologous. □

4. Group extensions

We return to our stated problem of how coarse structure is preserved in short exact sequences

$$1 \to K \xrightarrow{\iota} G \xrightarrow{\pi} H \to 1,$$

where K is a closed normal subgroup of G and ι the inclusion map. As our goal is not to enter on a detailed study of group cohomology, we shall restrict ourselves to some basic cases that appear in practice. Oftentimes these are central extensions, but there are several examples of a more general case, which still admit a good theory, and we shall therefore develop a slightly wider framework. Indeed, the main setting will be when G is generated by K and its centraliser $C_G(K) = \{g \in G \mid \forall k \in K\ gk = kg\}$, i.e., $G = K \cdot C_G(K)$. Recall that

$$Z(K) = \{g \in K \mid \forall k \in K\ gk = kg\} = K \cap C_G(K)$$

denotes the centre of the group K.

Definition 4.18. *A* cocycle *from a group H to a group K is a map $\omega\colon H \times H \to Z(K)$ satisfying the cocycle equation*

$$\omega(h_1, h_2)\omega(h_1h_2, h_3) = \omega(h_2, h_3)\omega(h_1, h_2h_3)$$

for all $h_1, h_2, h_3 \in H$.

Applying the cocycle equation to the triples $(h_1, h_2, h_3) = (1, 1, x)$ and $(x, 1, 1)$, one immediately gets that

$$\omega(x, 1) = \omega(1, x) = \omega(1, 1)$$

for all $x \in H$. Requiring moreover that $\omega(1, 1) = 1$, we obtain the *normalised* cocycles, which correspond to sections ϕ with $\phi(1) = 1$. The presentation of normalised cocyles is somewhat simpler than the general case, but for us it will be convenient not to have this additional restriction on sections ϕ of quotient maps.

In any case, applying again the cocycle equation to the triple (x, x^{-1}, x), one obtains

$$\omega(x, x^{-1}) = \omega(x^{-1}, x).$$

Using this, we observe that, if $\omega\colon H \times H \to Z(K)$ is a cocycle, then one may define a group multiplication on the cartesian product $K \times H$ by

$$(k_1, h_1) \cdot (k_2, h_2) = \big(k_1k_2\omega(h_1, h_2)\omega(1, 1)^{-1}, h_1h_2\big)$$

with identity element $\big(1,1\big)$ and inverse operation

$$(k,h)^{-1} = \big(k^{-1}\omega(h,h^{-1})^{-1}\omega(1,1),h^{-1}\big).$$

We let $K \times_\omega H$ denote the group thus obtained.

Observe that K is naturally homomorphically embedded into $K \times_\omega H$ via the map $\iota(k) = (k,1)$, while $\pi\colon K \times_\omega H \to H$ given by $\pi(k,h) = h$ is an epimorphism with kernel $\iota[K]$. It follows that $K \times_\omega H$ is an extension of H by K with corresponding exact sequence

$$1 \to K \xrightarrow{\iota} K \times_\omega H \xrightarrow{\pi} H \to 1.$$

Unless ω is trivial, the section $\phi\colon H \to K \times_\omega H$ of the quotient map π given by $\phi(h) = \big(\omega(1,1),h\big)$ is only an injection and not an embedding of H into $K \times_\omega H$. Observe also that, while K may not be central in $K \times_\omega H$, we have

$$\iota(k)\phi(h) = \big(k\omega(1,1),h\big) = \phi(h)\iota(k)$$

and hence K is centralised by $\phi[H]$. Identifying K with its image in $K \times_\omega H$ via ι, we see that $K \times_\omega H = K \cdot \phi[H]$ and thus $K \times_\omega H = K \cdot C_{K\times_\omega H}(K)$ and $Z(K) \subseteq Z(K \times_\omega H)$. It follows also that K is central in $K \times_\omega H$ if and only if K is abelian. Moreover, observe that ω and ϕ are related via

$$\omega(h_1,h_2) = \phi(h_1h_2)^{-1}\phi(h_1)\phi(h_2)$$

or, more formally, via the equation $\iota\big(\omega(h_1,h_2)\big) = \phi(h_1h_2)^{-1}\phi(h_1)\phi(h_2)$.

Assume now conversely that

$$1 \to K \xrightarrow{i} G \xrightarrow{p} H \to 1$$

is an extension of a group H by a group K and that, identifying K with its image by i, we have $G = K \cdot C_G(K)$. Suppose also that $\phi\colon H \to C_G(K)$ is a section of the quotient map p and define $\omega\colon H \times H \to Z(K)$ by

$$\omega(h_1,h_2) = \phi(h_1h_2)^{-1}\phi(h_1)\phi(h_2),$$

whence $\omega(1,1) = \phi(1)$. Then the map $\alpha\colon K \times_\omega H \to G$ defined by $\alpha(k,h) = k\phi(h)\omega(1,1)^{-1}$ is an isomorphism so that the following diagram commutes

$$\begin{tikzcd}
K \arrow[r, "\iota"] \arrow[d, "\mathrm{id}"] & K \times_\omega H \arrow[r, "\pi"] \arrow[d, "\alpha"] & H \arrow[d, "\mathrm{id}"] \\
K \arrow[r, "i"] & G \arrow[r, "p"] & H
\end{tikzcd}$$

So, for a group K, cocycles $\omega\colon H \times H \to Z(K)$ and extensions G of H by K with $G = K \cdot C_G(K)$ are dual to each other via the above construction.

Now suppose that H and K are topological groups. Then, if $\omega\colon H\times H \to Z(K)$ is a continuous cocycle, the induced group multiplication on the cartesian product $K \times H$ is continuous and thus $K \times_\omega H$ is a topological group.

However, even if G is a central extension of H by K with continuous bonding maps i and p, then G need not be given as $K \times_\omega H$ for some continuous cocycle ω. Indeed, suppose ω is continuous and $\beta\colon K \times_\omega H \to G$ is a topological isomorphism so that the diagram

$$\begin{tikzcd} K \arrow[r, "\iota"] \arrow[d, "\mathrm{id}"] & K \times_\omega H \arrow[r, "\pi"] \arrow[d, "\beta"] & H \arrow[d, "\mathrm{id}"] \\ K \arrow[r, "i"] & G \arrow[r, "p"] & H \end{tikzcd}$$

commutes. Then $\psi(h) = \beta(1, h)$ is a continuous section $\psi\colon H \to G$ for the quotient map $p\colon G \to H$ and this need not exist in general. Consider, for example, the extension $\mathbb{Z} \to \mathbb{R} \to \mathbb{T}$.

Example 4.19 (Bartle–Graves selectors). Let us remark that, by the existence of Bartle–Graves selectors (see Corollary 7.56 [**26**]), every surjective bounded linear operator $T\colon X \to Z$ between Banach spaces admits a continuous modest section. Thus, if Y is a closed linear subspace of a Banach space X, then X is isomorphic to a *twisted sum* $Y \times_\omega X/Y$ as above.

Remark 4.20. While the cocycle ω and thus the description of an extension G of H by K as a product $K \times_\omega H$ depends on the specific choice of section $\phi\colon H \to C_G(K)$ for the quotient map $\pi\colon G \to H$, the function $\sigma\colon H \times H \to G$ defined by $\sigma(h_1, h_2) = [\phi(h_1), \phi(h_2)]$ is independent of ϕ. Also, if $H \cong G/K$ is given the quotient topology, in which case $\pi\colon G \to H$ is continuous and open, the map $\sigma\colon H \times H \to G$ becomes continuous.

5. External extensions

The problem of understanding the coarse structure of extensions splits into two tasks. Namely, on the one hand, we must analyse *external* extensions of topological groups K and H given as the skewed product $K \times_\omega H$ for some continuous cocycle $\omega\colon H \times H \to Z(K)$. On the other hand, there is the more involved task of *internal* extensions given by short exact sequences

$$1 \to K \xrightarrow{\iota} G \xrightarrow{\pi} H \to 1$$

of topological groups, where K is a closed normal subgroup of G and ι the inclusion map. Observe that, in this case, the section ϕ of the quotient map and the corresponding cocycle ω are not explicitly given and may not, in general, be chosen continuous. Again, in the case of internal extensions, we will mostly consider the case when we have $G = K \cdot C_G(K)$.

Let us stress that, in the case of both external and internal extensions, it is vital to keep track the range of the maps ϕ and ω when discussing their coarse qualities. For example, since $C_G(K)$ and K may not be coarsely embedded in G, ϕ could be bornologous as a map into G, but not as a map into $C_G(K)$ and, similarly, ω could be bornologous as a map into G, but not as a map into $Z(K)$ or even into K. Of course, when dealing with locally compact groups, this issue does not come up, since closed subgroups are automatically coarsely embedded. For general Polish groups, however, this makes computations substantially more delicate.

Nevertheless, in all computations, the only coarse structures that occur are those of K, G and H and hence the issue of when K is coarsely embedded in G becomes important. On the other hand, the coarse structures of the topological groups $C_G(K)$ and $Z(K)$ are irrelevant.

Recall also that K is coarsely embedded in G if the coarse structure on K, when K is viewed as a topological group in its own right, coincides with the coarse structure on G restricted to K, i.e., if a subset $A \subseteq K$ is coarsely bounded in K

exactly when it is coarsely bounded in G. In particular, the latter reformulation shows that being coarsely embedded is independent of whether we talk of the left or the right coarse structure on K and G (as long as we make the same choice for K and G).

Let us begin with the easier of the two tasks, namely, external extensions, where we will require the additional assumption of H being locally bounded. Observe first that, if $\omega\colon H\times H\to Z(K)$ is a cocycle, then the map $\omega'(h_1,h_2)=\omega(h_1,h_2)\omega(1,1)^{-1}$ is a normalised cocyle, i.e., with $\omega'(1,1)=1$. So to simplify the presentation, we would restrict ourselves to the latter.

Theorem 4.21. *Suppose $\omega\colon H\times H\to Z(K)$ is a continuous normalised cocycle from a locally bounded topological group H to the centre of a topological group K and define a section $\phi\colon H\to K\times_\omega H$ of the quotient map by $\phi(h)=(1,h)$.*

1. *Assume that $\omega\colon H\times H\to K$ is modest. Then $\phi\colon H\to K\times_\omega H$ is modest and K is coarsely embedded in $K\times_\omega H$. Moreover, a subset A is coarsely bounded in $K\times_\omega H$ if and only if the two projections $A_K=\mathrm{proj}_K(A)$ and $A_H=\mathrm{proj}_H(A)$ are coarsely bounded in K and H respectively.*
2. *Assume that $\omega[H\times B]$ is coarsely bounded in K for all coarsely bounded $B\subseteq H$. Then $\phi\colon H\to K\times_\omega H$ is bornologous and $K\times_\omega H$ is coarsely equivalent to the direct product $K\times H$ via the formal identity $(k,h)\mapsto(k,h)$.*
3. *Finally, if $\omega[H\times H]$ is coarsely bounded in K, then $\phi\colon H\to K\times_\omega H$ is a quasimorphism.*

Proof. Consider first the case that $\omega\colon H\times H\to K$ is modest and assume that $A\subseteq K\times_\omega H$ and $C\subseteq H$ are coarsely bounded. Let also $V\subseteq K$ be an arbitrary identity neighbourhood and $U\subseteq H$ a coarsely bounded identity neighbourhood. Pick finite sets $1\in E\subseteq K$, $F\subseteq H$ and an $m\geqslant 1$ so that $A\subseteq\big((E\times F)\cdot(V\times U)\big)^m$ and $C\subseteq(FU)^m$.

Then, since elements of $K\times\{1\}$ commute with elements of $\{1\}\times H$, we have

$$\begin{aligned}\big((E\times F)\cdot(V\times U)\big)^m&=\Big((E\times\{1\})\cdot(\{1\}\times F)\cdot(V\times\{1\})\cdot(\{1\}\times U)\Big)^m\\&=\Big((EV)^m\times\{1\}\Big)\cdot\Big((\{1\}\times F)\cdot(\{1\}\times U)\Big)^m.\end{aligned}$$

Also,

$$C\subseteq(FU)^m=\mathrm{proj}_H\Big[\big((\{1\}\times F)\cdot(\{1\}\times U)\big)^m\Big]$$

and, by the assumption on ω,

$$\big((\{1\}\times F)\cdot(\{1\}\times U)\big)^m\subseteq D\times(FU)^m$$

for some coarsely bounded subset D of K.

In particular,

$$A\subseteq\big((E\times F)\cdot(V\times U)\big)^m\subseteq(EV)^mD\times(FU)^m,$$

showing that the two projections $A_K=\mathrm{proj}_K(A)$ and $A_H=\mathrm{proj}_H(A)$ are coarsely bounded in K and H respectively.

Similarly,

$$\begin{aligned}\{1\}\times C&\subseteq(D^{-1}\times\{1\})\cdot\big((\{1\}\times F)\cdot(\{1\}\times U)\big)^m\\&\subseteq(D^{-1}\times\{1\})\cdot\big((E\times F)\cdot(V\times U)\big)^m.\end{aligned}$$

As $D^{-1} \times \{1\}$ is the homomorphic image of a coarsely bounded set in K, it is coarsely bounded in $K \times_\omega H$ and hence $\{1\} \times C$ is coarsely bounded too. Thus, if B is coarsely bounded in K, then $B \times C = (B \times \{1\}) \cdot (\{1\} \times C)$ is coarsely bounded in $K \times_\omega H$.

We have thus shown that a subset $A \subseteq K \times_\omega H$ is coarsely bounded if and only if the two projections $A_K = \mathrm{proj}_K(A)$ and $A_H = \mathrm{proj}_H(A)$ are coarsely bounded in K and H respectively. In particular, a subset $B \subseteq K$ is coarsely bounded if and only if $B \times \{1\}$ is coarsely bounded in $K \times_\omega H$ and so K is coarsely embedded in $K \times_\omega H$. Similarly, if $C \subseteq H$ is coarsely bounded, then $\phi[C] = \{1\} \times C$ is coarsely bounded in $K \times_\omega H$, showing that $\phi\colon H \to K \times_\omega H$ is modest.

Consider now instead the case that $\omega[H \times B]$ is coarsely bounded for every coarsely bounded set B in H. Then $\omega\colon H \times H \to K$ is modest and hence the coarsely bounded subsets of $K \times_\omega H$ are those contained in products $A \times B$ with A and B coarsely bounded in K and H respectively.

To see the formal identity $K \times H \to K \times_\omega H$ is bornologous, suppose that A and B coarsely bounded in K and H respectively. Then, if $k_1^{-1}k_2 \in A$ and $h_1^{-1}h_2 \in B$, also

$$(k_1, h_1)^{-1}(k_2, h_2) = (k_1^{-1}k_2\omega(h_1, h_1^{-1}h_2)^{-1}, h_1^{-1}h_2) \in A\omega[H \times B]^{-1} \times B.$$

As $A\omega[H \times B]^{-1}$ and B are both coarsely bounded, so is $A\omega[H \times B]^{-1} \times B$, as required.

Conversely, to see the inverse formal identity $K \times_\omega H \to K \times H$ is bornologous, suppose that A and B coarsely bounded in K and H respectively. Then, if

$$(k_1, h_1)^{-1}(k_2, h_2) = (k_1^{-1}k_2\omega(h_1, h_1^{-1}h_2)^{-1}, h_1^{-1}h_2) \in A \times B,$$

also $h_1^{-1}h_2 \in B$ and $k_1^{-1}k_2 \in A\omega(h_1, h_1^{-1}h_2) \subseteq A\omega[H \times B]$.

Thus, the formal identity is a coarse equivalence and, since $\phi\colon H \to K \times_\omega H$ is the composition of this with the embedding $H \to K \times H$, also ϕ is bornologous.

Finally, observe that if $\omega[H \times H]$ is coarsely bounded in K, then the defect of ϕ is coarsely bounded in $K \times_\omega H$ and so ϕ is a quasimorphism. $\square$

For good measure, let us point out that Theorem 4.21, item (1), does not imply that $K \times_\omega H$ is coarsely equivalent to the direct product of K and H; only that they have the same coarsely bounded sets under the natural identification.

Among the intended applications of Theorem 4.21 consider a cocycle $\omega\colon \Gamma \times \Gamma \to Z(K)$ defined on a countable discrete group Γ with values in a topological group K. Then ω is both continuous and modest, since it trivially maps finite sets to finite sets.

Corollary 4.22. *Suppose $\omega\colon \Gamma \times \Gamma \to Z(K)$ is a normalised cocycle defined on a countable discrete group Γ with values in the centre of a topological group K. Then K is coarsely embedded in $K \times_\omega \Gamma$. Moreover, a subset $A \subseteq K \times_\omega \Gamma$ is coarsely bounded exactly when contained in the product $B \times F$ of a coarsely bounded set B and a finite set F.*

Similarly, with stronger assumptions on Γ, we have the following.

Corollary 4.23. *Suppose $\omega\colon \Gamma \times \Gamma \to Z(K)$ is a normalised cocycle defined on a countable discrete group Γ with values in the centre of a topological group K and assume that $\omega[\Gamma \times F]$ is coarsely bounded for every finite set F. Then $K \times_\omega \Gamma$ is coarsely equivalent to $K \times \Gamma$.*

6. Internal extensions of Polish groups

We now come to the more delicate task of understanding internal extensions of Polish groups. Thus, in the present section, we consider a short exact sequence of Polish groups with continuous bonding maps

$$1 \to K \xrightarrow{\iota} G \xrightarrow{\pi} H \to 1.$$

Observe that since $\iota[K] = \ker\pi$ is closed, by the open mapping theorem, ι is also open and thus a topological isomorphism between K and $\iota[K]$. Therefore, to simplify notation, we may identify K with its image under ι and thus assume ι to be the inclusion map. Throughout, we shall also assume that G is generated by K and its centraliser, i.e., that $G = K \cdot C_G(K)$.

EXAMPLE 4.24. An example of this setup is when a Polish group G is generated by a discrete normal subgroup $K = \Gamma$ and a connected subgroup F. Then the conjugation action of G on Γ defines a Borel measurable and thus continuous homomorphism $\mathtt{ad}\colon G \to \mathrm{Aut}(\Gamma)$. However, as $\mathrm{Aut}(\Gamma)$ is totally disconnected and F connected, it follows that F is contained in $\ker(\mathtt{ad})$, i.e., that $F \leqslant C_G(\Gamma)$. In other words, $G = \Gamma \cdot C_G(\Gamma)$ and we can let $H = G/\Gamma$.

Working instead exclusively from assumptions on G/Γ, we have the following familiar result.

LEMMA 4.25. *Suppose Γ is a discrete normal subgroup of a topological group G so that G/Γ has no proper open subgroups. Assume also that either Γ is finitely generated or that G/Γ is locally connected. Then $G = \Gamma \cdot C_G(\Gamma)$.*

PROOF. Let $\mathtt{ad}\colon G \to \mathrm{Aut}(\Gamma)$ be the continuous homomorphism defined by the conjugation action of G on Γ. Assume first that Γ is generated by a finite subset E and note that

$$C_G(\Gamma) = \bigcap_{x\in E} \{g \in G \mid \mathtt{ad}(g)(x) = x\}$$

is the intersection of finitely many open sets and thus an open subgroup of G. Since the quotient map $\pi\colon G \to G/\Gamma$ is open, it follows that $\pi\big[C_G(\Gamma)\big]$ is an open subgroup of G/Γ, i.e., $\pi\big[C_G(\Gamma)\big] = G/\Gamma$ and so $G = \Gamma \cdot C_G(\Gamma)$.

Assume now instead that G/Γ is locally connected. Then, as in the proof of Proposition 4.12, we map find an open identity neighbourhood U in G so that $\pi\colon U \to \pi[U]$ is a homeomorphism with an open identity neighbourhood in G/Γ. Shrinking $\pi[U]$ if necessary, we may suppose that $\pi[U]$ and thus also U are connected. It follows that $\mathtt{ad}[U] = \mathrm{id}$, i.e., that $U \subseteq C_G(\Gamma)$. So $C_G(\Gamma)$ is an open subgroup of G and, as before, $G = \Gamma \cdot C_G(\Gamma)$. □

Observe that, if G is a Polish group generated by commuting closed subgroups K and F, then $G = K \cdot F$ and the map

$$(k, f) \in K \times F \mapsto kf \in G$$

is a continuous epimorphism from the Polish group $K \times F$ whose kernel is the central subgroup $N = \{(x, x^{-1}) \in K \times F \mid x \in K \cap F\}$. Thus, by the open mapping theorem, we see that G is isomorphic to the quotient group $\frac{K\times F}{N}$.

We now come to the main result of this section, which establishes coarse embeddability of K from assumptions on $\omega_\phi\colon H \times H \to K$. As always, K will be a closed subgroup of a Polish group G so that $G = K \cdot C_G(K)$ and $\pi\colon G \to H$ is the quotient map to $H = G/K$.

THEOREM 4.26. *Assume that H is locally bounded and that $\phi\colon H \to C_G(K)$ is a C-measurable section for π. Then the following conditions are equivalent,*

(1) *$\phi\colon H \to G$ is modest and K is coarsely embedded in G,*
(2) *$\omega_\phi[B \times B]$ is coarsely bounded in K for all coarsely bounded $B \subseteq H$.*

Before commencing the proof, let us point out a crucial feature of Theorem 4.26. Namely, in (2), $\omega_\phi[B \times B]$ is assumed to be coarsely bounded in K and not just in G. On the one hand, unless we already know something about G, this would probably be our only way to show that $\omega_\phi[B \times B]$ is coarsely bounded in G, but, on the other hand, it is formally stronger and accounts for K being coarsely embedded in G. Notwithstanding this, the non-trivial implication from (2) to (1) allows us to gather important structural information about G and the position of K within G just from knowledge of K, H and an appropriate map between them.

PROOF. Suppose first that $\phi\colon H \to G$ is modest and K is coarsely embedded in G. Then, for every coarsely bounded $B \subseteq H$, we have

$$\omega_\phi[B \times B] \subseteq \big(\phi[B^2]^{-1}\phi[B]^2\big) \cap K,$$

whence $\omega_\phi[B \times B]$ is coarsely bounded in G and thus also in K, as the latter is coarsely embedded. This verifies (1)⇒(2).

Assume instead that (2) holds. Set $\beta = \phi \circ \pi$ and let $\alpha(g) = g\beta(g)^{-1}$, whence $g = \alpha(g)\beta(g)$ is the canonical factorisation of any $g \in G$ into $\alpha(g) \in K$ and $\beta(g) \in C_G(K)$. In particular, $\alpha(k) = k\beta(k)^{-1} = k\phi(1)^{-1}$ for all $k \in K$. Note then that, for all $x, y \in G$, we have

$$\alpha(xy)\beta(xy) = xy = \alpha(x)\beta(x)\alpha(y)\beta(y)$$

and hence

$$\begin{aligned} 1 &= \beta(xy)^{-1}\alpha(xy)^{-1}\alpha(x)\beta(x)\alpha(y)\beta(y) \\ &= \alpha(xy)^{-1}\alpha(x)\alpha(y) \cdot \beta(xy)^{-1}\beta(x)\beta(y) \\ &= \omega_\alpha(x,y) \cdot \omega_\beta(x,y). \end{aligned}$$

In other words, $\omega_\alpha(x,y) = \omega_\beta(x,y)^{-1}$ and so also

$$\alpha(xy) = \alpha(x)\alpha(y) \cdot \omega_\beta(x,y)$$

for all $x, y \in G$. Observe that $\beta\colon G \to G$ and also hence $\alpha\colon G \to K$ are C-measurable.

Note that, for all $x, y \in G$,

$$\begin{aligned} \omega_\beta(x,y) &= \beta(xy)^{-1}\beta(x)\beta(y) \\ &= \phi(\pi(xy))^{-1}\phi(\pi(x))\phi(\pi(y)) \\ &= \phi(\pi(x)\pi(y))^{-1}\phi(\pi(x))\phi(\pi(y)) \\ &= \omega_\phi\big(\pi(x), \pi(y)\big). \end{aligned}$$

So if $B \subseteq G$ is so that $\pi[B]$ is coarsely bounded in H, then by (2) the set $\omega_\beta[B \times B]$ is coarsely bounded in K.

To see that K is coarsely embedded in G, suppose that $A \subseteq K$ is coarsely bounded in G. We must show that A is also coarsely bounded in K. So fix an identity neighbourhood W in K. Then K is covered by countably many left-translates of W and G is covered by the inverse images by α of these. As $\alpha\colon G\to K$ is C-measurable and thus Baire measurable, it follows from the Baire category theorem that there is some $V=kW$ with $k\in K$ so that $\alpha^{-1}(V)$ is comeagre in a non-empty open set $U\subseteq G$. As H is locally bounded, by shrinking U, we may assume that $\pi[U]$ is coarsely bounded in H. Since $\alpha^{-1}(V)$ is comeagre in U, we have by Pettis' lemma [**57**] that

$$U^2=\big(\alpha^{-1}(V)\cap U\big)^2$$

and so $\alpha[U^2]\subseteq V^2\cdot\omega_\beta[U\times U]$.

Also, as A is coarsely bounded in G, there is a finite set $E\subseteq G$ and an m so that $A\subseteq (EU^2)^m$. Set $B=(E\cup U\cup\{1\})^{3m}$ and note that $\pi[B]$ is coarsely bounded in H. Moreover, for all $e_i\in E$ and $u_i\in U^2$,

$$\begin{aligned}\alpha(e_1u_1\cdots e_mu_m)&=\alpha(e_1)\alpha(u_1\cdots e_mu_m)\cdot\omega_\beta(e_1,u_1\cdots e_mu_m)\\&=\dots\\&=\alpha(e_1)\alpha(u_1)\cdots\alpha(e_m)\alpha(u_m)\cdot\omega_\beta(e_m,u_m)\omega_\beta(u_{m-1},e_mu_m)\\&\quad\ \cdots\omega_\beta(u_1,e_2\cdots e_mu_m)\omega_\beta(e_1,u_1\cdots e_mu_m)\\&\in\alpha(e_1)\alpha(u_1)\cdots\alpha(e_m)\alpha(u_m)\cdot\omega_\beta[B\times B]^{2m-1}\\&\subseteq\big(\alpha[E]V^2\omega_\beta[B\times B]\big)^m\cdot\omega_\beta[B\times B]^{2m-1}\\&=\big(\alpha[E]V^2\big)^m\cdot\omega_\beta[B\times B]^{3m-1},\end{aligned}$$

where the last equality follows from the fact that ω_β and α take values in $C_G(K)$ and K respectively. Thus, $\alpha[A]\subseteq\big(\alpha[E]V^2\big)^m\cdot\omega_\beta[B\times B]^{3m-1}$. As $V=kW$ is the translate of an arbitrary identity neighbourhood in K and $\omega_\beta[B\times B]$ is coarsely bounded in K, this shows that $\alpha[A]$ and hence also $A=\alpha[A]\phi(1)$ are coarsely bounded in K. So K is coarsely embedded in G.

Finally, that $\phi\colon H\to G$ is modest follows directly from Proposition 4.3. □

Combining now Theorem 4.26 with Corollary 4.15, we have the following.

COROLLARY 4.27. *Assume that K and H are locally bounded and that $\phi\colon H\to C_G(K)$ is a C-measurable section for π so that $\omega_\phi[B\times B]$ is coarsely bounded in K for all coarsely bounded $B\subseteq H$. Then G is locally bounded, K is coarsely embedded in G and $\phi\colon H\to G$ is modest.*

Similarly to Theorem 4.26, we also obtain a criterion for bornologous sections.

COROLLARY 4.28. *Assume that H is locally bounded and that $\phi\colon H\to C_G(K)$ is a C-measurable section for π. Then the following conditions are equivalent,*

(1) *$\phi\colon H\to G$ is bornologous and K is coarsely embedded in G,*
(2) *$\omega_\phi[H\times B]$ is coarsely bounded in K for all coarsely bounded $B\subseteq H$.*

PROOF. Suppose first that $\phi\colon H\to G$ is bornologous and $B\subseteq H$ is coarsely bounded. Then, by Lemma 4.4, $\omega_\phi[H\times B]$ is coarsely bounded in G. So, if also K is coarsely embedded in G, then $\omega_\phi[H\times B]$ is coarsely bounded in K too.

Conversely, assume $\omega_\phi[H\times B]$ is coarsely bounded in K for all coarsely bounded $B\subseteq H$. Then, by Theorem 4.26, $\phi\colon H\to G$ is modest and K is coarsely embedded in G. Applying Lemma 4.4 once again, we find that $\phi\colon H\to G$ is actually bornologous. □

Occasionally, there may be other ways to see that K is coarsely embedded in G rather than employing the cocycle ω_ϕ. Also, in various cases, it may be more natural to deal with subgroups F of the centraliser $C_G(K)$ instead of all of it.

LEMMA 4.29. *Suppose G is a Polish group generated by commuting closed subgroups K and F, i.e., $[K,F]=1$. Assume also that either*

(1) *K is locally bounded and $K\cap F$ is coarsely embedded in both K and F, or*
(2) *$K\cap F$ is well-embedded in F, that is, that there is a continuous left-invariant metric d on F so that $d|_{K\cap F}$ is coarsely proper on $K\cap F$.*

Then K is coarsely embedded in G.

PROOF. As K and F commute and generate G, the map $(k,f)\in K\times F\mapsto kf\in G$ is a continuous epimorphism between Polish groups and thus, by the open mapping theorem, also open. It follows that G has a neighbourhood basis at the identity consisting of sets VW for open $V\subseteq K$ and $W\subseteq F$. As $K\cap F$ is also central in G, we observe that, if ∂ and d are continuous left-invariant metrics on K and F respectively, we may define a continuous left-invariant metric on G by setting, for $k_i\in K$ and $f_i\in F$,

$$D(k_1f_1,k_2f_2)=\inf_{x\in K\cap F}\partial(k_1,k_2x)+d(f_1x,f_2).$$

Assume first that K is locally bounded with coarsely proper metric ∂ and that $K\cap F$ coarsely embedded in K and in F. Suppose towards a contradiction that K is not coarsely embedded in G. We may then find a sequence $h_n\in K$, which is coarsely bounded in G, but so that $\lim_n\partial(h_n,1)=\infty$. In particular, if D_1 is the metric on G given by

$$D_1(k_1f_1,k_2f_2)=\inf_{x\in K\cap F}\partial(k_1,k_2x),$$

then $\{h_n\}_n$ is D_1-bounded and hence $\sup_n\partial(h_n,x_n)<\infty$ for some sequence of elements $x_n\in K\cap F$. Thus, on the one hand, as ∂ is coarsely proper, the set $\{h_n^{-1}x_n\}_n$ is coarsely bounded in K and hence in G. Therefore, as $\{x_n\}\subseteq\{h_n\}_n\cdot\{h_n^{-1}x_n\}_n$, we see that $\{x_n\}_n$ is coarsely bounded in G. On the other hand, $\lim_n\partial(x_n,1)=\infty$, whereby $\{x_n\}_n$ is coarsely unbounded in K and hence also in $K\cap F$. But, as $K\cap F$ is coarsely embedded in F, $\{x_n\}_n$ will be coarsely unbounded in F and therefore $\sup_n d(x_n,1)=\infty$ for some continuous left-invariant metric d on F.

Let now D_2 be the continuous left-invariant metric on G given by

$$D_2(k_1f_1,k_2f_2)=\inf_{y\in K\cap F}\partial(k_1,k_2y)+d(f_1y,f_2).$$

As $\{x_n\}_n$ is coarsely bounded in G, we have $\sup_n D_2(1,x_n)<\infty$ and thus

$$\sup_n\big(\partial(1,y_n)+d(y_n,x_n)\big)<\infty$$

for some $y_n\in K\cap F$. Since $\sup_n d(x_n,1)=\infty$ and thus also $\sup_n d(y_n,1)=\infty$, it follows that $\{y_n\}_n$ is coarsely unbounded in F and hence also in $K\cap F$ and in K. This, in turn, contradicts that $\sup_n\partial(1,y_n)<\infty$.

Let us now consider the case that $K\cap F$ is well-embedded in F, as witnessed by some continuous left-invariant metric on F. Suppose for a contradiction that K is not coarsely embedded in G. Then there is a sequence $h_n\in K$ so that $\lim_n\partial(h_n,1)=\infty$ for some compatible left-invariant metric ∂ on K, but so that

$\{h_n\}_n$ is coarsely bounded in G. Define a continuous left-invariant metric D_3 on G by setting, for $k_i \in K$ and $f_i \in F$,

$$D_3(k_1f_1, k_2f_2) = \inf_{x \in K \cap F} \partial(k_1, k_2x) + d(f_1x, f_2).$$

As $\{h_n\}_n$ is coarsely bounded in G, it has finite D_3-diameter and we can therefore find $x_n \in K \cap F$ so that $\sup_n \big(\partial(h_n, x_n) + d(x_n, 1)\big) < \infty$. As the metric d is coarsely proper on $K \cap F$, it follows that $\{x_n\}_n$ is coarsely bounded in $K \cap F$ and hence in K, whereby

$$\infty = \sup_n \big(\partial(h_n, 1) - \partial(x_n, 1)\big) \leqslant \sup_n \partial(h_n, x_n) < \infty,$$

which is absurd. $\square$

While in Corollary 4.15 modest sections were used to prove local boundedness of G, thus far we have not employed the added strength of having a bornologous section. This is done in the following.

Proposition 4.30. *Let $\pi\colon G \to H$ be a continuous epimorphism between topological groups. Assume also that the kernel $K = \ker \pi$ is coarsely embedded in G and that $\phi\colon H \to C_G(K)$ is a section for π, which is bornologous as a map $H \xrightarrow{\phi} G$. Then G is coarsely equivalent to $K \times H$.*

Proof. Observe that, since ϕ takes values only in $C_G(K)$, we automatically have $G = K \cdot C_G(K)$. Also, since both $G \xrightarrow{\pi} H$ and $H \xrightarrow{\phi} G$ are bornologous, so is the composition $\beta = \phi \circ \pi$. For simplicity of notation, we let $\alpha(g) = g \cdot \beta(g)^{-1}$, whereby $g = \alpha(g) \cdot \beta(g)$ is the canonical decomposition of any element $g \in G$ with $\alpha(g) \in K$ and $\beta(g) \in C_G(K)$. As in the proof of Theorem 4.26, we see that $\omega_\alpha(x, y) = \omega_\beta(x, y)^{-1}$.

Suppose first that $A \subseteq G$ is coarsely bounded. Then $\beta[A]$ is coarsely bounded in G and $(A \cdot \beta[A]^{-1}) \cap K$ is coarsely bounded in G and hence also in K, since the latter is coarsely embedded in G. As $\alpha[A] \subseteq (A \cdot \beta[A]^{-1}) \cap K$, this shows that $\alpha\colon G \to K$ is modest.

Assume again that $A \subseteq G$ is coarsely bounded. Since $\beta\colon G \to G$ is bornologous, Lemma 4.4 implies that $\omega_\beta[G \times A]$ is coarsely bounded in G and so $\omega_\alpha[G \times A] = \omega_\beta[G \times A]^{-1}$ is coarsely bounded in G and hence also in K. As $\alpha\colon G \to K$ is also modest, another application of Lemma 4.4 shows that $\alpha\colon G \to K$ is bornologous.

Note now that the map $\Theta\colon K \times H \to G$ defined by $\Theta(k, h) = k\phi(h)$ is the composition of the bornologous map $\mathrm{id}_K \times \phi\colon K \times H \to K \times C_G(K)$ and the continuous homomorphism $K \times C_K(G) \to G$, $(k, x) \mapsto kx$. Therefore, Θ is a bornologous map with bornologous inverse $g \in G \mapsto (\alpha(g), \pi(g)) \in K \times H$. It follows that Θ is a coarse equivalence between $K \times H$ and G as claimed. $\square$

Combing now Corollary 4.28 and Proposition 4.30, we arrive at the second main result of this section.

Theorem 4.31. *Suppose $\pi\colon G \to H$ be a continuous epimorphism between Polish groups with kernel K. Assume also that H is locally bounded and that $\phi\colon H \to C_G(K)$ is a C-measurable section so that $\omega_\phi[H \times B]$ is coarsely bounded in K for every coarsely bounded set $B \subseteq H$. Then G is coarsely equivalent to $K \times H$.*

Focusing instead on extensions by discrete groups, we establish the following result.

THEOREM 4.32. *Let G be a Polish group generated by a discrete normal subgroup Γ and a connected closed subgroup F. Assume also that $\Gamma \cap F$ is coarsely embedded in F and that $\phi\colon G/\Gamma \to G$ is a bornologous lift of the quotient map with $\operatorname{im}\phi \subseteq C_G(\Gamma)$. Then G is coarsely equivalent with $\Gamma \times G/\Gamma$.*

PROOF. Observe first that by Example 4.24, F will automatically be a subgroup of the centraliser $C_G(\Gamma)$. Being discrete, Γ is also locally bounded and thus Lemma 4.29 applies to show that Γ is coarsely embedded in G. By Proposition 4.30, we now see that G is coarsely equivalent to $\Gamma \times G/\Gamma$. □

Note that the choice of dealing with the connected subgroup $F \leqslant C_G(K)$ allows us some additional flexibility, which may facilitate the verification that $K \cap F$ is coarsely embedded in F.

7. A further computation for general extensions

We now provide some computations for general extensions of Polish groups, i.e., without assuming centrality.

In a short exact sequence of topological groups

$$1 \to K \xrightarrow{\iota} G \xrightarrow{\pi} H \to 1$$

it is natural to conjecture that, if one of the two groups K and H is trivial, the middle term G should essentially be equal to the other of K and H. In case K is the trivial term, this is verified for coarse structure by the following simple fact.

PROPOSITION 4.33. *Suppose K is a normal subgroup of a topological group G and assume that K is coarsely bounded in G. Then the quotient map*

$$\pi\colon G \to G/K$$

is a coarse equivalence.

PROOF. Since π is a continuous homomorphism, it is bornologous and evidently also cobounded. It thus suffices to show that π is expanding, which, since π is a homomorphism, is equivalent to π being coarsely proper.

So suppose $A \subseteq G$ is not coarsely bounded in G and fix a continuous left-invariant écart d on G so that A has infinite d-diameter. Then the Hausdorff distance d_H on the quotient group G/K, defined by

$$d_H(gK, fK) = \max\big\{\sup_{a\in gK}\inf_{b\in fK} d(a,b), \sup_{b\in fK}\inf_{a\in gK} d(a,b)\big\},$$

is a continuous left-invariant écart and, moreover, satisfies

$$d_H(gK, fK) = \inf_{k\in K} d(g, fk) = \inf_{k\in K} d(gk, f).$$

Since K is coarsely bounded in G, it has finite d-diameter, $\operatorname{diam}_d(K) = C$. Also, as A has infinite d-diameter, there are $x_n \in A$ with $d(x_n, x_n) > n$, whence

$$\begin{aligned} d_H(x_nK, x_1K) &= \inf_{k\in K} d(x_n, x_1k) \\ &\geqslant \inf_{k\in K}\big(d(x_n, x_1) - d(x_1, x_1k)\big) \\ &= \inf_{k\in K}\big(d(x_n, x_1) - d(1, k)\big) \\ &\geqslant n - C. \end{aligned}$$

So $\pi[A]$ has infinite d_H-diameter and therefore fails to be coarsely bounded in G/K. $\square$

8. Coarse structure of covering maps

In the following we fix a path-connected, locally path-connected and locally compact, metrisable space X. We moreover assume that X is *semilocally simply connected*, i.e., that every point $x \in X$ has a neighbourhood V so that any loop lying in V is nullhomotopic in X.

Assume also that Γ is a finitely generated group acting freely and cocompactly by homeomorphisms on X. Furthermore, suppose that the action is *proper*, i.e., that the set $\{a \in \Gamma \mid a \cdot K \cap K \neq \emptyset\}$ is finite for every compact set $K \subseteq X$. We let $M = X/\Gamma$ denote the compact metrisable quotient space and define

$$X \xrightarrow{p} M$$

to be the corresponding covering map. Observe that, as p is locally a homeomorphism, M is also path-connected, locally path-connected and semilocally simply connected. In particular, as X and M are path-connected, the fundamental groups are independent up to isomorphism of the choice of base point.

By a result of R. Arens [**2**], since X is locally connected, the homeomorphism group of the locally compact space X is a topological group when equipped with the compact-open topology, i.e., given by the subbasic open sets

$$O_{C,U} = \{g \in \mathrm{Homeo}(X) \mid g[C] \subseteq U\},$$

where $C \subseteq X$ is compact and $U \subseteq X$ open. This is simply the induced topology on $\mathrm{Homeo}(X)$ when viewed as a closed subgroup of $\mathrm{Homeo}(\hat{X})$, where $\hat{X}$ is the Alexandroff one-point compactification and $\mathrm{Homeo}(\hat{X})$ is equipped with the compact-open topology of $\hat{X}$. Similarly, $\mathrm{Homeo}(M)$ will be equipped with the compact-open topology. As Γ acts freely on X, we may identify Γ with its image in $\mathrm{Homeo}(X)$. Observe that, since Γ acts properly on X, it will be a discrete subgroup of $\mathrm{Homeo}(X)$.

Recall that a homeomorphism $\tilde{h} \in \mathrm{Homeo}(X)$ is a *lift* of a homeomorphism $h \in \mathrm{Homeo}(M)$ provided that

$$\begin{tikzcd} X \arrow[r, "\tilde{h}"] \arrow[d, "p"] & X \arrow[d, "p"] \\ M \arrow[r, "h"] & M \end{tikzcd}$$

commutes. As X may not be simply connected, some $h \in \mathrm{Homeo}(M)$ may not admit a lift. For example, if $X = \mathbb{R} \times S^1$ is the open annulus and $M = S^1 \times S^1$ the torus with $p(t,x) = (e^{2\pi ti}, x)$, then the homeomorphism $h(x,y) = (y,x)$ of M evidently has no lift to X.

However, if $\tilde{h}$ is the lift of some homeomorphism h, then $\pi(\tilde{h}) = h$ is uniquely defined from $\tilde{h}$ by $hp = p\tilde{h}$. Moreover, as the set of all lifts of homeomorphisms of M form a subgroup L of $\mathrm{Homeo}(X)$, this gives us a homomorphism $\pi\colon L \to \mathrm{Homeo}(M)$.

We claim that $\ker \pi = \Gamma$. Indeed, that $\Gamma \leqslant L$ and $\pi(a) = \mathrm{id}_M$ for all $a \in \Gamma$ is immediate. Conversely, suppose that $f \in \ker \pi$. Then, as Γ acts freely on X, the closed sets $X_a = \{x \in X \mid f(x) = a(x)\}$ for $a \in \Gamma$ form a partition of X. However, as Γ also acts properly, each X_a will be open. Thus, as X is connected, we have $X = X_a$ for some $a \in \Gamma$, i.e., $f = a \in \Gamma$.

As $\Gamma = \ker \pi$ is normal in L, this shows that L is contained in the normaliser $N_{\mathrm{Homeo}(X)}(\Gamma)$ of Γ inside $\mathrm{Homeo}(X)$. Conversely, if $f \in \mathrm{Homeo}(X)$ normalises Γ, we may define a homeomorphism $h \in \mathrm{Homeo}(X)$ by $hp(x) = pf(x)$ having f as a lift, showing that $f \in L$. In other words, $L = N_{\mathrm{Homeo}(X)}(\Gamma)$. Observe also that, as Γ is discrete in $\mathrm{Homeo}(X)$, it normaliser is closed.

LEMMA 4.34. *The set*

$$Q = \{h \in \mathrm{Homeo}(M) \mid h \textit{ admits a lift } \tilde{h} \in \mathrm{Homeo}(X)\}$$

is an open subgroup of $\mathrm{Homeo}(M)$.

PROOF. We first show that there is an identity neighbourhood V in $\mathrm{Homeo}(M)$ so that, for any loop $\sigma\colon S^1 \to M$ and homeomorphism $f \in V$, the two maps

$$\sigma\colon S^1 \to M \quad \text{and} \quad f\sigma\colon S^1 \to M$$

are homotopic. In particular, by the lifting of homotopies, σ admits a lift $\tilde{\sigma}\colon S^1 \to X$ if and only if $f\sigma$ admits a lift $\widetilde{f\sigma}\colon S^1 \to X$.

Indeed, let d be a compatible metric on M and fix a covering $\mathcal{U}$ of M by path-connected open sets U so that any loop in U is nullhomotopic in M. Let also $\epsilon > 0$ be a Lebesgue number for $\mathcal{U}$, i.e., so that any set of diameter ϵ is contained in some $U \in \mathcal{U}$. Fix a covering $\mathcal{W}$ of M by path-connected open sets of diameter $< \frac{\epsilon}{2}$ and let $\delta > 0$ be a Lebesgue number for $\mathcal{W}$. We set

$$V = \{f \in \mathrm{Homeo}(M) \mid \sup_{x \in M} d(f(x), x) < \delta\}.$$

Now, suppose that $\sigma\colon S^1 \to M$ and $f \in V$. Find a finite subset $D \subseteq S^1$ so that, for neighbours $x, y \in D$ in the induced circular ordering, the σ-image of the shortest circular arc $I_{x,y} \subseteq S^1$ from x to y has diameter $< \delta$. For each $x \in D$, observe that $d(\sigma(x), f\sigma(x)) < \delta$ and so $\sigma(x)$ and $f\sigma(x)$ belong to some common $W \in \mathcal{W}$, which means that there is a path $\gamma_x \subseteq W$ from $\sigma(x)$ to $f\sigma(x)$. Then, if $x, y \in D$ are neighbours in the circular ordering and $I_{x,y}$ is as above,

$$\gamma_x \centerdot f\sigma[I_{x,y}] \centerdot \overline{\gamma_y} \centerdot \overline{\sigma[I_{x,y}]}$$

is a loop of diameter $< \delta + \frac{\epsilon}{2} < \epsilon$ and thus contained in some $U \in \mathcal{U}$, whence nullhomotopic. As σ and $f\sigma$ are the concatenations of the $\sigma[I_{x,y}]$ and $f\sigma[I_{x,y}]$ respectively, this shows that σ and $f\sigma$ are homotopic in M.

Observe now that, since $X \xrightarrow{p} M$ is a regular covering, i.e., Γ acts transitively on each fibre of p, the criterion for lifting of homeomorphisms of M to X can be formulated as follows. A homeomorphism $h \in \mathrm{Homeo}(M)$ admits a lift $\tilde{h} \in \mathrm{Homeo}(X)$ if and only if, for every loop $\sigma\colon S^1 \to X$, the loop

$$hp\sigma\colon S^1 \to M$$

admits a lift $\widetilde{hp\sigma}\colon S^1 \to X$.

Now suppose $h \in \mathrm{Homeo}(M)$ admits a lift and $f \in V$. Then also fh admits a lift. For, if $\sigma\colon S^1 \to X$, then $hp\sigma\colon S^1 \to M$ admits a lift $\widetilde{hp\sigma}\colon S^1 \to X$ and so, by what was established above, also $fhp\sigma\colon S^1 \to M$ admits a lift $\widetilde{fhp\sigma}\colon S^1 \to X$.

Since V is symmetric, this show that, for $f \in V$, a homeomorphism $h \in \mathrm{Homeo}(M)$ admits a lift if and only if fh does. Thus, as V is open, the set of homeomorphisms with lifts is an open subgroup of $\mathrm{Homeo}(M)$. $\square$

We therefore have a natural epimorphism $\pi\colon N_{\mathrm{Homeo}(X)}(\Gamma) \to Q$ with kernel Γ and hence a short exact sequence of Polish groups

$$1 \to \Gamma \to N_{\mathrm{Homeo}(X)}(\Gamma) \xrightarrow{\pi} Q \to 1.$$

Now, as Γ is finitely generated, its automorphism group $\mathrm{Aut}(\Gamma)$ is countable and so the kernel of the representation by conjugation

$$\mathrm{ad}\colon N_{\mathrm{Homeo}(X)}(\Gamma) \to \mathrm{Aut}(\Gamma)$$

is a countable index open normal subgroup, namely the centraliser $\ker(\mathrm{ad}) = C_{\mathrm{Homeo}(X)}(\Gamma)$ of Γ in $\mathrm{Homeo}(X)$. We set

$$Q_0 = \pi\big[C_{\mathrm{Homeo}(X)}(\Gamma)\big],$$

which, as π is open, is an open normal subgroup of Q. Observe that, since $C_{\mathrm{Homeo}(X)}(\Gamma)$ is open in $N_{\mathrm{Homeo}(X)}(\Gamma)$, also $\Gamma \cdot C_{\mathrm{Homeo}(X)}(\Gamma)$ is an open normal subgroup of $N_{\mathrm{Homeo}(X)}(\Gamma)$, which gives us the exact sequence

$$1 \to \Gamma \to \Gamma \cdot C_{\mathrm{Homeo}(X)}(\Gamma) \xrightarrow{\pi} Q_0 \to 1.$$

Pick now a relatively compact fundamental domain $D \subseteq X$ for the action of Γ and fix a point $x_0 \in D$. Suppose also that ρ is a left-invariant proper metric on Γ. We define an écart d on X by simply letting

$$d(aD \times bD) = \rho(a, b)$$

for $a, b \in \Gamma$. That is, for $x, y \in D$ and $a, b \in \Gamma$, we set $d\big(a(x), b(y)\big) = \rho(a, b)$. Clearly, d is invariant under the action by Γ on X. Moreover, for elements $g, f \in C_{\mathrm{Homeo}(X)}(\Gamma)$,

$$\sup_{x\in X} d\big(g(x), f(x)\big) = \sup_{x\in D} \sup_{a\in\Gamma} d(ga(x), fa(x)) = \sup_{x\in D} d(g(x), f(x)) < \infty,$$

since $a \in \Gamma$ commutes with g, f and acts by isometries. It thus follows that

$$d_\infty(g, f) = \sup_{x\in X} d\big(g^{-1}(x), f^{-1}(x)\big)$$

is a (generally discontinuous) left-invariant écart on $C_{\mathrm{Homeo}(X)}(\Gamma)$.

Observe that the centre $Z(\Gamma)$ of Γ is given by

$$Z(\Gamma) = \Gamma \cap C_{\mathrm{Homeo}(X)}(\Gamma)$$

and suppose henceforth that there is a bornologous section

$$\psi\colon \Gamma/Z(\Gamma) \longrightarrow \Gamma$$

for the quotient map. For example, this happens if $Z(\Gamma)$ is either finite or has finite index in Γ. Then, by Proposition 4.30, the map

$$(a, s) \in Z(\Gamma) \times \Gamma/Z(\Gamma) \mapsto a\psi(s) \in \Gamma$$

is a coarse equivalence and hence

$$a\psi(s) \mapsto a$$

is a bornologous map from $\Gamma = Z(\Gamma) \cdot \mathrm{im}(\psi)$ onto $Z(\Gamma)$.

We claim that every $h \in Q_0$ has a unique lift $\phi(h) \in C_{\mathrm{Homeo}(X)}(\Gamma)$ so that

$$\phi(h)^{-1}(x_0) \in \mathrm{im}(\psi) \cdot D.$$

Indeed, if $\tilde{h} \in C_{\mathrm{Homeo}(X)}(\Gamma)$ is any lift of h, note that, since D is a fundamental domain for the action $\Gamma \curvearrowright X$, there are $a \in Z(\Gamma)$ and $s \in \Gamma/Z(\Gamma)$ so that $\tilde{h}^{-1}(x_0) \in$

$a\psi(s)\cdot D$, whence we may let $\phi(h)=\tilde{h}a$. Uniqueness follows from the fact that any two lifts of $h\in Q_0$ in $C_{\mathrm{Homeo}(X)}(\Gamma)$ differ by an element of $Z(\Gamma)$ and that $\mathrm{im}(\psi)\cdot D$ is a fundamental domain for the action of $Z(\Gamma)$ on X.

LEMMA 4.35. *For every constant c, there is a finite set $A\subseteq Z(\Gamma)$ so that*

$$d_\infty\big(\phi(h),\phi(g)a\big)\leqslant c \quad\Rightarrow\quad a\in A$$

for all $h,g\in Q_0$ and $a\in Z(\Gamma)$.

PROOF. Fix the constant c. Then, as ρ is a proper left-invariant metric on Γ and the map $a\psi(s)\mapsto a$ from $\Gamma=Z(\Gamma)\cdot\mathrm{im}(\psi)$ to $Z(\Gamma)$ is bornologous, there is a finite set $A\subseteq Z(\Gamma)$ so that

$$\rho\big(a\psi(s),b\psi(t)\big)\leqslant c \;\;\Rightarrow\;\; b^{-1}a\in A$$

for all $a,b\in Z(\Gamma)$ and $s,t\in\Gamma/Z(\Gamma)$.

Suppose now that $h,g\in Q_0$ and $a\in Z(\Gamma)$ with $d_\infty\big(\phi(h),\phi(g)a\big)\leqslant c$. Choose $s,t\in\Gamma/Z(\Gamma)$ so that $\phi(h)^{-1}(x_0)\in\psi(s)D$ and $\phi(g)^{-1}(x_0)\in\psi(t)D$. Then $a\phi(h)^{-1}(x_0)$ and $a\psi(s)(x_0)$ both belong to $a\psi(s)\cdot D$ and thus have d-distance 0. Similarly, $d\big(\psi(t)(x_0),\phi(g)^{-1}(x_0)\big)=0$. It follows that

$$\begin{aligned}\rho\big(a\psi(s),\psi(t)\big)&=d\big(a\psi(s)(x_0),\psi(t)(x_0)\big)\\&=\;d\big(a\phi(h)^{-1}(x_0),\phi(g)^{-1}(x_0)\big)\\&\leqslant\;d_\infty\big(\phi(h)a^{-1},\phi(g)\big)\\&=\;d_\infty\big(\phi(h),\phi(g)a\big)\\&\leqslant c\end{aligned}$$

and so $a\in A$ as required. □

Suppose now that H is a subgroup of Q_0, which is Polish in some finer group topology τ_H. For example, M could be a manifold and H the symmetries of some additional structure of M, e.g., a volume form, a differentiable or symplectic structure, and τ_H a canonical topology defined from this additional structure. Note that H will be closed in $\mathrm{Homeo}(M)$ if and only if its Polish topology coincides with that induced from $\mathrm{Homeo}(M)$. Let also $G=\pi^{-1}(H)$ be the group of all lifts of elements in H, whence

$$\Gamma\leqslant G\leqslant\Gamma\cdot C_{\mathrm{Homeo}(X)}(\Gamma).$$

As in Section 3, G is given its canonical topology lifted from H.

We now arrive at the main result of this section, with antecedents in an earlier result of K. Mann and the author (Theorem 30 [**46**]). The latter dealt exclusively with the fundamental group $\Gamma=\pi_1(M)$ of a compact manifold M acting by deck-transformations on the universal cover $X=\tilde{M}$. However, even in this case, the assumptions on Γ were slightly different from those below, since one required a bornologous section $\phi\colon\Gamma/A\to\Gamma$, where A is a specific geometrically defined central subgroup of Γ. Moreover, in that result, the subgroup H was simply $\mathrm{Homeo}_0(M)$ itself.

So, to state the theorem, let us briefly summarise the setup. We are given a proper, free and cocompact action $\Gamma\curvearrowright X$ of a finitely generated group Γ on a path-connected, locally path-connected and semilocally simply connected, locally compact metrisable space X. Then the centraliser $C_{\mathrm{Homeo}(X)}(\Gamma)$ is an open subgroup of the normaliser $N_{\mathrm{Homeo}(X)}(\Gamma)$, which, in turn, is the group of all lifts of homeomorphisms of $M = X/\Gamma$ to X. Let

$$N_{\mathrm{Homeo}(X)}(\Gamma) \xrightarrow{\pi} \mathrm{Homeo}(M)$$

be the corresponding quotient map and let

$$Q_0 = \pi\big[C_{\mathrm{Homeo}(X)}(\Gamma)\big]$$

be the open subgroup of Homeo(M) consisting of homeomorphisms admitting lifts in $C_{\mathrm{Homeo}(X)}(\Gamma)$.

THEOREM 4.36. *Suppose that there is a bornologous section $\psi\colon \Gamma/Z(\Gamma) \to \Gamma$ for the quotient map. Assume that H is a subgroup of Q_0, which is Polish in some finer group topology, and let $G = \pi^{-1}(H)$ be the group of lifts of elements of H with the topology lifted from H, whence the exact sequence*

$$1 \to \Gamma \to G \xrightarrow{\pi} H \to 1.$$

Then there is a bornologous section $\phi\colon H \to C_G(\Gamma)$ for the quotient map π and we have the following coarse equivalences

$$G \approx_{\mathrm{coarse}} H \times \Gamma, \qquad C_G(\Gamma) \approx_{\mathrm{coarse}} H \times Z(\Gamma).$$

PROOF. Suppose that $h \in H$. Then, since $\phi(h) \in C_{\mathrm{Homeo}(X)}(\Gamma)$ is a lift of h and $\ker\pi = \Gamma \leqslant G$, we see that $\phi(h)$ belongs to the closed subgroup $C_G(\Gamma) = G \cap C_{\mathrm{Homeo}(X)}(\Gamma)$ of G.

We claim that $\phi\colon H \to C_G(\Gamma)$ and a fortiori $\phi\colon H \to G$ is bornologous. To see this, suppose that $B \subseteq H$ is coarsely bounded and $V \subseteq C_G(\Gamma)$ is an identity neighbourhood. We must find a finite set $F \subseteq C_G(\Gamma)$ and an $n \geqslant 1$ so that, for $h, g \in H$,

$$h^{-1}g \in B \quad\Rightarrow\quad \phi(h)^{-1}\phi(g) \in (FV)^n.$$

Let first $U \subseteq X$ be a relatively compact open set containing $\overline{D}$, where D is the fundamental domain for the action $\Gamma \curvearrowright X$. Thus, by the properness of the action, the set $\{a \in \Gamma \mid a \cdot D \cap U \neq \emptyset\}$ is finite and so U has finite d-diameter. Also, as the topology on G and thus also on $C_G(\Gamma)$ refine the compact-open topology from the action on X, the set $W = \{f \in C_G(\Gamma) \mid f^{-1}[\overline{D}] \subseteq U\}$ is an identity neighbourhood in $C_G(\Gamma)$. Observe that, if $f \in W$, then

$$d_\infty(f, \mathrm{id}) = \sup_{x \in D} d(f^{-1}x, x) \leqslant \mathrm{diam}_d(U).$$

That is, W has finite d_∞-diameter.

Now, $V \cap W$ is a identity neighbourhood in $C_G(\Gamma)$, so, as $\pi\colon C_G(\Gamma) \to H$ is a continuous epimorphism and therefore an open map, $\pi[V \cap W]$ is an identity neighbourhood in H. Thus, as B is coarsely bounded, there is a finite set $E \subseteq H$ and some n so that $B \subseteq \big(E \cdot \pi[V \cap W]\big)^n$. Write $E = \pi[F]$ for some finite set $F \subseteq C_G(\Gamma)$, whereby

$$B \subseteq \big(\pi[F] \cdot \pi[V \cap W]\big)^n = \pi\big[(E \cdot V \cap W)^n\big].$$

So, as W has finite d_∞-diameter and d_∞ is left-invariant, also $c = \mathrm{diam}_{d_\infty}\big((E \cdot V \cap W)^n\big) < \infty$. We let $A \subseteq Z(\Gamma)$ be the finite set associated with c by Lemma 4.35.

Now, suppose $h, g \in H$ with $h^{-1}g \in B$. Then, as $\phi(h)^{-1}\phi(g) \in C_G(\Gamma)$ is a lift of $h^{-1}g$, there is some $a \in \Gamma \cap C_G(\Gamma) = Z(\Gamma)$ so that $\phi(h)^{-1}\phi(g)a \in (E \cdot V \cap W)^n$, whereby

$$d_\infty\big(\phi(h), \phi(g)a\big) \leqslant c$$

and so $a \in A$. It follows that, for $h, g \in H$,

$$h^{-1}g \in B \;\Rightarrow\; \phi(h)^{-1}\phi(g) \in (E \cdot V \cap W)^n A^{-1},$$

showing that $\phi\colon H \to C_G(\Gamma)$ is bornologous.

We now show that $Z(\Gamma) = \Gamma \cap C_G(\Gamma)$ is coarsely embedded in $C_G(\Gamma)$. To see this, note that the set W above is an identity neighbourhood in $C_G(\Gamma)$ of finite d_∞-diameter. As d_∞ is a left-invariant écart on $C_G(\Gamma)$, it follows that $(FW)^n$ has finite d_∞-diameter for all finite sets F and $n \geqslant 1$. So every coarsely bounded set in $C_G(\Gamma)$ has finite d_∞-diameter, whence every infinite subset of $Z(\Gamma)$ has infinite d_∞-diameter, i.e., $Z(\Gamma)$ is coarsely embedded in $C_G(\Gamma)$.

Now, as Γ is locally bounded and $Z(\Gamma) = \Gamma \cap C_G(\Gamma)$ is coarsely embedded in both Γ and $C_G(\Gamma)$, by Lemma 4.29 (1), we conclude that also Γ is coarsely embedded in G. Therefore, we may now apply Proposition 4.30 to see that $C_G(\Gamma)$ is coarsely equivalent to $Z(\Gamma) \times H$, while G is coarsely equivalent to $\Gamma \times H$. □

Suppose a compact manifold M is given with universal cover $X = \tilde{M}$. Then $\Gamma = \pi_1(M)$ acts freely, properly and cocompactly by deck-transformations on X and $M = X/\Gamma$. Moreover, as X is simply connected, every homeomorphism of M lifts to X and so $Q = \mathrm{Homeo}(M)$. Furthermore, since the group $\mathrm{Homeo}_0(M)$ of isotopically trivial homeomorphisms by definition is path-connected, it will be contained in the open subgroup Q_0 of $\mathrm{Homeo}(M)$.

THEOREM 4.37. *Suppose M is a compact manifold, H is a subgroup of* $\mathrm{Homeo}_0(M)$*, which is Polish in some finer group topology, and let G be the group of all lifts of elements in H to homeomorphisms of the universal cover $\tilde{M}$. Assume that the quotient map*

$$\pi_1(M) \longrightarrow \pi_1(M)/Z\big(\pi_1(M)\big)$$

admits a bornologous section. Then G is coarsely equivalent to the direct product group $\pi_1(M) \times H$.

Perhaps equally important is the fact that the coarse equivalence between G and $\pi_1(M) \times H$ is given by a bornologous section $\phi\colon H \to G$ of the quotient map, where ϕ is the map defined above.

CHAPTER 5

Polish groups of bounded geometry

1. Gauges and groups of bounded geometry

In [**60**], J. Roe considers the coarse spaces of bounded geometry, which are a natural generalisation of metric spaces of bounded geometry. For this, suppose $(X,\mathcal{E})$ is a coarse space and $E\in\mathcal{E}$ a symmetric entourage. Then, in analogy with A. N. Kolmogorov's notions of metric entropy and capacity [**42, 43**], we define the E-capacity and E-entropy of a subset $A\subseteq X$ by

$$\mathrm{cap}_E(A)=\sup\big(k\;\big|\;\exists a_1,\dots,a_k\in A\colon (a_i,a_j)\notin E \text{ for } i\neq j\big)$$

and

$$\mathrm{ent}_E(A)=\min\big(|B|\;\big|\;A\subseteq E[B]\big)=\min\big(|B|\;\big|\;\forall a\in A\;\exists b\in B\;(a,b)\in E\big).$$

The following inequalities are then straightforward to verify

$$\mathrm{cap}_{\mathrm{E\circ E}}\leqslant \mathrm{ent}_E\leqslant \mathrm{cap}_E.$$

Also, if $E\subseteq E'$, then clearly $\mathrm{ent}_{E'}\leqslant\mathrm{ent}_E$.

Definition 5.1. *A coarse space $(X,\mathcal{E})$ has* bounded geometry[1] *if there is a symmetric entourage $E\in\mathcal{E}$ so that, for every entourage $F\in\mathcal{E}$,*

$$sup_{x\in X}\;\mathrm{ent}_E(F_x)<\infty,$$

where $F_x=\{y\in X\;\big|\;(y,x)\in F\}$.

For example, a metric space (X,d) has bounded geometry if and only if there is a finite diameter α with the property that, for every β, there is a k so that every set of diameter $\leqslant\beta$ can be covered by k sets of diameter α.

When dealing with discrete metric spaces, it is often useful to deal instead with a slightly stronger notion, which unfortunately is also denoted bounded geometry in the literature. To avoid ambiguity, we keep a separate terminology.

Definition 5.2. *A metric space (X,d) is* locally finite *if every set of finite diameter is finite. Also, (X,d) is* uniformly locally finite *if, for every diameter β, there is a K so that subsets of diameter $\leqslant\beta$ have cardinality at most K.*

Also, by inspection, one sees that bounded geometry is a *coarse invariant*, i.e., that any coarse space $(Y,\mathcal{F})$, coarsely equivalent to a coarse space $(X,\mathcal{E})$ of bounded geometry, is itself coarsely bounded.

Now, if G is a topological group, the basic entourages of the form $E_A=\{(x,y)\;\big|\;x^{-1}y\in A\}$ are cofinal in the coarse structure on G and, with this observation, the following lemma is straightforward to verify.

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[1] The actual definition given by Roe in [**60**] is, for various reasons, more complicated and expressed in terms of the capacity, but can easily be checked to be equivalent to ours.

LEMMA 5.3. *A topological group G has bounded geometry if and only if there is a coarsely bounded set $A \subseteq G$ that covers every other coarsely bounded set B by finitely many left-translates, i.e., so that $B \subseteq FA$ for some finite set $F \subseteq G$.*

It will be useful to have a name for the sets A appearing in Lemma 5.3, as these will appear throughout the paper.

DEFINITION 5.4. *A subset A of a topological group G is said to be a* gauge *for G if A is coarsely bounded, symmetric, $1 \in A$ and, for every coarsely bounded set $B \subseteq G$, there is a finite set F so that $B \subseteq FA$.*

Of course the quintessential example of a Polish group with bounded geometry is a locally compact group. In fact, in a locally compact Polish group G, every symmetric relatively compact identity neighbourhood V is a gauge, since every relatively compact set can be covered by finitely many left translates of V.

Our first observation is that Polish groups with bounded geometry are automatically locally bounded, though not necessarily locally compact.

LEMMA 5.5. *Polish groups with bounded geometry are locally bounded.*

PROOF. Let A be a gauge for a Polish group G. Replacing A by its closure, we may assume that A is closed, whereby all powers A^k are analytic sets and thus have the property of Baire. Obviously, we may also assume that G is uncountable and hence a perfect space.

We claim that A^4 is a coarsely bounded identity neighbourhood. To see this, observe that, if A^2 is non-meagre, then by Pettis' lemma [**57**] $A^4 = (A^2)^{-1}A^2$ is an identity neighbourhood. On the other hand, if A^2 is meagre, then, since the mapping $(x,y) \in G \times G \mapsto x^{-1}y \in G$ is surjective continuous and open, the subset $\{(x,y) \in G \times G \mid x^{-1}y \notin A^2\}$ is comeagre in $G \times G$. By Mycielski's Independence Theorem [**53**], we can thus find a homeomorphic copy $C \subseteq G$ of Cantor space so that $x^{-1}y \notin A^2$ and hence $xA \cap yA = \emptyset$ for all distinct $x, y \in C$.

Now, since C is compact and hence coarsely bounded in G, there is a finite subset $F \subseteq G$ so that $C \subseteq FA$. There are thus distinct $x, y \in C$ belonging to some fA with $f \in F$, i.e., $f \in xA \cap yA$, contradicting the assumption on C. □

By Lemma 5.5, every Polish group of bounded geometry admits an open gauge. Indeed, if A is a gauge and V is a coarsely bounded symmetric open identity neighbourhood, which exists by Lemma 5.5, then VAV is also a gauge. Also, as mentioned above, every symmetric relatively compact identity neighbourhood in a locally compact Polish group is a gauge. Coversely, if G is a Polish group with an identity neighbourhood U that can be covered by finitely many left-translates by any smaller identity neighbourhood, then U will in fact be relatively compact and thus G is locally compact. So gauges in general Polish groups of bounded geometry can neither be too big, nor too small.

Suppose G is a Polish group generated by a coarsely bounded set. Then G will have bounded geometry if and only if G is generated by a coarsely bounded set A with the property that $A^2 \subseteq FA$ for some finite F. Indeed, if G has bounded geometry, then we may find some gauge A generating G. And conversely, if A is a coarsely bounded generating set so that $A^2 \subseteq FA$ for some finite F, then also $A^n \subseteq F^{n-1}A$ and hence, as the A^n are cofinal among coarsely bounded sets in G, A covers any coarsely bounded set by a finite number of left-translates.

Now recall that a Polish group G is locally bounded if and only if it admits a coarsely proper metric. Therefore, if it has bounded geometry, it has a coarsely proper metric d and after rescaling this, we can suppose that the open unit ball is a gauge for G.

DEFINITION 5.6. *A compatible left-invariant metric metric d on a topological group G is a* gauge metric *if it is coarsely proper and the open unit ball is a gauge for G.*

2. Dynamic and geometric characterisations of bounded geometry

Postposting for the moment the presentation of specific examples of groups of bounded geometry, we instead give alternate descriptions of these and thus point to how they might arise.

Recall that a continuous action $G \curvearrowright X$ of a topological group G on a locally compact Hausdorff space X is said to be *cocompact* if there is a compact set $K \subseteq X$ with $X = G \cdot K$.

LEMMA 5.7. *Suppose $G \curvearrowright X$ is a continuous cocompact action of a topological group G on a locally compact Hausdorff space X and assume that, for some $x \in X$ and every subset $A \subseteq G$,*

$$A \cdot x \text{ is relatively compact } \Leftrightarrow A \text{ is coarsely bounded.}$$

Then G has bounded geometry.

PROOF. Let $K \subseteq X$ be a compact set with $G \cdot K = X$ and $x \in K$. As K is compact and X locally compact, we may find a relatively compact open set $U \supseteq K$. Let also $A = \{g \in G \mid gx \in U\}$, which is coarsely bounded.

Assume that $B \subseteq G$ is coarsely bounded. Then $\overline{Bx}$ is compact, so, as $X = G \cdot K = G \cdot U$, there is a finite set $F \subseteq G$ with $\overline{Bx} \subseteq F \cdot U$. But then, for $g \in B$, there is $f \in F$ with $gx \in fU$, i.e., $f^{-1}g \in A$, showing that $B \subseteq FA$. So A covers every coarsely bounded set by finitely many left translates and G has bounded geometry. □

Assume that $G \curvearrowright X$ is a continuous action of a topological group G on a locally compact Hausdorff space X. We say that the action is *coarsely proper* if the set $\{g \in G \bigm| g \cdot K \cap K \neq \emptyset\}$ is coarsely bounded in G for all compact subsets $K \subseteq X$. Conversely, the action is *modest* if $B \cdot K$ is relatively compact for all compact $K \subseteq X$ and coarsely bounded sets $B \subseteq G$.

EXAMPLE 5.8. Suppose $G \curvearrowright X$ is a coarsely proper and modest continuous action of a topological group G on a locally compact Hausdorff space X. Then the sets

$$E_K = \{(g, f) \in G \times G \mid gK \cap fK \neq \emptyset\}$$

form a basis for the coarse structure on G as K varies over compact subsets of X.

To see this, consider first the coarse entourage $E_A = \{(g, f) \in G \times G \mid g^{-1}f \in A\}$, where A is some coarsely bounded set. Pick $x \in X$ and let $K = \{x\} \cup \overline{A \cdot x}$. Then, if $g^{-1}f \in A$, also $g^{-1}f \cdot x \in K$ and so $gK \cap fK \neq \emptyset$, i.e., $E_A \subseteq E_K$. Conversely, if $K \subseteq X$ is compact, then the set $A = \{h \in G \mid hK \cap K \neq \emptyset\}$ is coarsely bounded and $E_K \subseteq E_A$.

Recall that, if G is a Polish group with a compatible left-invariant metric d, the Roelcke uniformity on G is that induced by the metric

$$d_\wedge(g,f)=\inf_{h\in G} d(g,h)+d(h^{-1},f^{-1}).$$

Also, a subset $A\subseteq G$ is Roelcke precompact if it is $d_\wedge$-totally bounded or, alternatively, if, for every identity neighbourhood $V\subseteq G$, there is a finite set $F\subseteq G$ so that $A\subseteq VFV$.

Now, a Polish group G is locally Roelcke precompact if it has a Roelcke precompact identity neighbourhood. For these, a fundamental result by J. Zielinski [**83**] states that a subset is coarsely bounded if and only if it is Roelcke precompact and so, in particular, the Roelcke precompact sets are closed under multiplication. Moreover, from this, he shows that the completion $X=\overline{(G,d_\wedge)}$ of a locally Roelcke precompact Polish group with respect to its Roelcke uniformity is in fact locally compact.

Before stating the next result, we must recall that a metric space (X,d) is *proper* if all closed sets of finite diameter are compact.

LEMMA 5.9. *Suppose G is a locally Roelcke precompact Polish group and let X be the Roelcke completion of G. Then the left and right-shift actions $\lambda\colon G\curvearrowright G$ and $\rho\colon G\curvearrowright G$ have unique extensions to commuting, coarsely proper, modest, continuous actions $\lambda\colon G\curvearrowright X$ and $\rho\colon G\curvearrowright X$.*

PROOF. It is well-known and easy to check that the two shift-actions extend uniquely to commuting continuous actions on the completion X, so we must verify that they are modest and coarsely proper. We do only the argument for the left-shift action as the argument for ρ is symmetric.

To see that the action is modest, suppose $K\subseteq X$ is compact and $B\subseteq G$ is coarsely bounded. Then there is a relatively compact open set $U\subseteq X$ containing K, whence, as G is dense in X, we have $K\subseteq\overline{U\cap G}$. Moreover, since $U\cap G$ is Roelcke precompact in G, it must be coarsely bounded, whereby $B\cdot\big(U\cap G\big)$ is coarsely bounded and thus Roelcke precompact. As

$$B\cdot K\subseteq B\cdot\overline{U\cap G}\subseteq\overline{B\cdot\big(U\cap G\big)},$$

also $B\cdot K$ is relatively compact in X.

Similarly, to see that the action is coarsely proper, let $K\subseteq X$ be compact and pick a relatively compact open set $U\subseteq X$ containing K. Assume $f\in G$ and $fK\cap K\neq\emptyset$. Then $fU\cap U\neq\emptyset$, so, as $U\cap G$ is dense in U, we have $f\cdot\big(U\cap G\big)\cap U\neq\emptyset$ and thus also $f\cdot\big(U\cap G\big)\cap\big(U\cap G\big)\neq\emptyset$. In other words, $f\in\big(U\cap G\big)\big(U\cap G\big)^{-1}$, showing that

$$\{f\in G\mid fK\cap K\neq\emptyset\}\subseteq\big(U\cap G\big)\big(U\cap G\big)^{-1}.$$

As the latter set is coarsely bounded in G, the action $\lambda\colon G\curvearrowright X$ is coarsely proper. □

In order to ensure cocompactness of actions, we must assume the acting group has bounded geometry.

LEMMA 5.10. *Suppose G is a coarsely embedded closed subgroup of a locally Roelcke precompact Polish group H and assume that G has bounded geometry. Let Y denote the closure of G inside the Roelcke completion X of H. Then the left and* *right-shift actions* $\lambda\colon G \curvearrowright G$ *and* $\rho\colon G \curvearrowright G$ *extend to commuting, coarsely proper, modest, cocompact, continuous actions of* G *on* Y.

PROOF. Observe that, as G is coarsely embedded in H, the restriction of the coarsely proper modest actions $\lambda\colon H \curvearrowright X$ and $\rho\colon H \curvearrowright X$ to $G \leqslant H$ are coarsely proper and modest. Also, as Y is the closure of G in X, it is invariant under both G-actions, so the restrictions $\lambda\colon G \curvearrowright Y$ and $\rho\colon G \curvearrowright Y$ are well-defined, coarsely proper and modest.

We now check that $\lambda\colon G \curvearrowright Y$ is cocompact, the argument for $\rho\colon G \curvearrowright Y$ being similar. For this, let $U \subseteq G$ be an open gauge. Since U is coarsely bounded in G, it is Roelcke precompact in H and hence $\overline{U} \subseteq Y$ is compact. We show that $Y = G\cdot\overline{U}$.

Indeed, assume that $x \in Y = \overline{G}$ and pick a sequence $g_n \in G$ converging to x. Then (g_n) is Cauchy in the Roelcke uniformity on H, so, if V is a fixed coarsely bounded identity neighbourhood in H, we may without loss of generality assume that $g_n \in Vg_1V$ for all $n \geqslant 1$. But, as G is coarsely embedded in H, $Vg_1V \cap G$ is coarsely bounded in G and hence must be contained in some finite number of left-translates of U, i.e., $Vg_1V \cap G \subseteq FU$ for some finite $F \subseteq G$. Therefore, by passing to a subsequence, we can also suppose that $g_n \in fU$ for some $f \in F$ and all n.

As $g_n \underset{n\to\infty}{\longrightarrow} x$ in Y, we have, by continuity of the G-action, that $f^{-1}g_n \underset{n\to\infty}{\longrightarrow} f^{-1}x$. But, since $f^{-1}g_n \in U$, we conclude that $f^{-1}x \in \overline{U}$, i.e., $x \in f\cdot\overline{U} \subseteq G\cdot\overline{U}$. Thus, $Y = G\cdot\overline{U}$ as required. $\square$

THEOREM 5.11. *The following conditions are equivalent for a Polish group* G.

1. G *has bounded geometry,*
2. G *is coarsely equivalent to a metric space of bounded geometry,*
3. G *is coarsely equivalent to a proper metric space,*
4. G *admits a continuous, coarsely proper, modest and cocompact action* $G \curvearrowright X$ *on a locally compact metrisable space* X.

PROOF. That (1) and (2) are equivalent follow immediately from the fact that bounded geometry is invariant under coarse equivalences and that every Polish group of bounded geometry is locally bounded and hence has metrisable coarse structure.

Suppose that G has bounded geometry and let d be a gauge metric for G. Let also X be a maximal 2-discrete subset of G, i.e., maximal so that $d(x,y) \geqslant 2$ for all $x \neq y$ in X. Note that, if $\beta > 0$ is a fixed diameter, pick a finite set $F \subseteq G$ so that the ball $B_d(3\beta) = \{x \in G \mid d(x,1) < 3\beta\}$ is contained in $F\cdot B_d(1)$, whence every set of diameter at most β is contained in a single left-translate of $FB_d(1)$. However, since X is 2-discrete, two distinct points of X cannot belong to the same left-translate of $B_d(1)$, showing that subsets of X of diameter $\leqslant \beta$ have cardinality at most $|F|$.

It follows that (X,d) is a proper metric space of bounded geometry, which, being cobounded in G, is coarsely equivalent to G. Thus (1) implies (3).

Conversely, assume that G is coarsely equivalent to a proper metric space (X,d), which, by picking a discrete subset, we may assume to be locally finite, i.e., finite diameter subsets are finite. Fix a coarse equivalence $\phi\colon (X,d) \to G$. Since $\phi[X]$ is cobounded in G, there is a symmetric coarsely bounded set $1 \in A \subseteq G$ so that $G = \phi[X]\cdot A$.

We claim that A is a gauge for G and thus that G is of bounded geometry. Indeed, suppose $B \subseteq G$ is coarsely bounded. Then, as ϕ is a coarse equivalence and (X, d) is locally finite, the set

$$\phi^{-1}(BA) = \{x \in X \mid \phi(x)A \cap B \neq \emptyset\}$$

is bounded and thus finite. But $B \subseteq \phi[X] \cdot A$, so actually $B \subseteq \phi[\phi^{-1}(BA)] \cdot A$, showing that B is covered by finitely many left-translates of A. Thus (3) implies (1).

The implication from (4) to (1) follows directly from Lemma 5.7, so let us consider the other direction. Let G be a Polish group of bounded geometry. Then G is locally bounded by Proposition 5.5. Therefore, by Theorem 3.32, we may assume that G is a coarsely embedded closed subgroup of the isometry group $\mathrm{Isom}(\mathbb{U})$ of the Urysohn metric space. As this latter is locally Roelcke precompact by Proposition 3.8, the result follows from Lemma 5.10. $\square$

By consequence, we see that the class of Polish groups with bounded geometry is closed under passing to coarsely embedded closed subgroups. On the other hand, the universal Polish group $\mathrm{Homeo}([0,1]^{\mathbb{N}})$ has bounded geometry in virtue of being coarsely bounded in itself. So the class of groups with bounded geometry is not closed under passing to arbitrary closed subgroups.

3. Examples

As noted earlier, the obvious example of a Polish group with bounded geometry is a locally compact group. Also, a coarsely bounded group such as S_∞ or $\mathrm{Homeo}(\mathbb{S}^n)$ is automatically of bounded geometry since the entire group is a gauge for itself. However, apart from these and simple algebraic constructs over these basic examples, there are much more interesting examples.

Consider first the groups $\mathrm{Homeo}_+(\mathbb{S}^1)$, $\mathrm{Homeo}_+([0,1])$ and $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ of orientation preserving homeomorphisms of the circle $\mathbb{S}^1$, the interval $[0,1]$, respectively, homeomorphisms of the real line commuting with integral shifts, i.e., with the maps $\tau_n(x) = x + n$ for $n \in \mathbb{Z}$. Alternatively, $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ may be described as the group of orientation preserving homeomorphisms of $\mathbb{R}$ preserving the relation $|x - y| = 1$. Now, $\mathrm{Homeo}_+([0,1])$ can be seen as the isotropy subgroup of any point on the circle and we can therefore write

$$\mathrm{Homeo}_+(\mathbb{S}^1) = \mathbb{T} \cdot \mathrm{Homeo}_+([0,1]),$$

where $\mathbb{T} = \mathrm{SO}(2)$ is the group of rotations of $\mathbb{S}^1$. On the other hand, $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ can be seen as the group of lifts of homeomorphisms of the circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ to its universal cover $\mathbb{R}$ and thus as a central extension of $\mathrm{Homeo}_+(\mathbb{S}^1)$ by $\mathbb{Z}$,

$$0 \to \mathbb{Z} \to \mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}) \to \mathrm{Homeo}_+(\mathbb{S}^1) \to 1.$$

Also, if H denotes the isotropy subgroup of 0 inside $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$, then every element of H fixes the points of $\mathbb{Z}$ and is thus just a homeomorphism of $[0,1]$ replicated on each interval $[n, n+1]$. So H is isomorphic to $\mathrm{Homeo}_+([0,1])$ and $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ factors as

$$\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}) = \mathbb{R} \cdot H.$$

Nevertheless, as the factors $\mathbb{T}$, $\mathrm{Homeo}_+([0,1])$ and $\mathbb{R}$, H do not commute, the factorisations of $\mathrm{Homeo}_+(\mathbb{S}^1)$ and of $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ are not as direct products. We similarly point out that the central extension $\mathbb{Z} \to \mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}) \to \mathrm{Homeo}_+(\mathbb{S}^1)$ does not split, so $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ is not a semidirect product of $\mathbb{Z}$ and $\mathrm{Homeo}_+(\mathbb{S}^1)$.

PROPOSITION 5.12. *The group* $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ *is quasi-isometric to* $\mathbb{Z}$ *and thus of bounded geometry. Moreover, the action*

$$\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}) \curvearrowright \mathbb{R}$$

is transitive, modest and coarsely proper.

PROOF. We show that the quotient map $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}) \xrightarrow{\pi} \mathrm{Homeo}_+(\mathbb{S}^1)$ admits a Borel measurable section $\phi\colon \mathrm{Homeo}_+(\mathbb{S}^1) \to \mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ with finite defect

$$\Delta = \{\phi(gh)^{-1}\phi(g)\phi(h) \mid g, h \in \mathrm{Homeo}_+(\mathbb{S}^1)\}$$

in $\mathbb{Z}$. Indeed, for $h \in \mathrm{Homeo}_+(\mathbb{S}^1)$, simply let $\phi(h)$ be the unique lift of h to a homeomorphism of $\mathbb{R}$ so that $\phi(h)(0) \in [0,1[$. Then $\phi(gh)^{-1}\phi(g)\phi(h)$ is an integral translation, i.e., belongs to the subgroup $\mathbb{Z}$, and evaluating it at 0, we see that it has possible values 0 or 1. So $\Delta \subseteq \{-1,0,1\}$.

So ϕ is a Borel measurable section for the quotient map and, moreover, ϕ is a quasimorphism. As $\mathrm{Homeo}_+(\mathbb{S}^1)$ is coarsely bounded and thus, a fortiori, locally bounded, it follows from Theorem 4.31 that the product $\mathbb{Z} \times \mathrm{Homeo}_+(\mathbb{S}^1)$ is coarsely equivalent with $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ via the map $(n,h) \mapsto \tau_n\phi(h)$. However, since $\mathrm{Homeo}_+(\mathbb{S}^1)$ is coarsely bounded, it is a trivial factor in $\mathbb{Z} \times \mathrm{Homeo}_+(\mathbb{S}^1)$ and hence the inclusion $n \mapsto \tau_n$ is a coarse equivalence between $\mathbb{Z}$ and $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$.

Now, observe that the map $g \in \mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}) \xmapsto{\sigma} \tau_n \in \mathbb{Z}$, where $g = \tau_n h$ is the unique decomposition of g into an integral translation τ_n and an element $h \in \mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ satisfying $h(0) \in [0,1[$, is a coarse equivalence. In particular, a subset $A \subseteq \mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ is coarsely bounded if and only if the corresponding set $\sigma(A)$ of translations is finite. As elements $h \in \mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ satisfying $h(0) \in [0,1[$ move every $x \in \mathbb{R}$ by at most distance 1, it easily follows that the action is both modest and coarsely proper. □

A case of special interest is when a gauge can be taken to be an open subgroup. Namely, suppose that V is a open subgroup of a Polish group G and let $\mathrm{Sym}(G/V)$ be the group of all permutations of the countable homogeneous space G/V of left V-cosets equipped with the *permutation group topology*, i.e., by declaring pointwise stabilisers to be open. Set $\pi\colon G \to \mathrm{Sym}(G/V)$ to be the homomorphism corresponding to the left-translation action of G on G/V. Recall that the *commensurator* of V is G is the subgroup defined by

$$\mathrm{Comm}_G(V) = \{g \in G \mid [V\colon V \cap gVg^{-1}] < \infty \;\&\; [V\colon V \cap g^{-1}Vg] < \infty\}$$

and note that $V \leqslant \mathrm{Comm}_G(V) \leqslant G$.

PROPOSITION 5.13. *The following conditions are equivalent for a coarsely bounded subgroup* V *of a Polish group* G.

1. V *is a gauge for* G *and hence* G *has bounded geometry,*
2. $G = \mathrm{Comm}_G(V)$, *i.e., every double coset* VgV *is a finite union of left* V*-cosets,*
3. *the homomorphism* $\pi\colon G \to \mathrm{Sym}(G/V)$ *induced by the left-translation action of* G *on the homogeneous space* G/V *is a coarse equivalence with a locally compact subgroup* $\overline{\pi[G]} \leqslant \mathrm{Sym}(G/V)$.

Proof. Observe that the index $[V : V \cap gVg^{-1}]$ equals the number of left-cosets of V contained in VgV. Therefore, G equals the commensurator $\mathrm{Comm}_G(V)$ if and only if every double coset VgV is a union of finitely many left V-cosets. Now, every coarsely bounded set in G can be covered by some $(FV)^n$ with $F \subseteq G$ finite and $n \geqslant 1$. So, for V to be a gauge for G, it suffices by an inductive argument to check that each double coset VgV is a union of finitely many left V-cosets. Conversely, if V is a gauge, then as VgV is coarsely bounded it must be a union finitely many left V-cosets. This shows the equivalence of (1) and (2).

Now, if $G = \mathrm{Comm}_G(V)$, then $\overline{\pi[G]}$ is a locally compact subgroup of $\mathrm{Sym}(G/V)$. Indeed, if $U \leqslant \mathrm{Sym}(G/V)$ denote the pointwise stabiliser of the coset $1V$, then $\pi^{-1}(U) = V$ and the V-orbit of any coset gV is just the V-cosets contained in VgV, which is finite by the assumption $G = \mathrm{Comm}_G(V)$. So this shows that $U \cap \overline{\pi[G]}$ is a compact open subgroup of $\overline{\pi[G]}$ and the latter is therefore locally compact. Moreover, we claim that $\pi \colon G \to \overline{\pi[G]}$ is coarsely proper. For, if $B \subseteq G$ fails to be coarsely bounded, it must intersect infinitely many distinct left-cosets of V, whereby $B \cdot V$ is an infinite subset of G/V and $\pi[B]$ cannot be relatively compact. Thus, if $G = \mathrm{Comm}_G(V)$, then $\pi \colon G \to \overline{\pi[G]}$ is a coarse equivalence of G with a totally disconnected locally compact group.

Conversely, suppose that $\overline{\pi[G]}$ is locally compact. Then there is a finite number of left-cosets $g_1V, \ldots, g_nV$ so that the stabiliser $W = g_1Vg_1^{-1} \cap \ldots \cap g_nVg_n^{-1}$ of these orbits induces only finite orbits on G/V, i.e., so that, for every $f \in G$, WfV is a finite union of left V-cosets. It follows that, if $F \subseteq G$ is finite and $n \geqslant 1$, there are finite sets $F_1, \ldots, F_n \subseteq G$ so that

$$(WF)^n \subseteq (WF)^{n-1}WFV \subseteq (WF)^{n-1}F_1V \subseteq (WF)^{n-2}F_2V \subseteq \ldots \subseteq F_nV.$$

As every coarsely bounded set is covered by some $(WF)^n$, we conclude that V covers every coarsely bounded set by finitely many left-translates, i.e., that V is a gauge for G and thus $G = \mathrm{Comm}_G(V)$. □

In general though, even in the case of non-Archimedean Polish groups of bounded geometry, we should not expect to have open subgroups as gauges and must therefore develop other tools to analyse the structure of groups of bounded geometry.

Recall that, if G is a Polish group with a compatible left-invariant metric d, the Roelcke uniformity on G is that induced by the metric

$$d_\wedge(g, f) = \inf_{h \in G} d(g, h) + d(h^{-1}, f^{-1}).$$

Also, a subset $A \subseteq G$ is Roelcke precompact if it is $d_\wedge$-totally bounded or, alternatively, if, for every identity neighbourhood $V \subseteq G$, there is a finite set $F \subseteq G$ so that $A \subseteq VFV$.

Example 5.14. Let $\mathrm{Aut}_{\mathbb{Z}}(\mathbb{Q})$ be the group of order-preserving permutations of $\mathbb{Q}$ commuting with integral translations. With the permutation group topology, this is a non-Archimedean Polish group that we claim is coarsely equivalent to $\mathbb{Z}$.

Observe first that $\mathrm{Aut}_{\mathbb{Z}}(\mathbb{Q})$ is locally Roelcke precompact and thus, by a result of J. Zielinski [**83**], that the Roelcke precompact subsets of $\mathrm{Aut}_{\mathbb{Z}}(\mathbb{Q})$ coincide with the coarsely bounded sets and hence are closed under products. Indeed, since the linear order $(\mathbb{Q}, <)$ is $\aleph_0$-categorical, its automorphism group $\mathrm{Aut}(\mathbb{Q})$ is oligomorphic and hence Roelcke precompact. It thus suffices to observe that the identity neighbourhood $V = \{g \in \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Q}) \mid g(0) = 0\}$ is isomorphic to $\mathrm{Aut}(\mathbb{Q})$.

We now identify the Roelcke precompact subsets of $\mathrm{Aut}_{\mathbb{Z}}(\mathbb{Q})$ as the sets B so that

$$\sup\{|g(0)| \mid g \in B\} < \infty.$$

To see this, let τ_k denote the translation by $k \in \mathbb{Z}$ and σ the translation by $1/2$. Then , if $f \in \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Q})$ with $f(0) = k + \alpha$ for some $0 < \alpha < 1$, choose $g \in V_0$ so that $g(\frac{1}{2}) = \alpha$, whereby $f(0) = \tau_k g\sigma(0)$, i.e., $f \in \tau_k g\sigma V \subseteq \tau_k V\sigma V$. It follows that, if $B \subseteq \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Q})$ with $\sup\{|g(0)| \mid g \in B\} < N$ for some $N \in \mathbb{N}$, then $B \subseteq \{\tau_k \mid k < N\}V\sigma V$, showing B to be Roelcke precompact.

Conversely, suppose that $B \subseteq \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Q})$ is Roelcke precompact and find a finite set $F \subseteq \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Q})$ so that $B \subseteq VFV$. Then $\sup\{|g(0)| \mid g \in B\} \leqslant \sup\{|f(0)| + 1 \mid f \in F\} < \infty$ as claimed.

Therefore, extending every $g \in \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Q})$ to a homeomorphism of $\mathbb{R}$, we see that the corresponding action $\mathrm{Aut}_{\mathbb{Z}}(\mathbb{Q}) \curvearrowright \mathbb{R}$ is coarsely proper, modest and cocompact, while $\mathbb{Z}$ is a cobounded subgroup of $\mathrm{Aut}_{\mathbb{Z}}(\mathbb{Q})$.

Now suppose W is a Roelcke precompact open subgroup of $\mathrm{Aut}_{\mathbb{Z}}(\mathbb{Q})$. Let

$$x = \sup\{w(0) \mid w \in W\} \in \mathbb{R},$$

which exists since the orbit of 0 under the Roelcke precompact set W is bounded. Remark also that x is fixed by W under the extended homeomorphic action $\mathrm{Aut}_{\mathbb{Z}}(\mathbb{Q}) \curvearrowright \mathbb{R}$. Observe that, if $g, f \in \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Q})$ with $g(x) \neq f(x)$, then $gW \neq fW$. Note also that $\mathrm{Aut}_{\mathbb{Z}}(\mathbb{Q})$ acts transitively on $\mathbb{R} \setminus \mathbb{Q}$, since any irrational Dedekind cut in $\mathbb{Q}$ can be moved to any other. So x cannot be irrational, as otherwise W would have uncountable index in $\mathrm{Aut}_{\mathbb{Z}}(\mathbb{Q})$, contradicting that it is open. Thus, W fixes the rational point x and therefore, for every finite set $F \subseteq \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Q})$, we have that $FW(x)$ is a finite subset of $\mathbb{Q}$. But this shows that W cannot be a gauge for $\mathrm{Aut}_{\mathbb{Z}}(\mathbb{Q})$, since for example $V\sigma\tau_{-x}$ is Roelcke precompact, while $V\sigma\tau_{-x}(x) = \mathbb{Q}\cap\,]0,1[$ is infinite and hence $V\sigma\tau_{-x}$ cannot be contained in some FW with F finite.

Summing up, we have a locally Roelcke precompact non-Archimedean Polish group $\mathrm{Aut}_{\mathbb{Z}}(\mathbb{Q})$ of bounded geometry, so that no open subgroup is a gauge for $\mathrm{Aut}_{\mathbb{Z}}(\mathbb{Q})$.

We can also construct a coarsely proper isometric action of $\mathrm{Aut}_{\mathbb{Z}}(\mathbb{Q})$ on a countable graph by letting $\mathbb{Q}$ be the vertex set and connecting two vertices $x, y \in \mathbb{Q}$ if $|x - y| < 1$. The resulting graph is quasi-isometric to $\mathbb{Z}$, showing again, by the Milnor–Schwarz Lemma, Theorem 2.57, that $\mathrm{Aut}_{\mathbb{Z}}(\mathbb{Q})$ is quasi-isometric to $\mathbb{Z}$.

Example 5.15 (Absolutely continuous homeomorphisms). Recall that a map $f\colon [0,1] \to \mathbb{R}$ is *absolutely continuous* if, for every $\epsilon > 0$, there is $\delta > 0$ so that

$$\sum_k |f(y_k) - f(x_k)| < \epsilon$$

whenever $0 \leqslant x_1 < y_1 < x_2 < y_2 < \ldots < x_n < y_n \leqslant 1$ satisfy $\sum_k (y_k - x_k) < \delta$. Equivalently, f is absolutely continuous if f' exists almost everywhere, f' is Lebesgue integrable and $f(x) = f(0) + \int_0^x f'(t)\,dt$. W. Herndon [**31**] has investigated the subgroup $\mathrm{AC}^*_{\mathbb{Z}}(\mathbb{R})$ of $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ consisting of all h so that h and h^{-1} are absolutely continuous when restricted to $[0,1]$. This can be shown to be a Polish group in a finer topology and Herndon proved that the inclusion of $\mathbb{Z}$ into $\mathrm{AC}^*_{\mathbb{Z}}(\mathbb{R})$ is a coarse equivalence.

Whereas the above three examples, $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$, $\mathrm{Aut}_{\mathbb{Z}}(\mathbb{Q})$ and $\mathrm{AC}^*_{\mathbb{Z}}(\mathbb{R})$ are very different topologically, they are quite similar algebraically and in their coarse structure. Namely, they are all obtained as central extensions by $\mathbb{Z}$ of coarsely bounded Polish subgroups of the homeomophism group of the circle and, in particular, they are all coarsely equivalent to $\mathbb{Z}$. Thus, apart from locally compact groups, it is not quite clear in which contexts Polish groups of bounded geometry arise. It would be especially interesting to obtain new topologically simple non-locally compact examples.

PROBLEM 5.16. *Find topologically simple, non-locally compact, non-coarsely bounded, Polish groups of bounded geometry.*

EXAMPLE 5.17 (An augmented Heisenberg group). The discrete Heisenberg group $H_3(\mathbb{Z})$ is the group of upper triangular matrices

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

with integral coefficients a, b, c. As the matrix product gives

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+x & c+z+ay \\ 0 & 1 & b+y \\ 0 & 0 & 1 \end{pmatrix},$$

it follows that $H_3(\mathbb{Z})$ may be written as a central extension of $\mathbb{Z}^2$ by $\mathbb{Z}$. Namely, if the cocycle $\omega\colon \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{Z}$ is defined by $\omega\big((a,b),(x,y)\big) = ay$, then the Heisenberg group is isomorphic to the cartesian product $\mathbb{Z} \times \mathbb{Z}^2$ with a product skewed by ω,

$$\big(c,(a,b)\big) \cdot \big(z,(x,y)\big) = \big(c+z+\omega\big((a,b),(x,y)\big), (a+x, b+y)\big).$$

Conversely, if H is an arbitrary group and $\omega\colon H \times H \to Z(G)$ is a map into the centre of a group G satisfying

(1) $\omega(1,x) = \omega(x,1) = 1$,
(2) $\omega(y,z)\omega(x,yz) = \omega(x,y)\omega(xy,z)$,

then the formula

$$(g,x) \cdot (f,y) = (gf\omega(x,y), xy)$$

defines a group operation on $G \times H$ giving rise to an extension $G \xrightarrow{\iota} G \times_\omega H \xrightarrow{\pi} H$ with embedding $\iota(g) = (g,1)$ and epimorphism $\pi(g,x) = x$.

We may therefore construct an augmented Heisenberg group $\mathbb{H}_3(\mathbb{Z})$ by defining the cocycle $\omega\colon \mathbb{Z}^2 \times \mathbb{Z}^2 \to Z\big(\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})\big) = \mathbb{Z}$ as before, i.e., $\omega\big((a,b),(x,y)\big) = \tau_{ay}$, and then setting $\mathbb{H}_3(\mathbb{Z}) = \mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}) \times_\omega \mathbb{Z}^2$. Observe that, as ω is trivially continuous, $\mathbb{H}_3(\mathbb{Z}) = \mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}) \times_\omega \mathbb{Z}^2$ is a topological group in the product topology. Also, identifying $\mathbb{Z}$ with the subgroup of integral translations in $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$, we may canonically identify $H_3(\mathbb{Z}) = \mathbb{Z} \times_\omega \mathbb{Z}^2$ with a subgroup of $\mathbb{H}_3(\mathbb{Z}) = \mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}) \times_\omega \mathbb{Z}^2$.

We claim that the inclusion map of $H_3(\mathbb{Z})$ into $\mathbb{H}_3(\mathbb{Z})$ is a quasi-isometry between these groups. To see this, let

$$B = \big\{h \in \mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}) \bigm| h(0) \in [0,1[\,\big\}$$

and observe that, since B is coarsely bounded in $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$, its homomorphic image $B \times \{\mathbf{0}\}$ is coarsely bounded in $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}) \times_\omega \mathbb{Z}^2$. As, moreover, $\mathbb{Z} \cdot B = \mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$, we see that $H_3(\mathbb{Z}) \cdot (B \times \{\mathbf{0}\}) = \mathbb{H}_3(\mathbb{Z})$ and thus that $H_3(\mathbb{Z})$ is cobounded in $\mathbb{H}_3(\mathbb{Z})$.

Now, by Corollary 4.22, a subset $A \subseteq H_3(\mathbb{Z}) = \mathbb{Z} \times_\omega \mathbb{Z}^2$ is coarsely bounded in $\mathbb{H}_3(\mathbb{Z}) = \mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}) \times_\omega \mathbb{Z}^2$ if and only if the projection $\mathrm{proj}_{\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})}(A)$ is coarsely bounded in $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$, while $\mathrm{proj}_{\mathbb{Z}^2}(A)$ is a finite subset of $\mathbb{Z}^2$. However, in this case, as $\mathrm{proj}_{\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})}(A)$ is a subset of the coarsely embedded subgroup $\mathbb{Z}$ of $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})(A)$, also $\mathrm{proj}_{\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})}(A)$ is finite. In other words, a subset $A \subseteq H_3(\mathbb{Z})$ is coarsely bounded in $\mathbb{H}_3(\mathbb{Z})$ if and only if it is finite, i.e., if and only if it is coarsely bounded in $H_3(\mathbb{Z})$.

It follows that $H_3(\mathbb{Z})$ is a coarsely embedded, cobounded subgroup of $\mathbb{H}_3(\mathbb{Z})$ and thus that the inclusion map is a coarse equivalence. Since also $H_3(\mathbb{Z})$ is finitely generated, the map is, in fact, a quasi-isometry.

Thus, $\mathbb{H}_3(\mathbb{Z})$ is a Polish group of bounded geometry, quasi-isometric to the discrete Heisenberg group $H_3(\mathbb{Z})$.

As the above examples of Polish groups of bounded geometry are all coarsely equivalent to locally compact groups, it is natural to wonder whether this is necessarily the case.

Problem 5.18. *Is every Polish group of bounded geometry coarsely equivalent to a locally compact second countable group?*

Several questions of similar nature have been studied in the literature. Namely, A. Eskin, D. Fisher and K. Whyte [**25**], answering a question of W. Woess, established the first examples of locally compact second countable groups not coarsely equivalent to any countable discrete group. Moreover, their examples were of two kinds, Lie groups and totally disconnected locally compact. Thus, not every locally compact group G has a discrete model, but one could ask whether it always has a combinatorial model, i.e., whether it is coarsely equivalent to a vertex transitive, connected, locally finite graph $\mathbb{X}$ or, equivalently, to its automorphism group $\mathrm{Aut}(\mathbb{X})$. However, even this turns out not to be the case (examples can be found in Section 6.C of [**17**]).

Let us nevertheless note the following simple fact.

Proposition 5.19. *Let G be a Polish group generated by a coarsely bounded set. Then G is quasi-isometric to a vertex transitive countable graph.*

Proof. Suppose $V \ni 1$ is a coarsely bounded, symmetric open generating set for G. Let also $\Gamma \leqslant G$ be a countable dense subgroup of G and define a graph $\mathbb{X}$ by setting

$$\mathrm{Vert}\,\mathbb{X} = \Gamma$$

and, for $x, y \in \Gamma$, $x \neq y$,

$$(x, y) \in \mathrm{Edge}\,\mathbb{X} \Leftrightarrow x \in yV.$$

Then $\mathrm{Edge}\,\mathbb{X}$ is a symmetric relation on the vertex set Γ, which is invariant under the, evidently transitive, left-shift action of Γ on itself. So $\mathbb{X}$ is a vertex transitive graph.

Let also $\rho_{\mathbb{X}}$ denote the shortest-path metric on $\mathbb{X}$ and ρ_V denote the word metric on G given by the generating set V. We claim that $\rho_{\mathbb{X}}$ is simply the restriction of ρ_V to Γ. Indeed, the inequality

$$\rho_V \leqslant \rho_{\mathbb{X}}$$

is obvious. For the other direction, fix $x, y \in \Gamma$ and suppose that $\rho_V(x, y) \leqslant k$, i.e., $y \in xV^k$. Then there are $v_1, \ldots, v_k \in V$ so that $y = xv_1 \cdots v_k$. However, we may not have $xv_1 \cdots v_i \in \Gamma$, so this may not give us a path in $\mathbb{X}$. Instead, note that since $yV^{k-1} \cap xV \neq \emptyset$, using the density of Γ in G, there is some $z_1 \in \Gamma \cap yV^{k-1} \cap xV$. Similarly, choose $z_2 \in yV^{k-2} \cap z_1 V$, etc. This produces a path $x, z_1, z_2, \ldots, z_{k-1}, y$ in $\mathbb{X}$, showing that $\rho_{\mathbb{X}}(x, y) \leqslant k$, as required.

Since G is quasi-isometric to the metric space (G, ρ_V) and Γ is a cobounded subset of (G, ρ_V), it follows that G is quasi-isometric to $\mathbb{X}$. □

4. Topological couplings

We now arrive at the generalisation of Gromov's theorem on topological couplings to groups of bounded geometry. Observe first that, by Lemma 5.7 and Theorem 5.11, the class of groups that allow comcompact actions on locally compact spaces faithfully representing their geometry are exactly those of bounded geometry.

Definition 5.20. *A* topological coupling *of Polish groups G and F is a pair $G \curvearrowright X \curvearrowleft F$ of commuting, coarsely proper, modest, cocompact continuous actions on a locally compact Hausdorff space X.*

As mentioned earlier, the basic motivating example for this definition is the coupling
$$\mathbb{Z} \curvearrowright \mathbb{R} \curvearrowleft \mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}).$$
Let us first begin by noting the easy direction of our theorem.

Proposition 5.21. *Let $G \curvearrowright X \curvearrowleft F$ be a topological coupling of Polish groups. Then G and F are coarsely equivalent.*

Proof. Let $K \subseteq X$ be a compact set with $X = G \cdot K = F \cdot K$ and pick a point $x \in X$. We choose maps $\phi\colon G \to F$ and $\psi\colon F \to G$ so that $g^{-1}x \in \phi(g)K$, i.e., $\phi(g)^{-1}x \in gK$, and $f^{-1}x \in \psi(f)K$ for all $g \in G$ and $f \in F$.

We claim that $\psi \circ \phi$ is close to the identity on G. Indded, given $g \in G$, we have $\phi(g)^{-1}x \in gK$ and $\phi(g)^{-1}x \in \psi(\phi(g))K$, so $gK \cap \psi(\phi(g))K \neq \emptyset$, whence $g^{-1}\psi(\phi(g)) \in \{h \in G \mid hK \cap K \neq \emptyset\}$. As the G-action is coarsely proper, the latter set is coarsely bounded, so $\psi \circ \phi$ is close to the identity on G. Similarly, $\phi \circ \psi$ is close to the identity on F.

Also, ϕ is bornologous. Indeed, suppose $B \ni 1$ is a symmetric coarsely bounded set in G and assume that $g^{-1}h \in B$. Then $g, h \in gB$ and so both $\phi(h)K \cap Bg^{-1}x \neq \emptyset$ and $\phi(g)K \cap Bg^{-1}x \neq \emptyset$. It follows that $g^{-1}x \in B\phi(h)K \cap B\phi(g)K = \phi(h)BK \cap \phi(g)BK$, whereby $\phi(g)^{-1}\phi(h) \in \{f \in F \mid fBK \cap BK \neq \emptyset\}$. As the G-action is modest and the F-action coarsely proper, the latter set is coarsely bounded in F, showing that ϕ is bornologous. Similarly ψ is bornologous.

As ϕ and ψ are bornologous and their compositions are close to the identities on G and F respectively, we conclude that they are coarse equivalences between G and F. □

Theorem 5.22. *Two Polish groups of bounded geometry are coarsely equivalent if and only if they admit a topological coupling.*

Proof. Suppose $\phi\colon G \to H$ is a coarse equivalence between Polish groups of bounded geometry. Without loss of generality, we may suppose that H is a coarsely embedded closed subgroup of a locally Roelcke precompact Polish group $\mathbb{H}$. Fix also coarsely proper left-invariant metrics ∂ on G and d on $\mathbb{H}$, whence the restriction of d to H is coarsely proper too. By rescaling ∂, we may suppose that the open ball of radius 1 is a gauge for G and that $\phi(g)\neq\phi(f)$ whenever $\partial(g,f)\geqslant 1$.

We let κ_ϕ, ω_ϕ be respectively the compression and the expansion moduli of ϕ, defined by

$$\kappa_\phi(t)=\inf_{\partial(g,f)\geqslant t} d(\phi(g),\phi(f)).$$

and

$$\omega_\phi(t)=\sup_{\partial(g,f)\leqslant t} d(\phi(g),\phi(f)).$$

Since $\phi[G]$ is cobounded in H, we may choose some $R>\sup_{h\in H} d(h,\phi[G])$.

Let also $d_\wedge$ be the compatible metric for the Roelcke uniformity on $\mathbb{H}$ given by

$$d_\wedge(h,f)=\inf_{k\in\mathbb{H}} d(h,k)+d(k^{-1},f^{-1}).$$

By compatibility with the uniformity, $d_\wedge$ extends to a compatible metric on the Roelcke completion $\widehat{\mathbb{H}}$ of $\mathbb{H}$, which we shall still denote $d_\wedge$. Note that, for any $\alpha>0$ and $h\in\mathbb{H}$, we have $B_{d_\wedge}(h,\alpha)=B_d(1,\alpha)hB_d(1,\alpha)$, where $B_{d_\wedge}(h,\alpha)$ and $B_d(1,\alpha)$ are the $d_\wedge$ and d-balls of radius α centred at h, respectively at 1. So $B_{d_\wedge}(h,\alpha)$ is Roelcke precompact, showing that $d_\wedge$ is a proper metric on $\widehat{\mathbb{H}}$.

Let X denote the closure of H inside $\widehat{\mathbb{H}}$, whence X is locally compact and the restriction of $d_\wedge$ to X remains proper. Also, as H has bounded geometry, the left shift action of H on itself extends to a coarsely proper, modest, cocompact continuous action $H\curvearrowright X$.

Let now $\Gamma\subseteq G$ be a maximally 1-discrete subset, i.e., maximal so that $\partial(a,b)\geqslant 1$ for distinct $a,b\in\Gamma$. By maximality, Γ is 1-dense in G. For every $a\in\Gamma$, define $\theta_a\colon G\to[0,2]$ by $\theta_a(g)=\max\{0,2-\partial(g,a)\}$. Note that θ_a is 1-Lipschitz and $\theta_a\geqslant 1$ on a ball of radius 1 centred at a, while $\mathrm{supp}(\theta_a)$ is contained in the 2-ball around a. Thus, if M is the maximum size of a 1-discrete set in a ball of radius 2, we see that every $g\in G$ belongs to the support of at least 1 and at most M many distinct functions θ_a. It follows that

$$\Theta(h)=\sum_{a\in\Gamma}\theta_a(h)$$

is a Lipschitz function with $1\leqslant\Theta\leqslant 2M$. Therefore, setting $\lambda_a=\frac{\theta_a}{\Theta}$, we have a partition of unity $\{\lambda_a\}_{a\in\Gamma}$ by Lipschitz functions with some uniform Lipschitz constant C and each λ_a supported in the 2-ball centred at a. Also, for every $g\in G$, the set

$$S_g=\{a\in\Gamma\mid \lambda_a(g)>0\}$$

has diameter at most 4 and cardinality at most M.

Consider now the free $\mathbb{R}$-vector space $\mathbb{R}^X$ over X, i.e., with basis $\{\mathbf{1}_x\}_{x\in X}$, and let Δ be the subset consisting of all finite convex combinations $\sum_{i=1}^m\alpha_i\mathbf{1}_{x_i}$ of basis vectors. Let also $\mathbb{M}(X)$ be the subspace of molecules, i.e., $m\in\mathbb{R}^X$ so that $\sum_{x\in X}m(x)=0$. Alternatively, $\mathbb{M}(X)$ is the hyperplane in $\mathbb{R}^X$ consisting of all linear combinations of atoms $\mathbf{1}_x-\mathbf{1}_y$, $x,y\in X$. The space of molecules can be equipped with the Arens–Eells norm $\|\cdot\|_{\text{Æ}}$, given by

$$\|m\|_{\text{Æ}}=\inf\Big(\sum_{i=1}^n|t_i|d_\wedge(x_i,y_i)\;\Big|\;m=\sum_{i=1}^n t_i(\mathbf{1}_{x_i}-\mathbf{1}_{y_i})\Big).$$

A basic fact (see [**77**]) is that this norm may alternatively be computed by

$$\|m\|_{\text{Æ}} = \sup\Big(\sum_{x\in X} m(x)F(x) \;\Big|\; F\colon (X,d_\wedge)\to\mathbb{R} \text{ is 1-Lipschitz }\Big).$$

Note that, if F_1 and F_2 differ by a constant, then, since $\sum_{x\in X} m(x)=0$, we have

$$\sum_{x\in X} m(x)F_1(x) = \sum_{x\in X} m(x)F_2(x).$$

Thus, in the computation of the norm, one may additionally require the Lipschitz function F to be 0 at any single point $x_0\in X$ we choose. Finally, define a metric $d_{\text{Æ}}$ on Δ by letting $d_{\text{Æ}}(v,w)=\|v-w\|_{\text{Æ}}$ for $v,w\in\Delta$.

The distance $d_{\text{Æ}}(v,w)$ is also sometimes called the *Kantorovich* distance and measures an optimal transport between the source v and the sink w. Thus, for example,

$$d_\wedge(\mathrm{supp}(v),\mathrm{supp}(w)) \;\leqslant\; d_{\text{Æ}}(v,w) \;\leqslant\; \sup_{\substack{x\in\mathrm{supp}(v)\\ y\in\mathrm{supp}(w)}} d_\wedge(x,y),$$

where the first inequality is seen using the 1-Lipschitz function given by $F(x)=d_\wedge(x,\mathrm{supp}(v))$.

The continuous action $H\curvearrowright X$ extends naturally to an action on $\mathbb{R}^X$ leaving Δ invariant by $h\cdot\big(\sum_{i=1}^m t_i\mathbf{1}_{x_i}\big)=\sum_{i=1}^m t_i\mathbf{1}_{hx_i}$. A priori, the action $H\curvearrowright\Delta$ may not be continuous. Nevertheless, using this, we define commuting left and right actions $H\curvearrowright\Delta^G\curvearrowleft G$ by setting, for $g,f\in G$, $h\in H$ and $\xi\in\Delta^G$,

$$\big(h\cdot\xi\big)_f = h\cdot\xi_f \quad\text{ and }\quad \big(\xi\cdot g\big)_f=\xi_{gf}.$$

While the action by G is clearly by homeomorphisms, it may not be continuous.

We now define $\psi\in\Delta^G$ by

$$\psi_g=\sum_{a\in\Gamma}\lambda_a(g)\mathbf{1}_{\phi(a)}=\sum_{a\in S_g}\lambda_a(g)\mathbf{1}_{\phi(a)}$$

and observe that each ψ_g is a convex combination of at most M basis vectors. Also, for all $g\in G$,

$$\mathrm{supp}(\psi_g)\subseteq\phi[S_g]\subseteq\overline{B}_d(\phi(g),\omega_\phi(2))\subseteq\overline{B}_{d_\wedge}(\phi(g),\omega_\phi(2))$$

and thus, for all $g\in G$ and $h\in H$,

$$\mathrm{diam}_{d_\wedge}\big(\mathrm{supp}(h\psi_g)\big)\leqslant\mathrm{diam}_d\big(\mathrm{supp}(h\psi_g)\big)\leqslant\mathrm{diam}_d\big(h\cdot\phi[S_g]\big)\leqslant\omega_\phi(4)$$

and

$$d_\wedge\big(\mathrm{supp}(\psi_g),\mathrm{supp}(\psi_f)\big)\leqslant d_\wedge(\phi(g),\phi(f))+2\omega_\phi(2).$$

We define

$$Y=\{v\in\Delta\mid \mathrm{supp}(v)\text{ has cardinality at most } M \text{ and } d_\wedge\text{-diameter at most } \omega_\phi(4)\ \}.$$

Claim 5.23. *The metric $d_{\text{Æ}}$ is proper on Y.*

Proof. Fix a finite diameter r. It suffices to show that any sequence (v_n) in Y satisfying $d_{\text{Æ}}(v_n,v_0)\leqslant r$ for all n has a subsequence converging in Y. To see this, write $v_n=\sum_{i=1}^M\alpha_{i,n}\mathbf{1}_{x_{i,n}}$ for some $x_{i,n}\in X$ and $\alpha_{i,n}\geqslant 0$. As

$$d_\wedge(\mathrm{supp}(v_n),\mathrm{supp}(v_0))\leqslant d_{\text{Æ}}(v_n,v_0)\leqslant r$$

and $\mathrm{diam}_{d_\wedge}\big(\mathrm{supp}(v_n)\big)\leqslant\omega_\phi(4)$, we see that the $\mathrm{supp}(v_n)$ all lie in some $d_\wedge$-bounded and thus compact subset of X. Therefore, by passing to a subsequence, we may suppose that $\alpha_i = \lim_n \alpha_{i,n}$ and $x_i = \lim_n x_{i,n}$ exist for all $i = 1, \dots, M$. Thus $\mathrm{diam}_{d_\wedge}\big(\{x_1, \dots, x_M\}\big) \leqslant \omega_\phi(4)$ and hence $v = \sum_{i=1}^M \alpha_i \mathbf{1}_{x_i} \in Y$. Moreover, if $F\colon (X, d_\wedge) \to \mathbb{R}$ is 1-Lipschitz with $F(x_1) = 0$, then $|F|$ is bounded by $\omega_\phi(4)$ on $\{x_1, \dots, x_M\}$ and so

$$
\begin{aligned}
\sum_{x\in X} (v - v_n)(x)F(x) &\leqslant \sum_{i=1}^M |\alpha_i F(x_i) - \alpha_{i,n}F(x_{i,n})| \\
&\leqslant \sum_{i=1}^M \big(|\alpha_i - \alpha_{i,n}| \cdot |F(x_i)| + |\alpha_{i,n}| \cdot |F(x_i) - F(x_{i,n})|\big) \\
&\leqslant \omega_\phi(4) \cdot \sum_{i=1}^M \big(|\alpha_i - \alpha_{i,n}| + |F(x_i) - F(x_{i,n})|\big) \\
&\leqslant \omega_\phi(4) \cdot \sum_{i=1}^M \big(|\alpha_i - \alpha_{i,n}| + d_\wedge(x_i, x_{i,n})\big).
\end{aligned}
$$

Majorising over all such F, we find that

$$
d_{\text{Æ}}(v, v_n) \leqslant \omega_\phi(4) \cdot \sum_{i=1}^M \big(|\alpha_i - \alpha_{i,n}| + d_\wedge(x_i, x_{i,n})\big).
$$

By consequence, $d_{\text{Æ}}(v, v_n) \to 0$ as required. □

We now consider the closed H and G-invariant subspace $\Omega = \overline{H \cdot \psi \cdot G}$ of Y^G, where Y^G is equipped with the product topology.

Claim 5.24. *For all $\xi \in \Omega$ and $f_1, f_2 \in G$, we have*

$$
d_{\text{Æ}}\big(\xi_{f_1}, \xi_{f_2}\big) \leqslant 2MC \cdot \partial(f_1, f_2) \cdot \omega_\phi\big(\partial(f_1, f_2) + 4\big).
$$

It follows that the action $\Omega \curvearrowleft G$ is continuous.

Proof. We first verify this inequality for $\xi \in \Omega$ of the form $\xi = h\psi g$. Observe that, for such ξ, we have

$$
\begin{aligned}
\xi_{f_1} - \xi_{f_2} &= h\psi_{gf_1} - h\psi_{gf_2} \\
&= \sum_{a \in S_{gf_1} \cup S_{gf_2}} \big(\lambda_a(gf_1) - \lambda_a(gf_2)\big)\mathbf{1}_{h\phi(a)}.
\end{aligned}
$$

Assume now that $F\colon (X, d_\wedge) \to \mathbb{R}$ is 1-Lipschitz and 0 at some point of $h\phi[S_{gf_1}]$. Then $|F|$ is bounded by $\mathrm{diam}_{d_\wedge}\big(h\phi[S_{gf_1} \cup S_{gf_2}]\big)$ on $h\phi[S_{gf_1} \cup S_{gf_2}]$, whence

$$
\begin{aligned}
\sum_{x\in X} \big(\xi_{f_1} - \xi_{f_2}\big)(x) \cdot F(x) &= \sum_{a \in S_{gf_1} \cup S_{gf_2}} \big(\lambda_a(gf_1) - \lambda_a(gf_2)\big) \cdot F(h\phi(a)) \\
&\leqslant \sum_{a \in S_{gf_1} \cup S_{gf_2}} \big|\lambda_a(gf_1) - \lambda_a(gf_2)\big| \cdot |F(h\phi(a))| \\
&\leqslant \big|S_{gf_1} \cup S_{gf_2}\big| \cdot C \cdot \partial(gf_1, gf_2) \cdot \mathrm{diam}_{d_\wedge}\big(h\phi[S_{gf_1} \cup S_{gf_2}]\big) \\
&\leqslant 2MC \cdot \partial(f_1, f_2) \cdot \mathrm{diam}_d\big(h\phi[S_{gf_1} \cup S_{gf_2}]\big) \\
&\leqslant 2MC \cdot \partial(f_1, f_2) \cdot \omega_\phi\big(\mathrm{diam}_\partial(S_{gf_1} \cup S_{gf_2})\big) \\
&\leqslant 2MC \cdot \partial(f_1, f_2) \cdot \omega_\phi\big(\partial(f_1, f_2) + 4\big).
\end{aligned}
$$

By majorising over all such F, we find that

$$\begin{aligned} d_{Æ}\big(\xi_{f_1},\xi_{f_2}\big) &= \|\xi_{f_1}-\xi_{f_2}\|_{Æ} \\ &\leqslant 2MC\cdot\partial(f_1,f_2)\cdot\omega_\phi\big(\partial(f_1,f_2)+4\big). \end{aligned}$$

The case for general $\xi\in\Omega$ is now immediate by density of $H\cdot\psi\cdot G$ in Ω.

To check continuity at some pair $\xi\in\Omega$ and $g\in G$, it suffices to verify that, for any $f\in G$ and $\epsilon>0$, there are neighbourhoods V of ξ and W of g so that $d_{Æ}((\zeta\cdot k)_f,(\xi\cdot g)_f)<\epsilon$ whenever $\zeta\in V$ and $k\in W$.

For this, set $\zeta\in V$ if $d_{Æ}(\zeta_{gf},\xi_{gf})<\epsilon/2$ and let

$$W=\{k\in G\mid 2MC\cdot\partial(kf,gf)\cdot\omega_\phi\big(\partial(kf,gf)+4\big)<\epsilon/2\}.$$

Then, given $\zeta\in V$ and $k\in W$, we have

$$\begin{aligned} d_{Æ}\big((\zeta\cdot k)_f,(\xi\cdot g)_f\big) &\leqslant d_{Æ}(\zeta_{kf},\zeta_{gf})+d_{Æ}(\zeta_{gf},\xi_{gf}) \\ &\leqslant 2MC\cdot\partial(kf,gf)\cdot\omega_\phi\big(\partial(kf,gf)+4\big)+\epsilon/2 \\ &<\epsilon \end{aligned}$$

as required. □

Claim 5.25. *The space Ω is locally compact. In fact, for every $r>0$ and $\xi\in\Omega$, the set $\{\zeta\in\Omega\mid d_{Æ}(\xi_1,\zeta_1)\leqslant r\}$ is a compact neighbourhood of ξ.*

Proof. Suppose that $\zeta\in\Omega$ satisfies $d_{Æ}(\xi_1,\zeta_1)\leqslant r$. Then, for all $f\in G$, we have

$$\begin{aligned} d_{Æ}(\xi_f,\zeta_f) &\leqslant d_{Æ}(\xi_f,\xi_1)+d_{Æ}(\xi_1,\zeta_1)+d_{Æ}(\zeta_1,\zeta_f) \\ &\leqslant 4MC\cdot\partial(f,1)\cdot\omega_\phi\big(\partial(f,1)+4\big)+r. \end{aligned}$$

It therefore follows that

$$\{\zeta\in\Omega\mid d_{Æ}(\xi_1,\zeta_1)\leqslant r\}\subseteq\prod_{f\in G}\overline{B}_{d_{Æ}}\Big(\xi_f,4MC\cdot\partial(f,1)\cdot\omega_\phi\big(\partial(f,1)+4\big)+r\Big),$$

where, by properness of $d_{Æ}$ on Y, the latter product is compact. Thus, $\{\zeta\in\Omega\mid d_{Æ}(\xi_1,\zeta_1)\leqslant r\}$ is a compact neighbourhood of $\xi\in\Omega$. □

Claim 5.26. *The action $\Omega\curvearrowleft G$ is coarsely proper.*

Proof. By Claim 5.25, it suffices to show that, for all $\xi\in\Omega$ and $r>0$, the set

$$\{f\in G\mid \exists\zeta\in\Omega\quad d_{Æ}(\zeta_1,\xi_1)<r\ \&\ d_{Æ}(\zeta_f,\xi_1)<r\}$$

is bounded in G.

Suppose first that $U\subseteq X$ is open and relatively compact. Assume that $h\in H$ and $g,f\in G$ satisfy $\mathrm{supp}(h\psi_{gf})\cap U\neq\emptyset$ and $\mathrm{supp}(h\psi_g)\cap U\neq\emptyset$. Then

$$d(\phi(gf),\phi(g))=d(h\phi(gf),h\phi(g))\leqslant 2\omega_\phi(2)+\mathrm{diam}_d(U\cap H)$$

and thus $\kappa_\phi(\partial(f,1))=\kappa_\phi(\partial(gf,g))\leqslant 2\omega_\phi(2)+\mathrm{diam}_d(U\cap H)$. It thus follows that there is some constant c_U depending only on U so that $\partial(f,1)\leqslant c_U$. Therefore, if $\zeta\in\Omega$ and $f\in G$ satisfy $\mathrm{supp}(\zeta_1)\cap U\neq\emptyset$ and $\mathrm{supp}(\zeta_f)\cap U\neq\emptyset$, we also have $\partial(f,1)\leqslant c_U$, since otherwise ζ may be approximated by some $h\psi g$ with the same property.

Let now $U=\{x\in X\mid d_\wedge(x,\mathrm{supp}(\xi_1))<r\}$ and assume that $\zeta\in\Omega$ and $f\in G$ satisfy $d_{Æ}(\zeta_1,\xi_1)<r$ and $d_{Æ}(\zeta_f,\xi_1)<r$. Then

$$d_\wedge(\mathrm{supp}(\zeta_1),\mathrm{supp}(\xi_1))<r,\quad d_\wedge(\mathrm{supp}(\zeta_f),\mathrm{supp}(\xi_1))<r$$

and hence $\mathrm{supp}(\zeta_1) \cap U \neq \emptyset$ and $\mathrm{supp}(\zeta_f) \cap U \neq \emptyset$. We therefore conclude that $\partial(f, 1) \leqslant c_U$ as required. $\square$

CLAIM 5.27. *The action $\Omega \curvearrowleft G$ is modest.*

PROOF. To see that the action is modest, by Lemma 5.25 it suffices to show that, for any coarsely bounded set $B \subseteq G$, $\xi \in \Omega$ and $r > 0$, the set

$$\{\zeta \in \Omega \mid d_{Æ}(\zeta_1, \xi_1) \leqslant r\} \cdot B$$

is relatively compact in Ω. So let $s = \sup_{g\in B} \partial(g, 1)$ and observe that, for all $\zeta \in \Omega$ and $g \in B$,

$$d_{Æ}\big(\zeta_g, \zeta_1\big) \leqslant 2MC \cdot \partial(g, 1) \cdot \omega_\phi\big(\partial(g, 1) + 4\big) \leqslant 2MCs \cdot \omega_\phi\big(s + 4\big).$$

Thus, if $d_{Æ}(\zeta_1, \xi_1) \leqslant r$ and $g \in B$, we have

$$d_{Æ}\big((\zeta g)_1, \xi_1\big) = d_{Æ}\big(\zeta_g, \xi_1\big) \leqslant r + 2MCs \cdot \omega_\phi\big(s + 4\big).$$

In other words,

$$\{\zeta \in \Omega \mid d_{Æ}(\zeta_1, \xi_1) \leqslant r\} \cdot B \ \subseteq\ \{\eta \in \Omega \mid d_{Æ}(\eta_1, \xi_1) \leqslant r + 2MCs \cdot \omega_\phi\big(s + 4\big)\big\}.$$

As the latter set is compact, modesty follows. $\square$

CLAIM 5.28. *The action $\Omega \curvearrowleft G$ is cocompact.*

PROOF. Consider $\mathbf{1}_{1_H} \in Y$ and observe that, for all $h \in H$ and $g \in G$, there is some $f \in G$ so that $d_\wedge(h\phi(gf), 1_H) \leqslant d(h\phi(gf), 1_H) = d(\phi(gf), h^{-1}) < R$ and thus

$$d_{Æ}(h\psi_{gf}, \mathbf{1}_{1_H}) < R + \omega_\phi(2).$$

We will show that, for any $\xi \in \Omega$, there is $f \in G$ so that

$$d_{Æ}(\xi_f, \mathbf{1}_{1_H}) \leqslant R + \omega_\phi(2) + 2MC \cdot \omega_\phi(5),$$

which shows that the set

$$\{\zeta \in \Omega \mid d_{Æ}(\zeta_1, \mathbf{1}_{1_H}) \leqslant R + \omega_\phi(2) + 2MC \cdot \omega_\phi(5)\}$$

is a compact fundamental domain for the action $\Omega \curvearrowleft G$.

To see this, let

$$U = \{x \in X \mid d_\wedge(x, \mathrm{supp}(\xi_1)) < 1\},$$

set

$$r = \sup\{t \mid \omega_\phi(t) \leqslant R + d(1_H, U \cap H) + \mathrm{diam}_d(U \cap H) + \omega_\phi(2)\}$$

and let $f_1, \ldots, f_n$ to be 1-dense in $B_\partial(1_G, r)$. Assume towards a contradiction that

$$d_{Æ}(\xi_f, \mathbf{1}_{1_H}) > R + \omega_\phi(2) + 2MC \cdot \omega_\phi(5)$$

for all $f \in G$.

Choose then some $h\psi g \in H \cdot \psi \cdot G$ close enough to ξ so that $d_{Æ}((h\psi g)_1, \xi_1) < 1$ and

$$d_{Æ}((h\psi g)_{f_i}, \mathbf{1}_{1_H}) > R + \omega_\phi(2) + 2MC \cdot \omega_\phi(5)$$

for all i. Now, if $f \in B_\partial(1_G, r)$, we may find f_i with $\partial(f, f_i) \leqslant 1$ and thus

$$\begin{aligned} d_{Æ}\big((h\psi g)_f, (h\psi g)_{f_i}\big) &\leqslant 2MC \cdot \partial(f, f_i) \cdot \omega_\phi\big(\partial(f, f_i) + 4\big) \\ &\leqslant 2MC \cdot \omega_\phi(5). \end{aligned}$$

We conclude that $d_{Æ}((h\psi g)_f, \mathbf{1}_{1_H}) > R + \omega_\phi(2)$ for all $f \in B_\partial(1_G, r)$.

However, as $d_{Æ}((h\psi g)_1, \xi_1) < 1$, we have

$$d_\wedge\big(\mathrm{supp}\big((h\psi g)_1\big), \mathrm{supp}(\xi_1)\big) < 1$$

and $\mathrm{supp}\big((h\psi g)_1\big) \cap U \neq \emptyset$. Then, if we choose $f \in G$ satisfying $d(h\phi(gf), 1_H) = d(\phi(gf), h^{-1}) < R$ and thus also $d_{Æ}(h\psi_{gf}, \mathbf{1}_{1_H}) < R + \omega_\phi(2)$, we have

$$\begin{aligned}\kappa_\phi\big(\partial(f, 1_G)\big) &\leqslant d\big(h\phi(gf), h\phi(g)\big)\\ &\leqslant d\big(h\phi(gf), 1_H\big) + d\big(1_H, h\phi(g)\big)\\ &\leqslant R + d(1_H, U \cap H) + \mathrm{diam}_d(U \cap H) + \omega_\phi(2)\end{aligned}$$

and so $f \in B_\partial(1_G, r)$, which is absurd. $\square$

We finally verify that the action $H \curvearrowright \Omega$ is continuous, coarsely proper, modest and cocompact.

For coarse properness, note that, by Claim 5.25, every compact subset of Ω is contained in some set of the form $D = \{\zeta \in \Omega \mid d_{Æ}(\zeta_1, \mathbf{1}_{1_H}) \leqslant r\}$. So let

$$Z = \{x \in X \mid d_\wedge(x, 1_H) \leqslant r + \omega_\phi(2)\}$$

and observe that, as the action of H on X is coarsely proper, there is a coarsely bounded set $B \subseteq H$ so that $h \cdot Z \cap Z = \emptyset$ for all $h \in H \setminus B$. Thus, for any $h \in H \setminus B$ and $\zeta \in \Omega$ with

$$d_\wedge\big(\mathrm{supp}(\zeta_1), 1_H\big) \leqslant d_{Æ}(\zeta_1, \mathbf{1}_{1_H}) \leqslant r,$$

we have $\mathrm{supp}(\zeta_1) \subseteq Z$ and thus

$$d_{Æ}\big((h\zeta)_1, \mathbf{1}_{1_H}\big) \geqslant d_\wedge\big(h \cdot \mathrm{supp}(\zeta_1), 1_H\big) > r + \omega_\phi(2),$$

i.e., $h \cdot D \cap D = \emptyset$, showing that the action is coarsely proper.

Similarly, to verify modesty, suppose a compact set

$$D = \{\zeta \in \Omega \mid d_{Æ}(\zeta_1, \mathbf{1}_{1_H}) \leqslant r\}$$

and a coarsely bounded set $B \subseteq H$ are given. Again, let

$$Z = \{x \in X \mid d_\wedge\big(x, 1_H\big) \leqslant r + \omega_\phi(2)\}$$

and note that, if $\zeta \in D$, then $\mathrm{supp}(\zeta_1) \subseteq Z$. Thus, if $\zeta \in D$ and $h \in B$, then $\mathrm{supp}\big((h\zeta)_1\big) \subseteq B \cdot Z$. Letting $s = \sup_{x \in B \cdot Z} d_\wedge(x, 1_H)$, we see that

$$B \cdot D \subseteq \{\eta \in \Omega \mid d_{Æ}(\eta_1, \mathbf{1}_{1_H}) \leqslant s\},$$

establishing modesty.

Since the action of H on X is cocompact, pick a compact set $Z \subseteq X$ so that $X = H \cdot Z$. Then, for every $\xi \in \Omega$, there is some $h \in H$ with $\mathrm{supp}\big((h\xi)_1\big) = h \cdot \mathrm{supp}\big(\xi_1\big)$ intersects Z. Picking some $z \in Z$, we see that, for any $\xi \in \Omega$, there is $h \in H$ so that

$$d_{Æ}\big((h\xi)_1, \mathbf{1}_z\big) \leqslant \omega_\phi(4) + \mathrm{diam}_{d_\wedge}(Z).$$

Setting $D = \{\zeta \in \Omega \mid d_{Æ}(\zeta_1, \mathbf{1}_z) \leqslant \omega_\phi(4) + \mathrm{diam}_{d_\wedge}(Z)\}$, which is compact, we have $H \cdot D = \Omega$, thus verifying cocompactness.

We now check continuity of the action at a pair $h \in H$ and $\xi \in \Omega$. For this, for any $g \in G$ and $\epsilon > 0$, we must find neighbourhoods U of h and V of ξ so that

$$d_{Æ}\big(h\xi_g, k\zeta_g\big) = d_{Æ}\big((h\xi)_g, (k\zeta)_g\big) < \epsilon$$

for all $k \in U$ and $\zeta \in V$. Observe that, since

$$d_{Æ}\big((h\xi)_g, (k\zeta)_g\big) = d_{Æ}\big(h\xi_g, h\zeta_g\big) + d_{Æ}\big(h\zeta_g, k\zeta_g\big),$$

it suffices to find neighbourhhods U of h and W of ξ_g in Y, so that for $k \in U$ and $w \in W$, we have $d_{Æ}\big(h\xi_g, hw\big) < \epsilon/2$ and $d_{Æ}\big(hw, kw\big) < \epsilon/2$.

Since the action $H \curvearrowright X$ is continuous and $d_\wedge$ is proper, we may find a neighbourhood U of h so that $d_\wedge\big(h(x), k(x)\big) < \epsilon/2$ for all $x \in X$ within distance $\omega_\phi(4)+1$ of $\mathrm{supp}(\xi_g)$. Thus, if $w \in Y$ and

$$d_\wedge\big(\mathrm{supp}(w), \mathrm{supp}(\xi_g)\big) \leqslant d_{Æ}(w, \xi_g) < 1,$$

then all of $\mathrm{supp}(w)$ is within distance $\omega_\phi(4)+1$ of $\mathrm{supp}(\xi_g)$ and so $d_{Æ}\big(hw, kw\big) < \epsilon/2$ whenever $k \in U$.

Observe that, since the set $Z = \{x \in X \mid d_\wedge(x, \mathrm{supp}(\xi_g)) \leqslant \omega_\phi(4) + 1\}$ is compact, the restriction of h to Z is uniformly continuous and the displacement $r = \sup_{x,y\in Z} d_\wedge(h(x), h(y))$ is finite. So find $0 < \delta < \frac{4r}{\epsilon}$ small enough so that, for $x, y \in Z$,

$$d_\wedge(x, y) < \delta \;\;\Rightarrow\;\; d_\wedge(h(x), h(y)) < \epsilon/4.$$

Assume $w \in Y$ satisfies $d_{Æ}(w, \xi_g) = \|w - \xi_g\|_{Æ} < \frac{\epsilon\delta}{4r} < 1$. Then, by the definition of the norm, we can write $w = \sum_{i=1}^m \alpha_i \mathbf{1}_{x_i}$ and $\xi_g = \sum_{i=1}^m \alpha_i \mathbf{1}_{y_i}$, where $\alpha_i > 0$, $x_i \in Z$ and

$$\sum_{d_\wedge(x_i,y_i)\geqslant\delta} \alpha_i \;<\; \frac{1}{\delta}\cdot\frac{\epsilon\delta}{4r} = \frac{\epsilon}{4r}.$$

It follows that

$$\begin{aligned}
d_{Æ}(hw, h\xi_g) &= \|hw - h\xi_g\|_{Æ} \\
&\leqslant \sum_{i=1}^m \alpha_i d_\wedge(hx_i, hy_i) \\
&\leqslant \sum_{d_\wedge(x_i,y_i)\geqslant\delta} \alpha_i d_\wedge(hx_i, hy_i) + \sum_{d_\wedge(x_i,y_i)<\delta} \alpha_i d_\wedge(hx_i, hy_i) \\
&\leqslant \sum_{d_\wedge(x_i,y_i)\geqslant\delta} \alpha_i r + \sum_{d_\wedge(x_i,y_i)<\delta} \alpha_i\cdot\epsilon/4 \\
&< \epsilon/4 + \epsilon/4.
\end{aligned}$$

Letting $W = \{w \in Y \mid d_{Æ}(w, \xi_g) < \frac{\epsilon\delta}{4r}\}$, we conclude that $H \curvearrowright \Omega$ is continuous.

As both of the commuting actions $H \curvearrowright \Omega$ and $\Omega \curvearrowleft G$ are continuous, coarsely proper, modest and cocompact, we have a topological coupling of H and G. $\square$

5. Coarse couplings

We shall now consider a more algebraic notion of similarity between Polish groups strengthening coarse equivalence and reminiscent of the Gromov–Hausdorff distance between metric spaces. Recall first that the *Hausdorff distance* between two subsets A and B of a metric space (X, d) is given by

$$d_{\mathrm{Haus}}(A, B) = \max\{\sup_{a\in A}\inf_{b\in B} d(a, b), \sup_{b\in B}\inf_{a\in A} d(a, b)\}.$$

For example, if $G \curvearrowright (X, d)$ is an isometric group action, then the Hausdorff distance between any two orbits $G\cdot x$ and $G\cdot y$ is simply

$$d_{\mathrm{Haus}}(Gx, Gy) = \inf_{g\in G} d\big(x, g(y)\big).$$

Thus, if $G, F \curvearrowright (X, d)$ are isometric group actions having two orbits $G\cdot x$ and $F\cdot y$ with finite Hausdorff distance, then any two orbits of G and F have finite

Hausdorff distance. We define two isometric actions $G, F \curvearrowright (X, d)$ to be *proximal* if the Hausdorff distance between some two or, equivalently, any two G and F-orbits is finite.

More abstractly, the *Gromov–Hausdorff distance* between two metric spaces X and Y is given by the infimum of the Hausdorff distance $d_{\mathrm{Haus}}\big(i[X], j[Y]\big)$ where i and j vary over all isometric embeddings $i\colon X \to Z$ and $j\colon Y \to Z$ into a common metric space (Z, d).

In case of topological groups, we would like also to preserve the algebraic structure of the groups and are led to consider two levels of proximity. Thus, suppose $G \hookrightarrow H \hookleftarrow F$ is a pair of simultaneously isomorphic and coarse embeddings of Polish groups G and F into a Polish group H and identify G and F with their images in H. Consider the following two conditions.

(1) Both G and F are cobounded in $G \cup F$, i.e., $F \subseteq GB$ and $G \subseteq FB$ for some coarsely bounded set $B \subseteq H$,
(2) both G and F are cobounded in H, i.e., $H = GB = FB$ for some coarsely bounded set $B \subseteq H$.

Evidently, (2) implies (1), while, on the other hand, (1) implies that G and F are coarsely equivalent, as they are both coarsely equivalent to $G \cup F$ with the coarse structure induced from H. Moreover, if H has a coarsely proper left-invariant metric, then (1) is equivalent to requiring that G and F have finite Hausdorff distance in H. We spell this out in the following definition.

Definition 5.29. *A* coarse coupling *of Polish groups G and F is a pair of simultaneously isomorphic and coarse embeddings $G \hookrightarrow H \hookleftarrow F$ into a locally bounded Polish group H so that, when identifying G and F with their images in H,*

$$d_{\mathrm{Haus}}(G, F) < \infty$$

for any coarsely proper metric d on H.

We now have the following basic equivalence characterising coarse couplings.

Theorem 5.30. *The following are equivalent for Polish groups G and F.*

(1) *G and F admit a coarse coupling $G \hookrightarrow H \hookleftarrow F$,*
(2) *G and F have proximal, coarsely proper, continuous, isometric actions on a metric space.*

If furthermore G and F have bounded geometry, these conditions imply that G and F admit a topological coupling.

Proof. Suppose $G, F \curvearrowright (X, d_X)$ are two proximal, coarsely proper, continuous, isometric actions on a metric space (X, d_X). Observe then that, for $x \in X$ fixed, the subspace $Y = \bigcup_n (FG)^n \cdot x$ is separable and simultaneously G and F-invariant. Let $\overline{Y}$ denote the completion of (Y, d_X) and $S_{\mathbb{U}}$ the Urysohn sphere, that is, a sphere of radius 1 in the Urysohn metric space. Since $\mathrm{Isom}(S_{\mathbb{U}})$ is a universal Polish group, G and F admit isomorphic embeddings into $\mathrm{Isom}(S_{\mathbb{U}})$ and we let $G, F \curvearrowright S_{\mathbb{U}}$ denote the corresponding isometric actions. Then we can let G and F act diagonally on $Z = \overline{Y} \times S_{\mathbb{U}}$, which preserves the sum d_Z of the two metrics on $\overline{Y}$ and $S_{\mathbb{U}}$.

Using a construction of M. Katětov [**37**], we may find an isometric embedding ι of (Z, d_Z) into the Urysohn metric space $\mathbb{U}$ and an isomorphic embedding $\theta\colon \mathrm{Isom}(Z, d_Z) \to \mathrm{Isom}(\mathbb{U})$ so that, for each $h \in \mathrm{Isom}(Z, d_Z)$, the following diagram commutes

$$\begin{tikzcd} Z \arrow[r, "h"] \arrow[d, "\iota"] & Z \arrow[d, "\iota"] \\ \mathbb{U} \arrow[r, "\theta(h)"] & \mathbb{U} \end{tikzcd}$$

As G and H are isomorphically embedded in $\mathrm{Isom}(S_{\mathbb{U}})$, their actions on Z induce isomorphic embeddings into $\mathrm{Isom}(Z, d_Z)$ and thus further into $\mathrm{Isom}(\mathbb{U})$ via θ. Observe now that, as the actions of G and F on $\overline{Y}$ are proximal and $S_{\mathbb{U}}$ has finite diameter, so are the actions on $Z = \overline{Y} \times S_{\mathbb{U}}$. Moreover, as Z is isometrically embedded in $\mathbb{U}$ and the diagram above commutes, we conclude that also the actions of G and F on U are proximal. Similarly, the actions on X are coarsely proper and therefore the same applies to the actions on $\mathbb{U}$.

We may thus identify G and F with closed subgroups of $\mathrm{Isom}(\mathbb{U})$ whose actions on $\mathbb{U}$ are proximal and both coarsely proper. Now fix some $z \in \mathbb{U}$. Then the orbit map $h \in \mathrm{Isom}(\mathbb{U}) \mapsto h(z) \in \mathbb{U}$ is a coarse equivalence. It follows from the properness of the actions that G and F are coarsely embedded in $\mathrm{Isom}(\mathbb{U})$ and by the proximality of the actions that G and F are cobounded in $G \cup F$.

Conversely, if $G \hookrightarrow H \hookleftarrow F$ is a coarse coupling, fix a compatible left-invariant coarsely proper metric d on H. Then, as G and F are coarsely embedded, d is coarsely proper on each of them, so the left-shift actions $G, F \curvearrowright (H, d)$ are coarsely proper. Moreover, as $G \subseteq FB$ and $F \subseteq GB$ for some coarsely bounded set $B \subseteq H$, the orbits $G \cdot 1$ and $F \cdot 1$ have finite Hausdorff distance and so the actions are proximal. □

6. Representations on reflexive spaces

For the following result, we should recall some facts about proper affine isometric actions. Namely, suppose that G is a locally compact Polish group. Then, if G is amenable, it admits a continuous proper affine isometric action on a Hilbert space [**4**] (see also [**14**]). On the other hand, if G is non-amenable, it can in general only be shown to have a continuous proper affine isometric action on a reflexive Banach space [**12**].

For general Polish groups, these results are no longer valid and the obstructions seem to be of two different kinds; harmonic analytic and large scale geometric. For the harmonic analytic obstructions, note there are Polish groups such as $\mathrm{Homeo}_+([0,1])$ [**48**] and $\mathrm{Isom}(\mathbb{U})$ with no non-trivial continuous linear or affine isometric representations on reflexive spaces whatsoever. Moreover, these examples are in fact even amenable.

Eschewing such groups, one can restrict the attention to non-Archimedean groups, which all have faithful unitary representations. But even in this setting, large scale geometric obstructions appear. Indeed, there are amenable non-Archimedean Polish groups such as $\mathrm{Isom}(\mathbb{Z}\mathbb{U})$ all of whose affine isometric actions on reflexive Banach spaces have fixed points [**65**] and so aren't coarsely proper.

However, by [**12**], every metric space of bounded geometry is coarsely embeddable into a reflexive space and by combining that construction with an averaging result of F. M. Schneider and A. Thom [**67**] improving an earlier result in [**65**], we obtain the following.

THEOREM 5.31. *Let G be an amenable Polish group of bounded geometry. Then G admits a coarsely proper continuous affine isometric action on a reflexive Banach space.*

PROOF. Fix a gauge metric d on G and let $X \subseteq G$ be maximal so that $d(x,y) \geqslant 2$ for all distinct $x, y \in X$. It follows that, for every diameter n, there is an upper bound k_n on the cardinality of subsets of X of diameter n.

For $g \in G$, we now define a function $\phi_g^n \colon X \to [0,1]$ by

$$\phi_g^n(x) = \begin{cases} 1 - \frac{d(x,g)}{n} & \text{if } d(x,g) \leqslant n, \\ 0 & \text{otherwise.} \end{cases}$$

By construction of ϕ_g^n, we find that, for all $g, f \in G$,

$$\|\phi_g^n - \phi_f^n\|_\infty \leqslant \frac{d(g,f)}{n},$$

while

$$\|\phi_g^n - \phi_f^n\|_\infty \geqslant \frac{n-4}{n},$$

whenever $d(g,f) \geqslant n$. Since the supports of ϕ_g^n and ϕ_f^n have diameter $\leqslant 2n$ and thus cardinality $\leqslant k_{2n}$, there is some sufficiently large coefficient $p_n < \infty$ so that

$$\|\phi_g^n - \phi_f^n\|_\infty \leqslant \|\phi_g^n - \phi_f^n\|_{p_n} \leqslant 2\|\phi_g^n - \phi_f^n\|_\infty \leqslant \frac{2}{n} \cdot d(g,f)$$

for all $g, f \in G$.

Thus, $g \in G \mapsto \phi_g^n \in \ell^{p_n}(X)$ is a $\frac{2}{n}$-Lipschitz map satisfying $\|\phi_g^n - \phi_f^n\|_{p_n} \geqslant \frac{n-4}{n}$ whenever $d(g,f) \geqslant n$. As G is amenable, it follows from Theorem 6.1 in [**67**] that G admits a continuous isometric linear representation $\pi_n \colon G \curvearrowright V_n$ on a separable Banach space V_n finitely representable in $L^{p_n}(\ell^{p_n}(X)) = L^{p_n}$ along with a $\frac{2}{n}$-Lipschitz cocycle $b_n \colon G \to V_n$ for π_n satisfying $\|b_n(g) - b_n(f)\|_{V_n} \geqslant \frac{n-4}{n}$ whenever $d(g,f) \geqslant n$. As V_n is finitely representable in L^{p_n}, we have that V_n is reflexive.

Now let $E = \big(\bigoplus_n V_n\big)_2$ be the ℓ^2-sum of the V_n and let $\pi \colon G \curvearrowright E$ be the product of the linear representations π_n. We claim that $b(g) = (b_1(g), b_2(g), \dots)$ defines a continuous cocycle $b \colon G \to E$ for π. To see this, note that

$$\|b(g) - b(f)\|_E^2 = \sum_n \|b_n(g) - b_n(f)\|_{V_n}^2 \leqslant \Big(\sum_n \frac{4}{n^2}\Big) \cdot d(g,f)^2.$$

Similarly, if $d(g,f) \geqslant n$, then $\|b_m(g) - b_m(f)\|_{V_m} \geqslant \frac{m-4}{m} \geqslant \frac{1}{2}$ for all $m = 8, \dots, n$ and so

$$\|b(g) - b(f)\|^2 \geqslant \sum_{m=8}^{n} \|b_m(g) - b_m(f)\|_{V_m}^2 \geqslant \frac{n-7}{4},$$

which shows that b is coarsely proper. □

We should point out that the above proof also shows a stronger result. Namely, that amenable Polish groups of bounded geometry admit a continuous coarsely proper left-invariant stable écart ∂. Indeed, since each of the V_n are stable Banach spaces, by virtue of being finitely representable in the super-stable space L^{p_n}, we may simply let $\partial(g,f) = \|b(g) - b(f)\|_E$. That this is a stronger result follows from the fact that every Polish group with a continuous coarsely proper left-invariant stable écart has a coarsely proper continuous affine isometric action on a reflexive Banach space [**65**].

Let us also observe that bounded geometry in itself is not sufficient for Theorem 5.31. For this, consider the group $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ and let H_0 and $H_{\frac{1}{2}}$ denote the isotropy subgroups of 0 and $\frac{1}{2}$ respectively. Then it is fairly easy to see that every translation τ_α of $\mathbb{R}$ of amplitude $|\alpha| < \frac{1}{6}$ can be written as $\tau_\alpha = gf$ for $g \in H_0$ and $f \in H_{\frac{1}{2}}$. As also every element of $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ is a product of some translation τ_β and an element of H_0, we see that $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ is generated by the two subgroups H_0 and $H_{\frac{1}{2}}$. As each of these is isomorphic to the group $\mathrm{Homeo}_+([0,1])$ that has no non-trivial continuous isometric linear and thus also affine representations on reflexive Banach spaces [**48**], we conclude that the same holds for $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$.

7. Compact G-flows and unitary representations

In this section, we shall consider the impact of bounded geometry on the topological dynamics of a Polish group. Let us recall that a *compact G-flow* of a topological group G is a continuous action $G \curvearrowright K$ on a compact Hausdorff space K. Such a flow is *minimal* provided that all orbits are dense. Among all minimal compact G-flows, there is a flow $G \curvearrowright M$ of which all other flows are a factor, i.e., so that for any other minimal compact G-flow $G \curvearrowright K$ there is a continuous G-equivariant map $\phi\colon M \to K$. Moreover, up to isomorphism of G-flows, the flow $G \curvearrowright M$ is unique and is therefore denoted the *universal minimal flow* of G.

For a topological group G, let $\mathrm{LUC}(G)$ denote the commutative $\mathbb{C}^*$-algebra of all bounded left-uniformly continuous complex valued functions on G with the supremum norm. Then the right regular representation $\rho\colon G \curvearrowright \mathrm{LUC}(G)$ is continuous and thus induces a continuous action ρ^* on the Gelfand spectrum $\mathfrak{A}(G) = \mathrm{spec}\big(\mathrm{LUC}(G)\big)$ via

$$\big(\rho^*(g)\omega\big)\phi = \omega\big(\rho(g)\phi\big) = \omega\big(\phi(\,\cdot\, g)\big)$$

for $\omega \in \mathfrak{A}(G)$ and $\phi \in \mathrm{LUC}(G)$.

While the compact flow $G \curvearrowright \mathfrak{A}(G)$ is not necessarily minimal, it does have a dense orbit $G\cdot\omega$. Indeed, observe that each $g \in G$ defines an element $\omega_g \in \mathfrak{A}(G)$ by $\omega_g(\phi) = \phi(g)$ for $\phi \in \mathrm{LUC}(G)$. The map $g \mapsto \omega_g$ is a homeomorphic embedding of G into $\mathfrak{A}(G)$ with dense image and, as $\rho^*(g)\omega_f = \omega_{fg}$, we see that ω_{1_G} has a dense G-orbit in $\mathfrak{A}(G)$. Moreover, if $G \curvearrowright K$ is any compact G-flow with a dense orbit $G \cdot x$, then there is a unique continuous G-equivariant map $\phi\colon \mathfrak{A}(G) \to K$ so that $\phi(\omega_{1_G}) = x$. The space $\mathfrak{A}(G)$ is called the *greatest ambit* of G and every minimal subflow of $G \curvearrowright \mathfrak{A}(G)$ is a realisation of the universal minimal flow of G.

While the greatest ambit is general is a very large non-metrisable compact space, a much investigated and fairly common phenomenon among Polish groups G is that the universal minimal flow is metrisable or even reduces to a single point. In the latter case, we say that G is *extremely amenable*, since this is equivalent to every compact G-flow having a fixed point and thus implies amenability. On the other hand, as shown by R. Ellis [**24**] for discrete groups and later W. Veech [**76**] in all generality, every locally compact group acts freely on its greatest ambit and thus fails to be extremely amenable unless trivial. Our first goal is to indicate that this result is really geometric, rather than topological, by giving an appropriate generalisation to Polish groups of bounded geometry.

Proposition 5.32. *Suppose G is a Polish group of bounded geometry with gauge metric d. Then, for every constant α, there is a uniform entourage U_α in* *the greatest ambit* $\mathfrak{A}(G)$ *of* G*, so that*

$$\big(\rho^*(g)\omega, \rho^*(f)\omega\big) \notin U_\alpha$$

whenever $\omega \in \mathfrak{A}(G)$ *and* $g, f \in G$ *with* $9 \leqslant d(g, f) \leqslant \alpha$.

In particular, every $g \in G$ *with* $d(g, 1) \geqslant 9$ *acts freely on* $\mathfrak{A}(G)$.

Proof. Let $X \subseteq G$ be a maximal 2-discrete subset and suppose α is given. Then (X, d) has bounded geometry and can therefore be partitioned into finitely many $2(\alpha + 4)$-discrete sets $X = \bigcup_{i=1}^m X_i$. Define bounded 1-Lipschitz functions $\phi_i \colon (G, d) \to \mathbb{R}$ by $\phi_i(x) = \min\{\alpha + 4, d(x, X_i)\}$ and let

$$U_\alpha = \{(\omega, \eta) \in \mathfrak{A}(G) \times \mathfrak{A}(G) \mid |\omega(\phi_i) - \eta(\phi_i)| < 1, \forall i\}.$$

To see that U_α is as required, assume towards a contradiction that $\omega \in \mathfrak{A}(G)$ and $g, f \in G$ with $9 \leqslant d(g, f) \leqslant \alpha$ satisfy $(\rho^*(g)\omega, \rho^*(f)\omega) \in U_\alpha$, i.e., that

$$\max_i \Big|\big(\rho^*(g)\omega\big)(\phi_i) - \big(\rho^*(f)\omega\big)(\phi_i)\Big| < 1.$$

Since the ω_x with $x \in G$ are dense in $\mathfrak{A}(G)$, there is some $x \in G$ so that also

$$\max_i \Big|\phi_i(xg) - \phi_i(xf)\Big| = \max_i \Big|\big(\rho^*(g)\omega_x\big)(\phi_i) - |\big(\rho^*(f)\omega_x\big)(\phi_i)\Big| < 1.$$

Find now $a, b \in X$ with $d(xg, a) < 2$, $d(xf, b) < 2$, whence $|d(a, b) - d(g, f)| = |d(a, b) - d(xg, xf)| < 4$ and thus $d(a, b) < d(g, f) + 4 \leqslant \alpha + 4$. Assume that $a \in X_i$ and observe that since X_i is $2(\alpha + 4)$-discrete, a is the unique point in X_i within distance $\alpha + 4$ of b, whereby $\phi_i(b) = d(b, X_i) = d(b, a)$. Thus, as ϕ_i is 1-Lipschitz and $\phi_i(a) = 0$,

$$\big|\phi_i(xg) - \phi_i(xf)\big| \geqslant \big|\phi_i(a) - \phi_i(b)\big| - 4 = d(a, b) - 4 \geqslant d(g, f) - 8 \geqslant 1,$$

contradicting our assumption on x. $\square$

To put this result into perspective, recall that $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ is generated by two isomorphic copies of the extremely amenable group $\mathrm{Homeo}_+([0, 1])$. Both of these copies are thus coarsely bounded in $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ and, in fact, every extremely amenable subgroup must be contained in some fixed coarsely bounded set $B \subseteq \mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$.

Elaborating on the result of Ellis and Veech, in [**39**] it was shown that every non-compact locally compact group has non-metrisable universal minimal flow. While the proof of this breaks down for Polish groups of bounded geometry, we are able to instead rely on a recent analysis due to I. Ben Yaacov, J. Melleray and T. Tsankov [**7**], who analysed metrisable universal minimal flows of Polish groups.

Proposition 5.33. *Let* G *be Polish group of bounded geometry with metrisable universal minimal flow. Then* G *is coarsely bounded.*

Proof. By the main result of [**7**], if G has metrisable universal minimal flow, then G contains an extremely amenable, closed, co-precompact subgroup H. Here be *co-precompact* means that, for every identity neighbourhood V in G, there is a finite set $F \subseteq G$ so that $G = VFH$. However, observe that, by Proposition 5.32, the extremely amenable subgroup H is coarsely bounded in G. Thus, if we let V be a coarsely bounded identity neighbourhood and choose F finite so that $G = VFH$, then G is a product of three coarsely bounded sets and therefore coarsely bounded itself. $\square$

As an example of a non-compact Polish group with metrisable universal minimal flow, one can of course take any extremely amenable group. For a more interesting case, consider $\mathrm{Homeo}_+(\mathbb{S}^1)$ whose tautological action on $\mathbb{S}^1$ is its universal minimal flow. While $\mathrm{Homeo}_+(\mathbb{S}^1)$ is coarsely bounded, the central extension $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ on the other hand is only of bounded geometry and thus has non-metrisable universal minimal flow.

Proposition 5.34. *Let G be an amenable Polish group of bounded geometry with gauge metric d. Then there are a strongly continuous unitary representation $\pi\colon G \curvearrowright \mathcal{H}$ and unit vectors $\xi_n \in \mathcal{H}$ so that*

$$\big\|\pi(g)\xi_n - \xi_n\big\| = \sqrt{2},$$

whenever $9 \leqslant d(g,1) \leqslant n$.

Proof. For each n, let U_n be the uniform entourage in $\mathfrak{A}(G)$ given by Proposition 5.32. Then, as $\mathfrak{A}(G)$ is compact, there is a finite covering $\mathfrak{A}(G) = \bigcup_{i=1}^k W_i$ by open sets satisfying $W_i \times W_i \subseteq U_n$. Observe that, if $\omega \in W_i$ and $9 \leqslant d(g,1) \leqslant n$, then $(\rho^*(g)\omega, \omega) \notin U_n \supseteq W_i \times W_i$, i.e., $\rho^*(g)\omega \notin W_i$ and so $\rho^*(g)[W_i] \cap W_i = \emptyset$.

Since G is amenable, it fixes a regular Borel probability measure μ on $\mathfrak{A}(G)$. So pick some i so that W_i has positive μ-measure and let $\xi_n = \frac{1}{\sqrt{\mu(W_i)}}\chi_{W_i} \in L^2(\mathfrak{A}(G),\mu)$. Then, whenever $9 \leqslant d(g,1) \leqslant n$, we see that $\pi(g)\xi_n$ and ξ_n are disjointly supported normalised L^2-functions and so $\big\|\pi(g)\xi_n - \xi_n\big\| = \sqrt{2}$. $\square$

8. Efficiently contractible groups

The complexities of the proof of Theorem 5.22 indicate the utility of when an arbitrary bornologous map may be replaced by a uniformly continuous map. A standard trick using partitions of unity (see for example Lemma A.1 [**18**]), shows that this may be accomplished provided the domain is locally compact and the range a Banach space. We will extend this further using generalised convex combinations in contractible groups.

Definition 5.35. *A Polish group G is said to be* efficiently contractible *if it admits a contraction $(R_\alpha)_{\alpha\in[0,1]}\colon G \to G$ onto 1_G so that, for every coarsely bounded set A, the restriction $R\colon [0,1]\times A \to G$ is uniformly continuous.*

Example 5.36. Suppose G is a contractible locally compact Polish group. Then the contraction $R\colon [0,1]\times G \to G$ will be uniformly continuous whenever restricted to $[0,1]\times A$, where $A \subseteq G$ is compact.

Example 5.37. If X is a Banach space, then the usual contraction $R_\alpha(x) = (1-\alpha)x$ is uniformly continuous when restricted to the bounded sets $[0,1]\times nB_X$.

Example 5.38. Consider $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ and let H be the subgroup fixing $\mathbb{Z}$ pointwise. Then H is isomorphic to $\mathrm{Homeo}_+([0,1])$, which admits a uniformly continuous contraction (see Example 1.19 [**63**]). Also, by shifting on the right by translations τ_α, one easily obtains an efficient contraction of $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ to H. So by composition we obtain an efficient contraction of $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$.

We shall now use the contraction R to define generalised convex combinations in the group G. First, for $x \in G$, let $S(1,x) = x$. We set $\Delta_n = \{(\lambda_i) \in [0,1]^n \mid$

$\sum_{i=1}^n \lambda_i = 1\}$ and, for $(\lambda_i) \in \Delta_n$ and $(x_i) \in G^n$, define

$$S(\lambda_1, x_1, \ldots, \lambda_n, x_n) = \begin{cases} x_n & \text{if } \lambda_n = 1, \\ x_n R_{\lambda_n}\big(x_n^{-1} S(\frac{\lambda_1}{1-\lambda_n}, x_1, \ldots, \frac{\lambda_{n-1}}{1-\lambda_n}, x_{n-1})\big) & \text{otherwise.} \end{cases}$$

We think of $S(\lambda_1, x_1, \ldots, \lambda_n, x_n)$ as being akin to the "convex combination" of the points $x_1, \ldots, x_n$ with coefficients $\lambda_1, \ldots, \lambda_n$. However, since the construction is not associative, S will in general not be symmetric in its variables, i.e., if $\pi \in \mathrm{Sym}(n)$, then we may have

$$S(\lambda_{\pi(1)}, x_{\pi(1)}, \ldots, \lambda_{\pi(n)}, x_{\pi(n)}) \neq S(\lambda_1, x_1, \ldots, \lambda_n, x_n).$$

Nevertheless, variables x_k with coefficient $\lambda_k = 0$ are redundant, in the sense that

$$S(\lambda_1, x_1, \ldots, 0, x_k, \ldots \lambda_n, x_n) = S(\lambda_1, x_1, \ldots, \widehat{0}, \widehat{x_k}, \ldots, \lambda_n, x_n),$$

where $\widehat{\ }$ indicates that the term has been omitted. This is easily seen by induction on n.

Lemma 5.39. *Assume d is a coarsely proper metric on G and fix K and n. Then, whenever $\mathrm{diam}_d\big(\{x_1, x_2, \ldots, x_n\}\big) \leqslant K$, the function*

$$(\lambda_i) \in \Delta_n \mapsto S(\lambda_1, x_1, \ldots, \lambda_n, x_n) \in G$$

is uniformly continuous where the modulus of uniform continuity is independent of the (x_i).

Proof. The proof is by induction on $n \geqslant 0$, with the case $n = 0$ being trivial.

So suppose that the result holds for some n and let $\epsilon > 0$ be given. Then, as Δ_n is compact, there is some constant θ so that

$$\begin{aligned} d\big(S(\lambda_1, x_1, \ldots, \lambda_n, x_n), x_n\big) &= d\big(S(\lambda_1, x_1, \ldots, \lambda_n, x_n), S(0, x_1, \ldots, 0, x_{n-1}, 1, x_n)\big) \\ &\leqslant \theta \end{aligned}$$

whenever $\mathrm{diam}_d\big(\{x_1, x_2, \ldots, x_n\}\big) \leqslant K$. Since R is uniformly continuous on bounded sets, we may pick $\delta > 0$ so that $d(R_\lambda(y), R_\sigma(z)) < \frac{\epsilon}{2}$, whenever $d(y, 1), d(z, 1) \leqslant K + \theta$, $d(y, z) < \delta$ and $|\lambda - \sigma| < 3\delta$. In particular,

$$d(R_\lambda(y), 1) = d(R_\lambda(y), R_1(y)) < \frac{\epsilon}{2}$$

assuming $d(y, 1) \leqslant K + \theta$ and $\lambda > 1 - 3\delta$.

Choose also $\eta > 0$ so that

$$d\big(S(\lambda_1, x_1, \ldots, \lambda_n, x_n), S(\sigma_1, x_1, \ldots, \sigma_n, x_n)\big) < \delta,$$

whenever $\mathrm{diam}_d\big(\{x_1, x_2, \ldots, x_n\}\big) \leqslant K$ and $|\lambda_i - \sigma_i| < \frac{3\eta}{\delta^2}$.

Assume now that $\mathrm{diam}_d\big(\{x_1, x_2, \ldots, x_{n+1}\}\big) \leqslant K$ and that $(\lambda_i), (\sigma_i) \in \Delta_{n+1}$ satisfy $|\lambda_i - \sigma_i| < \min\{\delta, \eta\}$. We must show that

$$d\big(S(\lambda_1, x_1, \ldots, \lambda_{n+1}, x_{n+1}), S(\sigma_1, x_1, \ldots, \sigma_{n+1}, x_{n+1})\big) < \epsilon.$$

Suppose first that $\lambda_{n+1} > 1 - 2\delta$. Then

$$d\big(S(\frac{\lambda_1}{1-\lambda_{n+1}}, x_1, \ldots, \frac{\lambda_n}{1-\lambda_{n+1}}, x_n), x_{n+1}\big) \leqslant \theta + K,$$

whereby either $\lambda_{n+1} = 1$ and thus $S(\lambda_1, x_1, \dots, \lambda_{n+1}, x_{n+1}) = x_{n+1}$ or we have

$$\begin{aligned} d\big(S(\lambda_1, x_1, \dots, & \lambda_{n+1}, x_{n+1}), x_{n+1}\big) \\ &= d\Big(x_{n+1} R_{\lambda_{n+1}}\big(x_{n+1}^{-1} S(\frac{\lambda_1}{1-\lambda_{n+1}}, x_1, \dots, \frac{\lambda_n}{1-\lambda_{n+1}}, x_n)\big), x_{n+1}\Big) \\ &= d\Big(R_{\lambda_{n+1}}\big(x_{n+1}^{-1} S(\frac{\lambda_1}{1-\lambda_{n+1}}, x_1, \dots, \frac{\lambda_n}{1-\lambda_{n+1}}, x_n)\big), 1\Big) \\ &< \frac{\epsilon}{2}. \end{aligned}$$

Moreover, also $\sigma_{n+1} > 1 - 3\delta$, so by the same reasoning

$$d\big(S(\sigma_1, x_1, \dots, \sigma_{n+1}, x_{n+1}), x_{n+1}\big) < \frac{\epsilon}{2}$$

and thus

$$d\big(S(\lambda_1, x_1, \dots, \lambda_{n+1}, x_{n+1}), S(\sigma_1, x_1, \dots, \sigma_{n+1}, x_{n+1})\big) < \epsilon.$$

Suppose instead that $\lambda_{n+1} \leqslant 1 - 2\delta$, whence both $\frac{1}{1-\lambda_{n+1}} \leqslant \frac{1}{\delta}$ and $\frac{1}{1-\sigma_{n+1}} \leqslant \frac{1}{\delta}$ and thus

$$\Big|\frac{\lambda_i}{1-\lambda_{n+1}} - \frac{\sigma_i}{1-\sigma_{n+1}}\Big| < \frac{3\eta}{\delta^2}$$

for all $i = 1, \dots, n$. By the choice of η, it follows that

$$d\Big(S(\frac{\lambda_1}{1-\lambda_{n+1}}, x_1, \dots, \frac{\lambda_n}{1-\lambda_{n+1}}, x_n), S(\frac{\sigma_1}{1-\sigma_{n+1}}, x_1, \dots, \frac{\sigma_n}{1-\sigma_{n+1}}, x_n)\Big) < \delta$$

and so

$$\begin{aligned} d\big(S(\lambda_1, x_1, \dots, & \lambda_{n+1}, x_{n+1}), S(\sigma_1, x_1, \dots, \sigma_{n+1}, x_{n+1})\big) \\ =& d\Big[x_{n+1} R_{\lambda_{n+1}}\Big(x_{n+1}^{-1} S(\frac{\lambda_1}{1-\lambda_{n+1}}, x_1, \dots, \frac{\lambda_n}{1-\lambda_{n+1}}, x_n)\Big), \\ & x_{n+1} R_{\sigma_{n+1}}\Big(x_{n+1}^{-1} S(\frac{\sigma_1}{1-\sigma_{n+1}}, x_1, \dots, \frac{\sigma_n}{1-\sigma_{n+1}}, x_n)\Big)\Big] \\ =& d\Big[R_{\lambda_{n+1}}\Big(x_{n+1}^{-1} S(\frac{\lambda_1}{1-\lambda_{n+1}}, x_1, \dots, \frac{\lambda_n}{1-\lambda_{n+1}}, x_n)\Big), \\ & R_{\sigma_{n+1}}\Big(x_{n+1}^{-1} S(\frac{\sigma_1}{1-\sigma_{n+1}}, x_1, \dots, \frac{\sigma_n}{1-\sigma_{n+1}}, x_n)\Big)\Big] \\ <& \frac{\epsilon}{2}. \end{aligned}$$

This finishes the inductive step and thus the proof of the lemma. □

Theorem 5.40. *Suppose $\phi\colon H \to G$ is a bornologous map from a Polish group H of bounded geometry to an efficiently contractible locally bounded Polish group G. Then ϕ is close to a uniformly continuous map $\psi\colon H \to G$.*

Proof. Fix a gauge metric ∂ on H, a coarsely proper metric d on G and let $X \subseteq H$ be a maximally 2-discrete subset, i.e., $\partial(x,y) \geqslant 2$ for distinct $x, y \in X$. Then X is 2-dense in H and (X, ∂) has bounded geometry. Fix also a linear ordering $\prec$ of X, set

$$K = \sup\big(d(\phi(x), \phi(y)) \;\big|\; \partial(x,y) < 8\big)$$

and let M be the maximum cardinality of a diameter 8 subset of X.

For every $x \in X$, define $\theta_x\colon H \to [0,3]$ by $\theta_x(h) = \max\{0, 3 - \partial(h,x)\}$. Note that θ_x is 1-Lipschitz and $\theta_x \geqslant 1$ on a ball of radius 2 centred at x, while $\mathrm{supp}(\theta_x)$

is contained in the 3-ball around x. Since X is 1-dense and has bounded geometry, it follows that

$$\Theta(h) = \sum_{x \in X} \theta_x(h)$$

is a bounded Lipschitz function with $\Theta \geqslant 1$. It follows that setting $\lambda_x = \frac{\theta_x}{\Theta}$, we have a partition of unity $\{\lambda_x\}_{x\in X}$ by Lipschitz functions with some uniform Lipschitz constant C and each λ_x supported in the 3-ball centred at x.

Let now $h \in H$ be given and suppose $x_1 \prec \ldots \prec x_n$ are the elements $x \in X$ so that $\lambda_x(h) \neq 0$. We define

$$\psi(h) = S\big(\lambda_{x_1}(h), \phi(x_1), \ldots, \lambda_{x_n}(h), \phi(x_n)\big).$$

As the λ_x are only supported in the 3-ball centred at x, the x_i are within distance 3 of h, whereby $\mathrm{diam}_d\big(\{\phi(x_1), \ldots, \phi(x_n)\}\big) \leqslant K$ and $n \leqslant M$. Thus,

$$d(\psi(h), \phi(h)) \leqslant d\Big(S\big(\lambda_{x_1}(h), \phi(x_1), \ldots, \lambda_{x_n}(h), \phi(x_n)\big), \phi(x_1)\Big) + d(\phi(x_1), \phi(h)),$$

which, by Lemma 5.39 and the fact that ϕ is bornologous, is bounded independently of h. Therefore, ψ is close to ϕ.

Let us now verify that $\psi\colon H \to G$ is uniformly continuous. So let $\epsilon > 0$ be given and choose $\delta > 0$ small enough so that

$$d\big(S(\lambda_1, a_1, \ldots, \lambda_n, a_n), S(\sigma_1, a_1, \ldots, \sigma_n, a_n)\big) < \epsilon,$$

whenever $n \leqslant M$, $\mathrm{diam}_d\big(\{a_1, \ldots, a_n\}\big) \leqslant K$ and $|\lambda_i - \sigma_i| < \delta$.

Suppose that $h, g \in H$ with $\partial(h,g) < \min\{1, \frac{\delta}{C}\}$, whence $|\lambda_x(h) - \lambda_x(g)| < \delta$ for all $x \in X$. Let also $x_1 \prec \ldots \prec x_n$ list $B_\partial(h, 4) \cap X$, whence $n \leqslant M$ and $\mathrm{diam}_d\big(\{\phi(x_1), \ldots, \phi(x_n)\}\big) \leqslant K$.

Since $\partial(h,g) < 1$, we see that, if $\lambda_x(h) \neq 0$ or $\lambda_x(g) \neq 0$, then x is among the $x_1, \ldots, x_n$. From this and by adding redundant variables, we see that, for some subsequences $1 \leqslant i_1 < \ldots < i_p \leqslant n$ and $1 \leqslant j_1 < \ldots < j_q \leqslant n$,

$$\begin{aligned}\psi(h) &= S\big(\lambda_{x_{i_1}}(h), \phi(x_{i_1}), \ldots, \lambda_{x_{i_p}}(h), \phi(x_{i_p})\big)\\ &= S\big(\lambda_{x_1}(h), \phi(x_1), \ldots, \lambda_{x_n}(h), \phi(x_n)\big)\end{aligned}$$

and

$$\begin{aligned}\psi(g) &= S\big(\lambda_{x_{j_1}}(g), \phi(x_{j_1}), \ldots, \lambda_{x_{j_q}}(g), \phi(x_{j_q})\big)\\ &= S\big(\lambda_{x_1}(g), \phi(x_1), \ldots, \lambda_{x_n}(g), \phi(x_n)\big).\end{aligned}$$

In particular, $d(\psi(h), \psi(g)) < \epsilon$ as desired. □

Corollary 5.41. *Suppose H is a Polish group of bounded geometry and G an efficiently contractible, locally bounded Polish group. If H is coarsely embeddable into G, then there is a uniformly continuous coarse embedding of G into H.*

9. Entropy, growth rates and metric amenability

The class of Polish groups of bounded geometry is very special in the sense that it will allow us to transfer a number of concepts of finitude from locally compact groups. Some of this can be done even in the larger category of coarse spaces of bounded geometry, e.g., [**60**], but we shall restrict our attention to Polish groups.

9.1. Entropy. The concept of a gauge of course allows to do some very basic counting and measuring. For this, we recall A. N. Kolmogorov's notions of metric entropy and capacity [**42, 43**] in metric spaces. So suppose (X,d) is a metric space of bounded geometry. Then $\alpha>0$ is said to be a *gauge* for (X,d) if, for every $\beta<\infty$, there is a K_β so that every set of diameter $\leqslant\beta$ can be covered by K_β many open balls $B_d(x,\alpha)=\{y\in X\mid d(y,x)<\alpha\}$ of radius α. In this case, we define the *α-entropy*, $\mathrm{ent}_\alpha(A)$, of a bounded set $A\subseteq X$ to be the minimal number of open balls of radius α covering A. Similarly, the *α-capacity*, $\mathrm{cap}_\alpha(A)$, is the largest size of a α-discrete subset D contained in A, i.e., so that $d(x,y)\geqslant\alpha$ for distinct points in D.

We observe that

$$\mathrm{cap}_{2\alpha}(A)\leqslant\mathrm{ent}_\alpha(A)\leqslant\mathrm{cap}_\alpha(A).$$

Also, if d is a gauge metric on a Polish group G of bounded geometry, then, since the open unit ball $B_d(1_G,1)$ centred at 1_G is a gauge for G, we see that the distance $\alpha=1$ is a gauge for the metric space (G,d) with $\mathrm{ent}_1=\mathrm{ent}_{B_d(1_G,1)}$ and $\mathrm{cap}_1=\mathrm{cap}_{B_d(1_G,1)}$. For suggestiveness of notation, we shall then write ent_d and cap_d for ent_1 and cap_1 respectively.

Now, suppose that A and A' are two gauges on a Polish group G of bounded geometry and let $N=\max\big\{\mathrm{ent}_A(A'),\mathrm{ent}_{A'}(A)\big\}$. Then

$$\frac{1}{N}\mathrm{ent}_A\leqslant\mathrm{ent}_{A'}\leqslant N\cdot\mathrm{ent}_A,$$

i.e., the entropy functions associated to any two gauges are bi-Lipschitz equivalent.

Also, if A is a subset of a metric space (X,d) and β a constant, we let

$$(A)_\beta=\{x\in X\;\big|\;d(x,A)<\beta\}$$

and

$$\partial_\beta A=(A)_\beta\setminus A.$$

For future reference, let us observe the following basic fact.

LEMMA 5.42. *Suppose that G is a locally compact second countable group with left Haar measure λ and compatible left-invariant proper metric d. Then, for any subset $A\subseteq G$,*

$$\frac{\lambda(A)}{\lambda\big(B_d(1_G,1)\big)}\leqslant\mathrm{ent}_d(A)\leqslant\mathrm{cap}_d(A)\leqslant\frac{\lambda\big((A)_{\frac12}\big)}{\lambda\big(B_d(1_G,\frac12)\big)}.$$

PROOF. This is evident, since A is covered by $\mathrm{ent}_d(A)$ many left-translates of $B_d(1_G,1)$ and $(A)_{\frac12}$ contains $\mathrm{cap}_d(A)$ many disjoint translates of $B_d(1_G,1/2)$. □

9.2. Growth rates. Assume now that G is a Polish group of bounded geometry generated by a coarsely bounded set A. Then, by increasing A, we may suppose that A is also a gauge for G and thus consider the corresponding increasing *growth function* for G given by

$$\mathfrak{g}_A(n)=\mathrm{ent}_A(A^n).$$

Assume now that B is a different gauge also generating G. Then, as observed above, the entropy function ent_B is bi-Lipschitz equivalent to that of A, say $\frac{1}{N}\mathrm{ent}_A\leqslant\mathrm{ent}_B\leqslant N\cdot\mathrm{ent}_A$ for some N. Moreover, as both A and B are symmetric, coarsely bounded generating sets containing 1, there is some sufficiently large M so that $A \subseteq B^M$ and $B \subseteq A^M$. It thus follows that

$$\mathfrak{g}_A(n) = \mathrm{ent}_A(A^n) \leqslant N \cdot \mathrm{ent}_B(A^n) \leqslant N \cdot \mathrm{ent}_B(B^{Mn}) = N \cdot \mathfrak{g}_B(Mn)$$

and similarly $\mathfrak{g}_B(n) \leqslant N \cdot \mathfrak{g}_A(Mn)$.

Definition 5.43. *Two increasing functions $\mathfrak{g}, \mathfrak{f} \colon \mathbb{N} \to \mathbb{R}_+$ are said to have* equivalent growth rates *if there is some constant λ so that*

$$\mathfrak{g}(n) \leqslant \lambda \cdot \mathfrak{f}(\lambda n + \lambda) + \lambda \quad \textit{and} \quad \mathfrak{f}(n) \leqslant \lambda \cdot \mathfrak{g}(\lambda n + \lambda) + \lambda$$

for all n.

Thus, in view of the above, we see that the functions $\mathfrak{g}_A$ and $\mathfrak{g}_B$ have equivalent growth rates and thus define a natural invariant of G.

Moreover, the growth rate is quasi-isometry invariant. To see this, suppose that A and B are gauges generating Polish groups G and H respectively and let ρ_A and ρ_B be the associated word metrics. Suppose also that $\phi \colon G \to H$ is a quasi-isometry, say $\frac{1}{K}\rho_A(x, y) - K \leqslant \rho_B(\phi x, \phi y) \leqslant K\rho_A(x, y) + K$ for all $x, y \in G$. Without loss of generality, we may assume that $\phi(1_G) = 1_H$, whence $\phi(A^n) \subseteq B^{Kn+K}$ for all n. Then, as the ρ_A-diameters of the inverse images $\phi^{-1}(hB)$ of left-translates of B are uniformly bounded, so are their A-entropies, i.e., $N = \sup_{h\in H} \mathrm{ent}_A\big(\phi^{-1}(hB)\big) < \infty$. It follows that

$$\mathrm{ent}_A(\phi^{-1}(C)) \leqslant N \cdot \mathrm{ent}_B(C)$$

for all $C \subseteq H$ and hence that

$$\mathfrak{g}_A(n) = \mathrm{ent}_A(A^n) \leqslant \mathrm{ent}_A\big(\phi^{-1}(B^{Kn+K})\big) \leqslant N \cdot \mathrm{ent}_B(B^{Kn+K}) = N \cdot \mathfrak{g}_B(Kn + K).$$

By symmetry, we find that $\mathfrak{g}_A$ and $\mathfrak{g}_B$ have equivalent growth.

Example 5.44 (Growth rates of locally compact groups). For a compactly generated, locally compact second countable group G, the growth rate is usually expressed as the volume growth, i.e., $\mathfrak{g}(n) = \lambda(A^n)$, where A is a compact generating set and λ left Haar measure. However, by Lemma 5.42, one sees that this gives equivalent growth rate to that given by the entropy function ent_A.

9.3. Metric amenability. We now consider the concept of metric amenability due to J. Block and S. Weinberger [**11**], see also [**55**]. This was originally introduced as a notion of amenability for metric spaces, but was expanded to coarse spaces of bounded geometry in [**60**]. To avoid confusion with the usual notion of amenability of topological groups, to which it is not equivalent, we shall use the more descriptive termininology of metric amenability.

In direct analogy with E. Følner's isoperimetric reformulation of amenability of discrete groups [**29**], we have the following concept of metric amenability [**11**].

Definition 5.45. *Let (X, d) be a metric space of bounded geometry with gauge α. Then (X, d) is said to be* metrically amenable *if, for all $\epsilon > 0$ and $\beta < \infty$, there is a coarsely bounded set A with*

$$\mathrm{ent}_\alpha(\partial_\beta A) < \epsilon \cdot \mathrm{ent}_\alpha(A).$$

A few words are in order with respect to this definition. Namely, observe first that it is independent of the specific choice of gauge α for (X, d). That is, (X, d) is metrically amenable with respect one if and only if it is with respect to another.

Moreover, metric amenability is a coarse invariant of metric spaces of bounded geometry.

The failure of metric amenability has a useful reformulation in terms of the growth of expansions of sets. Namely, (X, d) with gauge α fails to be metrically amenable if and only if, for all K, there is some σ so that

$$\mathrm{ent}_\alpha\big((A)_\sigma\big) \geqslant K \cdot \mathrm{ent}_\alpha(A)$$

for all bounded sets A.

Also, if G is a Polish group of bounded geometry, we say that G is *metrically amenable* when (G, d) is. Again, by the bi-Lipschitz equivalence of the entropy functions associated to different gauges, we see that this definition is independent of the choice of gauge metric d on G. In fact, suppose A is any choice of gauge for G. Then G is metrically amenable exactly when, for all $\epsilon > 0$ and coarsely bounded set C, there is a coarsely bounded set B so that

$$\mathrm{ent}_A(BC \setminus B) < \epsilon \cdot \mathrm{ent}_A(B).$$

Similarly, as gauge metrics are compatible metrics for the coarse structure on G, we note that metric amenability is a coarse invariant of Polish groups of bounded geometry.

Example 5.46 (Amenability versus metric amenability of locally compact groups). Suppose that G is a locally compact second countable group with left Haar measure λ, right Haar measure μ and compatible left-invariant proper metric d. Then, by the Lipschitz bounds of Lemma 5.42, one easily checks that G is metrically amenable if and only if, for all $\epsilon > 0$ and compact set K, there is a compact set A with

$$\lambda(AK \setminus A) < \epsilon \cdot \lambda(A).$$

This immediately implies that G is unimodular. Indeed, let Δ be the modular function, i.e., $\Delta(g) = \frac{\lambda(Ag)}{\lambda(A)}$ for a measurable set $A \subseteq G$, and suppose that G is metrically amenable. Then, for any $g \in G$ and $\epsilon > 0$, there is some compact A with $\lambda(Ag) \leqslant \lambda(A) + \lambda(Ag \setminus A) < (1 + \epsilon)\lambda(A)$ and so $\Delta(g) < 1 + \epsilon$. I.e., $\Delta(g) = 1$ for all $g \in G$.

On the other hand, G being amenable is equivalent to the condition that, for all $\epsilon > 0$ and compact K, there is a compact set A with

$$\mu(AK \setminus A) < \epsilon \cdot \mu(A).$$

As amenability and metric amenability clearly coincide in a unimodular group, where we may suppose that $\lambda = \mu$, we see that

$$G \text{ is metrically amenable } \Leftrightarrow\ G \text{ is amenable and unimodular},$$

as shown in [**72**]. In particular, for countable discrete groups, the two notions of amenability coincide, which is also part of Følner's theorem. However, for example, the metabelian locally compact group $\mathbb{R}_+ \ltimes \mathbb{R}$ is amenable, but fails to be unimodular and thus is not metrically amenable either. One may also note that $\mathbb{R}_+ \ltimes \mathbb{R}$ is coarsely equivalent to the metrically non-amenable hyperbolic plane $\mathbb{H}^2$.

In the context of Polish groups, metric amenability no longer implies amenability. Indeed, $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ is a Polish group of bounded geometry coarsely equivalent to $\mathbb{Z}$ and therefore metrically amenable. However, $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ is not amenable since, e.g., it acts continuously on the compact space $\mathbb{S}^1$ without preserving a measure.

10. Nets in Polish groups

Recall that a subset X of a topological group G is *cobounded* in G if $G = XB$ for some coarsely bounded set $B \subseteq G$. Also, X is *uniformly discrete* if uniformly discrete with respect to the left-uniformity on G, i.e., if there is an identity neighbourhood U so that $xU \cap yU = \emptyset$ for all distinct $x, y \in X$.

Definition 5.47. *A subset X of a topological group G is said to be a* net *in G if it is simultaneously cobounded and uniformly discrete.*

We claim that a Polish group G admits a net if and only if it is locally bounded. Indeed, observe that, if X is a net in G, then, being uniformly discrete, X is countable and, by coboundedness, G is covered by countably many coarsely bounded sets, whereby G is locally bounded. Conversely, if G is locally bounded, we choose a net in G by letting X be a maximal 1-discrete set with respect to some coarsely proper metric on G.

In case G is a Polish group of bounded geometry, we would like nets to reflect this fact. So we define a net $X \subseteq G$ to be *proper* if there is an open gauge $U \subseteq G$ so that $xU \cap yU = \emptyset$ for distinct $x, y \in X$. In this case, no two distinct points of X can belong to the same left-translate of U, which means that the entropy function ent_U, when restricted to subsets of X, is simply the counting measure.

As above, every Polish group of bounded geometry admits a proper net. The main quality of a proper net X that we shall be using is that, if d is a coarsely proper metric on G, then (X, d) is uniformly locally finite. Indeed, for every diameter r, there is a k so that every set $B \subseteq X$ of diameter r is covered by k left-translates of U. Since no two distinct points of X can belong to the same left-translate of U, this implies that such $B \subseteq X$ have cardinality at most k.

Also, if X is a net in a locally bounded Polish group G and d is a coarsely proper metric on G, then the isometric inclusion $(X, d) \hookrightarrow (G, d)$ is cobounded and thus a coarse equivalence. In other words, G is coarsely equivalent to the discrete metric space (X, d).

It is well-known that all nets in an infinite-dimensional Banach space are bi-Lipschitz equivalent. We generalise this to Polish groups of unbounded geometry.

Proposition 5.48. *Let G and H be Polish groups with unbounded geometry, d and ∂ coarsely proper metrics and X and Y nets in G and H respectively. Then G and H are coarsely equivalent if and only if there is a bijective coarse equivalence $\phi\colon (X, d) \to (Y, \partial)$.*

Proof. Clearly, if (X, d) and (Y, ∂) are coarsely equivalent, so are G and H.

Now, suppose conversely that G and H are coarsely equivalent, whereby also (X, d) and (Y, ∂) are coarsely equivalent by some map $\psi\colon X \to Y$. Assume that X is σ-cobounded in (G, d), i.e., $G = \bigcup_{x\in X} B(x, \sigma)$. Then, since G has unbounded geometry, there is some $\alpha > 0$ so that no ball $B_d(g, \alpha - \sigma)$ of diameter $\alpha - \sigma$ can be covered by finitely many balls of radius σ. It follows that every ball $B_d(g, \alpha)$ of radius α has infinite intersection with X. Similarly, we may suppose that, for every $h \in H$, the intersection $B_\partial(h, \alpha) \cap Y$ is infinite.

Pick also $\beta > 2\alpha$ large enough so that

$$d(x, x') \geqslant \beta \;\;\Rightarrow\;\; \partial(\psi(x), \psi(x')) \geqslant 2\alpha$$

and let $Z \subseteq X$ be a maximal β-discrete subset. Then $\psi\colon Z \to Y$ is injective and $\psi[Z]$ is cobounded and 2α-discrete in Y.

Pick a partition $\{P_z\}_{z\in Z}$ of X so that, for all $z\in Z$,

$$X\cap B_d(z,\alpha)\subseteq P_z\subseteq B_d(z,2\beta)$$

and, similarly, a partition $\{Q_z\}_{z\in Z}$ of Y so that, for some σ and all $z\in Z$,

$$Y\cap B_\partial(\psi(z),\alpha)\subseteq Q_z\subseteq B_\partial(\psi(z),\sigma).$$

Then each P_z and Q_z will be infinite, whereby we may extend ψ to a bijection ϕ between X and Y so that $\phi[P_z]=Q_z$ for all $z\in Z$. As ϕ and ψ are easily seen to be close maps, i.e., $\sup_{x\in X}\partial(\phi(x),\psi(x))<\infty$, it follows that ϕ is a coarse equivalence between (X,d) and (Y,∂) are required. □

A similar statement is also true for metrically non-amenable groups provided we restrict our attention to proper nets in G and H. The reasoning that follows underlies the investigations of K. Whyte in [**79**].

PROPOSITION 5.49. *Let G and H be metrically non-amenable Polish groups of bounded geometry with coarsely proper metrics d and ∂ and proper nets X and Y. Then G and H are coarsely equivalent if and only if there is a bijective coarse equivalence $\phi\colon (X,d)\to(Y,\partial)$.*

PROOF. Suppose that $\phi\colon G\to H$ is a coarse equivalence, whence

$$\sup_{h\in H}\mathrm{diam}_d(\phi^{-1}(h))<\infty.$$

As (X,d) is uniformly locally finite, we see that $k=\sup_{h\in H}|\phi^{-1}(h)\cap X|<\infty$. In other words, $\phi\colon X\to H$ is at most k-to-1.

Assume that Y is α-cobounded in H and that U is an open gauge for H so that $yU\cap y'U=\emptyset$ for distinct y,y' in Y, whence ent_U is just the counting measure when restricted to subsets of Y. Pick also N so that $\mathrm{ent}_U\leqslant N\cdot\mathrm{ent}_{(U)_\alpha}$ and, since H is metrically non-amenable, some σ so that $\mathrm{ent}_U\big((A)_\sigma\big)\geqslant Nk\cdot\mathrm{ent}_U(A)$ for all $A\subseteq H$. Note that, since Y is α-cobounded in H, if $(A)_\alpha\cap Y\subseteq\bigcup_{i=1}^n h_iU$, then $A\subseteq\bigcup_{i=1}^n h_i\cdot(U)_\alpha$, i.e., $\mathrm{ent}_{(U)_\alpha}(A)\leqslant\mathrm{ent}_U((A)_\alpha\cap Y)$ for any $A\subseteq H$.

Then, for any finite subset $A\subseteq H$,

$$\begin{aligned}|A|\leqslant\mathrm{ent}_U(A)&\leqslant\frac{1}{Nk}\mathrm{ent}_U\big((A)_\sigma\big)\leqslant\frac{1}{k}\mathrm{ent}_{(U)_\alpha}\big((A)_\sigma\big)\\&\leqslant\frac{1}{k}\mathrm{ent}_U\big((A)_{\sigma+\alpha}\cap Y\big)=\frac{1}{k}\big|(A)_{\sigma+\alpha}\cap Y\big|.\end{aligned}$$

In particular, if $D\subseteq X$ is finite, then

$$|D|\leqslant k\cdot\big|\phi[D]\big|\leqslant\big|(\phi[D])_{\sigma+\alpha}\cap Y\big|.$$

Define now a relation $\mathcal{R}\subseteq X\times Y$ by letting $x\mathcal{R}y\Leftrightarrow y\in(\phi(x))_{\sigma+\alpha}$. By the above, any finite subset $D\subseteq X$ is $\mathcal{R}$-related to at least $|D|$ many elements of Y. So, by Hall's marriage lemma, this implies that there is an injection $\zeta\colon X\to Y$ whose graph is contained in $\mathcal{R}$, i.e., so that $\partial(\zeta(x),\phi(x))<\sigma+\alpha$.

Similarly, if $\phi'\colon H\to G$ is a coarse inverse to ϕ, one may produce an injection $\zeta'\colon Y\to X$ with $d(\zeta'(y),\phi'(x))<\sigma'+\alpha'$, where α' and σ' correspond to α and σ in the construction above.

Finally, by the Schröder–Bernstein Theorem, there is a bijection $\eta\colon X\to Y$ so that, for all $x\in X$, either $\eta(x)=\zeta(x)$ or $x=\zeta'(\eta(x))$. In particular, η is close to the coarse equivalence ϕ and hence is a coarse equivalence itself. □

Let us note that in the proofs of Propositions 5.48 and 5.49, from a coarse equivalence $\psi\colon G\to H$, we produce a bijective coarse equivalence $\phi\colon X\to Y$, which is close to ψ. In fact, keeping track of the constants in the proof of Proposition 5.49, we can extract the following statement.

PROPOSITION 5.50. *Suppose G is a metrically non-amenable Polish group of bounded geometry, d is a coarsely proper metric and X is a proper net. Then there is a constant K so that, for every isometry $\psi\colon (G,d)\to(G,d)$, there is a permutation ϕ of X with $\sup_{x\in X} d(\psi(x),\phi(x))\leqslant K$.*

Suppose G is a metrically non-amenable Polish group of bounded geometry and pick a coarsely proper metric d and a proper net X. Let also K be the constant given by Proposition 5.50. Then every element $g\in G$ acts isometrically on (G,d) by left-translation and Proposition 5.50 thus provides an element $\phi(g)$ of the group $\mathrm{Sym}(X)$ of all permutations of X so that $\sup_{x\in X} d(gx,\phi(g)x)\leqslant K$. In particular, for all $g,h\in G$ and $x,y\in X$,

(1) $\big|d(\phi(g)x,\phi(g)y)-d(x,y)\big|\leqslant 2K$, while
(2) $d(\phi(g)\phi(h)x,\phi(gh)x)\leqslant 3K$.

Since (X,d) is uniformly locally finite, it follows from condition (2) that the *defect* of $\phi\colon G\to \mathrm{Sym}(X)$,

$$\Delta_\phi=\{\phi(gh)^{-1}\phi(g)\phi(h)\bigm| g,h\in G\}^{\pm},$$

is relatively compact, when $\mathrm{Sym}(X)$ is equipped with the permutation group topology whose basic identity neighbourhoods are the pointwise stabilisers of finite subsets of X.

11. Two-ended Polish groups

In the following, we aim to determine the structure of Polish groups coarsely equivalent to $\mathbb{R}$. In the case of finitely generated groups, these are of course simply the two-ended groups classified via a result of C. T. C. Wall; namely, every f.g. two-ended group contains a finite-index infinite cyclic subgroup (see [**49**] for a proof). We will establish a generalisation of this to all Polish groups stating, in particular, that any Polish group coarsely equivalent to $\mathbb{R}$ contains a cobounded undistorted copy of $\mathbb{Z}$.

It will be useful to keep a few examples in mind that will indicate the possible behaviours. Apart from simple examples $\mathbb{Z}$ and $\mathbb{R}$, we of course have groups such as $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ and $\mathrm{Aut}_{\mathbb{Z}}(\mathbb{R})$ that, though acting on $\mathbb{R}$, do not admit homomorphisms to $\mathbb{R}$. But one should also note the infinite dihedral group D_∞ of all isometries of $\mathbb{Z}$, which contains $\mathbb{Z}$ as an index 2 subgroup.

Since we will be passing to a subgroup of finite index, let us begin by observing that these are always coarsely embedded.

LEMMA 5.51. *Suppose H is a finite index open subgroup of a topological group G. Then H is coarsely embedded in G.*

PROOF. Suppose $H\curvearrowright X$ is a continuous isometric action on a metric space (X,d) and let Ω be the space of H-equivariant continuous maps $\xi\colon G\to X$, i.e., so that, for all $g\in G$ and $h\in H$,

$$h\cdot\xi(g)=\xi(hg).$$

Fix a transversal $1 \in T \subseteq G$ for the right-cosets of H in G and equip Ω with the metric

$$d_\infty(\xi,\zeta) = \sup_{g\in G} d\big(\xi(g),\zeta(g)\big) = \sup_{t\in T}\sup_{h\in H} d\big(\xi(ht),\zeta(ht)\big) = \sup_{t\in T} d\big(\xi(t),\zeta(t)\big) < \infty.$$

Then G acts continuously and isometrically on Ω via $g\cdot\xi = \xi(\,\cdot\, g)$.

Suppose now that $A\cdot x$ is unbounded for some $A\subseteq H$ and $x\in X$. Let $\xi\in\Omega$ be defined by $\xi(ht) = h(x)$ for all $h\in H$ and $t\in T$. Then

$$\sup_{h\in A} d_\infty\big(h\cdot\xi,\xi\big) \geqslant \sup_{h\in A} d\big((h\cdot\xi)(1),\xi(1)\big) \geqslant \sup_{h\in A} d\big(\xi(h),\xi(1)\big) \geqslant \sup_{h\in A} d\big(h(x),x\big) = \infty$$

and thus also $A\cdot\xi$ is unbounded in Ω. It follows that a subset $A\subseteq H$ is coarsely bounded in H if and only if it is coarsely bounded in G and hence H is coarsely embedded in G. $\square$

Lemma 5.52. *Let G be a Polish group that is the Zappa–Szép product of a closed subgroup H and a compact group K, i.e., $G = HK$ and $H\cap K = \{1\}$. Then H is coarsely embedded in G.*

Proof. The proof is very similar to the proof of Lemma 5.51. Namely, assuming $H \curvearrowright (X,d)$ is a continuous isometric action, Ω is again the set of H-equivariant continuous maps $\xi\colon G\to X$ with the metric

$$d_\infty(\xi,\zeta) = \sup_{g\in G} d\big(\xi(g),\zeta(g)\big) = \sup_{k\in K} d\big(\xi(k),\zeta(k)\big).$$

Suppose $x\in X$ is given. Recall that, by the structure theorem for Zappa–Szép products, Theorem A.3, the multiplication defines a homeomorphism $H\times K\to G$. This means that we can define $\xi\in\Omega$ by $\xi(hk) = h(x)$. As before we see that, for $A\subseteq H$, the set $A\cdot\xi$ is unbounded in Ω if $A\cdot x$ is unbounded in X. So H is coarsely embedded in G. $\square$

Note that in both cases above the inclusion mapping will be a coarse equivalence between H and G and thus a quasi-isometry whenever G is generated by a coarsely bounded set.

We do not know if Lemmas 5.51 and 5.52 admit a common generalisation.

Problem 5.53. *Suppose H is a cocompact closed subgroup of a Polish group G, i.e., $G = HK$ for some compact set $K\subseteq G$. Is H coarsely embedded in G?*

In the following, let G be a Polish group coarsely equivalent to $\mathbb{R}$, whence G is generated by a coarsely bounded set and thus admits a maximal metric d. Let $X\subseteq G$ be a maximal 1-discrete subset of G. Suppose also that $\psi\colon G\to\mathbb{R}$ is a quasi-isometry, whereby, as X is 1-discrete, the restriction $\psi\restriction_X$ is K-Lipschitz for some $K>0$. Thus, by the MacShane–Whitney extension theorem, there is a K-Lipschitz extension $\tilde\psi\colon (G,d)\to\mathbb{R}$ of $\psi\restriction_X$, namely,

$$\tilde\psi(y) = \inf_{x\in X} \psi(x) + Kd(y,x).$$

Also, as X is maximally 1-discrete, we see that ϕ remains a quasi-isometry and so,

$$\frac{1}{K'}d(x,y) - K' \leqslant |\tilde\psi(x) - \tilde\psi(y)| \leqslant K'd(x,y)$$

for all $x,y\in G$ and some $K'>0$. Setting $\phi = \frac{1}{K'}\tilde\psi$, we see that

$$\frac{1}{C}d(x,y) - C \leqslant |\phi x - \phi y| \leqslant d(x,y)$$

for some constant $C > 2$ so that $\phi[G]$ is C-cobounded in $\mathbb{R}$.

LEMMA 5.54. *Suppose that $x, y, z \in G$ and that*

$$\phi x + C^4 \leqslant \phi y \leqslant \phi z - C^4.$$

Then, for every $g \in G$, either

$$\phi(gx) < \phi(gy) < \phi(gz)$$

or

$$\phi(gx) > \phi(gy) > \phi(gz).$$

PROOF. Note first that, for all $u, v \in G$,

$$|\phi(gu) - \phi(gv)| \geqslant \frac{1}{C}d(gu, gv) - C = \frac{1}{C}d(u, v) - C \geqslant \frac{1}{C}|\phi u - \phi v| - C,$$

so $\phi(gu) \neq \phi(gv)$ provided $\phi u + C^4 \leqslant \phi u$. In particular, $\phi(gx), \phi(gy), \phi(gz)$ are all distinct.

Assume for a contradiction that

$$\phi(gx) < \phi(gz) < \phi(gy).$$

Then, as $\phi[G]$ is C-cobounded in $\mathbb{R}$ and $\phi x < \phi y$, we may find a sequence $x_0 = x, x_1, x_2, \ldots, x_n = y \in G$ so that

$$\phi x_0 < \phi x_1 < \ldots < \phi x_n = \phi y < \phi z - C^4$$

and $|\phi x_i - \phi x_{i+1}| \leqslant 2C$ for all i. In particular,

$$|\phi(gx_i) - \phi(gx_{i+1})| \leqslant d(gx_i, gx_{i+1}) = d(x_i, x_{i+1}) \leqslant C|\phi x_i - \phi x_{i+1}| + C^2 \leqslant 3C^2$$

and so, as

$$\phi(gx_0) = \phi(gx) < \phi(gz) < \phi(gy) = \phi(gx_n),$$

we have $|\phi(gz) - \phi(gx_i)| \leqslant \frac{3C^2}{2}$ for some i.

But then

$$C^4 \leqslant |\phi z - \phi x_i| \leqslant d(z, x_i) = d(gz, gx_i) \leqslant C|\phi(gz) - \phi(gx_i)| + C^2 \leqslant \frac{3C^3}{2} + C^2,$$

which is absurd since $C > 2$.

The arguments for all other cases are entirely analogous and are left to the reader. □

LEMMA 5.55. *Let*

$$H = \{g \in G \mid \phi(gx) < \phi(gy) \text{ whenever } \phi x + C^4 \leqslant \phi y\}$$

is an open subgroup of index at most 2. Moreover,

$$G \setminus H = \{g \in G \mid \phi(gx) > \phi(gy) \text{ whenever } \phi x + C^4 \leqslant \phi y\}.$$

PROOF. Observe that each $g \in G$ satisfies exactly one of the following conditions,

$$\phi(gx) < \phi(gy) \text{ whenever } \phi x + C^4 \leqslant \phi y \tag{1}$$

or

$$\phi(gx) > \phi(gy) \text{ whenever } \phi x + C^4 \leqslant \phi y. \tag{2}$$

If not, there are $g, x_0, x_1, y_0, y_1 \in G$ so that $\phi x_0 + C^4 \leqslant \phi y_0$ and $\phi x_1 + C^4 \leqslant \phi y_1$, while $\phi(gx_0) < \phi(gy_0)$ and $\phi(gx_1) > \phi(gy_1)$. But then we can pick some $u, z \in G$

so that $\phi u \leqslant \min\{\phi x_0, \phi x_1\} - C^4$ and $\max\{\phi y_0, \phi y_1\} + C^4 \leqslant \phi z$, whence by Lemma 5.54 both

$$\phi(gu) < \phi(gx_0) < \phi(gy_0) < \phi(gz)$$

and

$$\phi(gu) > \phi(gx_1) > \phi(gy_1) > \phi(gz),$$

which is absurd.

Now, to see that H is a subgroup, assume that $h, f \in H$, i.e., that $g = h$ and $g = f$ each satisfy condition (1) and pick $x, y \in G$ with $\phi x + C^6 \leqslant \phi y$. Then $\phi(hx) < \phi(hy)$ and

$$|\phi(hx) - \phi(hy)| \geqslant \frac{1}{C} d(x, y) - C \geqslant \frac{1}{C} |\phi x - \phi y| - C \geqslant C^5 - C \geqslant C^4.$$

Thus, $\phi(hx) + C^4 \leqslant \phi(hy)$, whence $\phi(fhx) < \phi(fhy)$. It follows that condition (2) fails for $g = fh$ and hence that instead $fh \in H$.

Similarly, if $h \in H$ and $f \notin H$, then $fh \notin H$ and, if $h, f \notin H$, then $fh \in H$. As $1 \in H$, this imples that H is a subgroup of index at most 2. Moreover, since $\phi\colon G \to \mathbb{R}$ is continuous, we see that H is open. $\square$

Theorem 5.56. *Suppose G is an amenable Polish group coarsely equivalent to $\mathbb{R}$. Then G contains an open subgroup H of index at most 2 and a coarsely proper continuous homomorphism $\phi\colon H \to \mathbb{R}$.*

Proof. Let H be the open subgroup given by Lemma 5.55 and, for $\xi\colon H \to \mathbb{R}$ and $h \in H$, set $\rho(h)\xi = \xi(\cdot\, h)$. Let also $\mathrm{LUC}(H)$ be the algebra of bounded real-valued left-uniformly continuous functions on H and define a cocycle $b\colon H \to \mathrm{LUC}(H)$ for the right-regular representation

$$\rho\colon H \curvearrowright \mathrm{LUC}(H)$$

by

$$b(h) = \phi - \rho(h)\phi.$$

Indeed, to see that $b(h) \in \mathrm{LUC}(H)$, note that

$$\|b(h)\|_\infty = \sup_{x \in H} |\phi(x) - \phi(xh)| \leqslant \sup_{x \in H} d(x, xh) = d(1, h).$$

And, to verify the cocycle equation, observe that

$$b(hf) = \phi - \rho(hf)\phi = (\phi - \rho(h)\phi) + \rho(h)\big(\phi - \rho(f)\phi\big) = b(h) + \rho(h)b(f).$$

As G is amenable and H has finite index in G, also H is amenable. So pick a ρ-invariant mean $\mathfrak{m} \in \mathrm{LUC}(H)^*$.

To define the homomorphism $\pi\colon H \to \mathbb{R}$, we simply set $\pi(h) = \mathfrak{m}\big(b(h)\big)$ and note that

$$\begin{aligned}\pi(hf) &= \mathfrak{m}\big(b(hf)\big)\\ &= \mathfrak{m}\big(b(h) + \rho(g)b(f)\big)\\ &= \mathfrak{m}\big(b(h)\big) + \mathfrak{m}\big(\rho(h)b(f)\big)\\ &= \mathfrak{m}\big(b(h)\big) + \mathfrak{m}\big(b(f)\big)\\ &= \pi(h) + \pi(f),\end{aligned}$$

i.e., π is a continuous homomorphism.

In order to verify that π is coarsely proper, fix $n \geqslant 4$ and suppose that $h \in H$ satisfies $d(h,1) \geqslant C^{n+2}$ and hence $|\phi(h) - \phi(1)| \geqslant C^n$. Suppose first that $\phi(h) + C^n \leqslant \phi(1)$. Then, by Lemma 5.54, we get

$$\phi(xh) + C^n \leqslant \phi(x)$$

for all $x \in H$, i.e.,

$$C^n \leqslant b(h) \leqslant d(h,1)$$

and so

$$C^n \leqslant \pi(h) = \mathfrak{m}\big(b(h)\big) \leqslant d(h,1).$$

Similarly, if $\phi(1) + C^n \leqslant \phi(h)$, then $-d(h,1) \leqslant \pi(h) \leqslant -C^n$, which thus shows that π is coarsely proper. $\square$

THEOREM 5.57. *Let G be a Polish group coarsely equivalent to $\mathbb{R}$. Then there is an open subgroup H of index at most 2 and a coarsely bounded set $A \subseteq H$ so that every $h \in H \setminus A$ generates a cobounded undistorted infinite cyclic subgroup.*

In other words, for every $h \in H \setminus A$, the map $n \in \mathbb{Z} \mapsto h^n \in G$ is a quasi-isometry between $\mathbb{Z}$ and G.

PROOF. Let ϕ and H be as above and suppose that $d(h,1) \geqslant C^6$ for some $h \in H$. Then either $\phi(1) + C^4 \leqslant \phi(h)$ or $\phi(h) + C^4 \leqslant \phi(1)$ and so, as $\langle h \rangle \subseteq H$, we have either

$$\ldots < \phi(h^{-2}) < \phi(h^{-1}) < \phi(1) < \phi(h) < \phi(h^2) < \ldots$$

or

$$\ldots < \phi(h^2) < \phi(h) < \phi(1) < \phi(h^{-1}) < \phi(h^{-2}) < \ldots.$$

Since also

$$C^4 \leqslant \frac{1}{C} d(h,1) - C \leqslant \frac{1}{C} d(h^{k+1}, h^k) - C \leqslant |\phi(h^{k+1}) - \phi(h^k)| \leqslant d(h,1)$$

for all $k \in \mathbb{Z}$, we see that $\{\phi(h^k)\}_{k\in\mathbb{Z}}$ is a linearly ordered bi-infinite sequence in $\mathbb{R}$ whose successive terms have distance between C^4 and $d(h,1)$. In particular, $\{\phi(h^k)\}_{k\in\mathbb{Z}}$ is cobounded in $\mathbb{R}$ and thus $\langle h \rangle$ must be cobounded in H.

Thus, $k \in \mathbb{Z} \mapsto h^k \in H$ is a quasi-isometry of $\mathbb{Z}$ and H and hence $\langle h \rangle$ is a cobounded undistorted infinite cyclic subgroup of H. It therefore suffices to set $A = \{h \in H \mid d(h,1) < C^6\}$. $\square$

CHAPTER 6

Automorphism groups of countable structures

1. Non-Archimedean Polish groups and large scale geometry

We now turn our attention to the coarse geometry of non-Archimedean Polish group, where G is *non-Archimedean* if there is a neighbourhood basis at the identity consisting of open subgroups of G.

One particular source of examples of non-Archimedean Polish groups are first-order model theoretical structures. Namely, if $\mathbf{M}$ is a countable first-order structure, e.g., a graph, a group, a field or a lattice, we equip its automorphism group $\mathrm{Aut}(\mathbf{M})$ with the *permutation group topology*, which is the group topology obtained by declaring the pointwise stabilisers

$$V_A = \{g \in \mathrm{Aut}(\mathbf{M}) \mid \forall x \in A \;\; g(x) = x\}$$

of all finite subsets $A \subseteq \mathbf{M}$ to be open. In this case, one sees that a basis for the topology on $\mathrm{Aut}(\mathbf{M})$ is given by the family of cosets fV_A, where $f \in \mathrm{Aut}(\mathbf{M})$ and $A \subseteq \mathbf{M}$ is finite.

Now, conversely, if G is a non-Archimedean Polish group, then, by considering its action on the left-coset spaces G/V, where V varies over open subgroups of G, one can show that G is topologically isomorphic to the automorphism group $\mathrm{Aut}(\mathbf{M})$ of some first order structure $\mathbf{M}$.

The investigation of non-Archimedean Polish groups via the interplay between the model theoretical properties of the structure $\mathbf{M}$ and the dynamical and topological properties of the automorphism group $\mathrm{Aut}(\mathbf{M})$ is currently very active as witnessed, e.g., by the papers [**39, 40, 73, 6**].

In the following, $\mathbf{M}$ denotes a countable first-order structure. We use $\overline{a}, \overline{b}, \overline{c}, \ldots$ as variables for finite tuples of elements of $\mathbf{M}$ and shall write $(\overline{a}, \overline{b})$ to denote the concatenation of the tuples $\overline{a}$ and $\overline{b}$. The automorphism group $\mathrm{Aut}(\mathbf{M})$ acts naturally on tuples $\overline{a} = (a_1, \ldots, a_n) \in \mathbf{M}^n$ via

$$g \cdot (a_1, \ldots, a_n) = (ga_1, \ldots, ga_n).$$

With this notation, the pointwise stabiliser subgroups $V_{\overline{a}} = \{g \in \mathrm{Aut}(\mathbf{M}) \mid g \cdot \overline{a} = \overline{a}\}$, where $\overline{a}$ ranges over all finite tuples in $\mathbf{M}$, form a neighbourhood basis at the identity in $\mathrm{Aut}(\mathbf{M})$. So, if $A \subseteq \mathbf{M}$ is the finite set enumerated by $\overline{a}$ and $\mathbf{A} \subseteq \mathbf{M}$ is the substructure generated by A, we have $V_{\mathbf{A}} = V_A = V_{\overline{a}}$. An *orbital type* $\mathcal{O}$ in $\mathbf{M}$ is simply the orbit of some tuple $\overline{a}$ under the action of $\mathrm{Aut}(\mathbf{M})$. Also, we let $\mathcal{O}(\overline{a})$ denote the orbital type of $\overline{a}$, i.e., $\mathcal{O}(\overline{a}) = \mathrm{Aut}(\mathbf{M}) \cdot \overline{a}$.

2. Orbital types formulation

If G is a non-Archimedean Polish group admitting a maximal metric, there is a completely abstract way of identifying its quasi-isometry type. For the special case of compactly generated non-Archimedean locally compact groups, H. Abels

(Beispiel 5.2 [**1**]) did this via constructing a vertex transitive and coarsely proper action on a countable connected graph. We may use the same idea in the general case exercising some caution while dealing with coarsely bounded rather than compact sets.

Namely, fix a symmetric open set $U \ni 1$ generating the group and being coarsely bounded in G. We construct a vertex transitive and coarsely proper action on a countable connected graph as follows. First, pick an open subgroup V contained in U and let $A \subseteq G$ be a countable set so that $VUV = AV$. Since $a \in VUV$ implies that also $a^{-1} \in (VUV)^{-1} = VUV$, we may assume that A is symmetric, whereby $AV = VUV = (VUV)^{-1} = V^{-1}A^{-1} = VA$ and thus also $(VUV)^k = (AV)^k = A^kV^k = A^kV$ for all $k \geqslant 1$. In particular, we note that A^kV is coarsely bounded in G for all $k \geqslant 1$.

The graph $\mathbb{X}$ is now defined to be the set G/V of left-cosets of V along with the set of edges $\big\{\{gV, gaV\} \bigm| a \in A \;\&\; g \in G\big\}$. Note that the left-multiplication action of G on G/V is a vertex transitive action of G by automorphisms of $\mathbb{X}$. Moreover, since $G = \bigcup_k (VUV)^k = \bigcup_k A^kV$, one sees that the graph $\mathbb{X}$ is connected and hence the shortest path distance ρ is a well-defined metric on $\mathbb{X}$.

We claim that the action $G \curvearrowright \mathbb{X}$ is coarsely proper. Indeed, note that, if $g_n \to \infty$ in G, then (g_n) eventually leaves every coarsely bounded subset of G and thus, in particular, leaves every A^kV. Since, the k-ball around the vertex $1V \in \mathbb{X}$ is contained in the set A^kV, one sees that $\rho(g_n \cdot 1V, 1V) \to \infty$, showing that the action is coarsely proper. Therefore, by the Milnor–Schwarz Lemma (Theorem 2.57), the mapping $g \mapsto gV$ is a quasi-isometry between G and $(\mathbb{X}, \rho)$.

However, this construction neither addresses the question of when G admits a maximal metric nor provides a very informative manner of defining this. For those questions, we need to investigate matters further. In the following, $\mathbf{M}$ will be a fixed countable first-order structure.

Lemma 6.1. *Suppose $\overline{a}$ is a finite tuple in $\mathbf{M}$ and $\mathcal{S}$ is a finite family of orbital types in $\mathbf{M}$. Then there is a finite set $F \subseteq \mathrm{Aut}(\mathbf{M})$ so that, whenever $\overline{a}_0, \ldots, \overline{a}_n \in \mathcal{O}(\overline{a})$, $\overline{a}_0 = \overline{a}$ and $\mathcal{O}(\overline{a}_i, \overline{a}_{i+1}) \in \mathcal{S}$ for all i, then $\overline{a}_n \in (V_{\overline{a}}F)^n \cdot \overline{a}$.*

Proof. For each orbital type $\mathcal{O} \in \mathcal{S}$, pick if possible some $f \in \mathrm{Aut}(\mathbf{M})$ so that $(\overline{a}, f\overline{a}) \in \mathcal{O}$ and let F be the finite set of these f. Now, suppose that $\overline{a}_0, \ldots, \overline{a}_n \in \mathcal{O}(\overline{a})$, $\overline{a}_0 = \overline{a}$ and $\mathcal{O}(\overline{a}_i, \overline{a}_{i+1}) \in \mathcal{S}$ for all i. Since the $\overline{a}_i$ are orbit equivalent, we can inductively choose $h_1, \ldots, h_n \in \mathrm{Aut}(\mathbf{M})$ so that $\overline{a}_i = h_1 \cdots h_i \cdot \overline{a}$. Thus, for all i, we have $(\overline{a}, h_{i+1}\overline{a}) = (h_1 \cdots h_i)^{-1} \cdot (\overline{a}_i, \overline{a}_{i+1})$, whereby there is some $f \in F$ so that $(\overline{a}, h_{i+1}\overline{a}) \in \mathcal{O}(\overline{a}, f\overline{a})$. It follows that, for some $g \in \mathrm{Aut}(\mathbf{M})$, we have $(g\overline{a}, gh_{i+1}\overline{a}) = (\overline{a}, f\overline{a})$, i.e., $g \in V_{\overline{a}}$ and $f^{-1}gh_{i+1} \in V_{\overline{a}}$, whence also $h_{i+1} \in V_{\overline{a}}FV_{\overline{a}}$. Therefore, $\overline{a}_n = h_1 \cdots h_n \cdot \overline{a} \in (V_{\overline{a}}FV_{\overline{a}})^n \cdot \overline{a} = (V_{\overline{a}}F)^n \cdot \overline{a}$. □

Lemma 6.2. *Suppose $\overline{a}$ is a finite tuple in $\mathbf{M}$ and $F \subseteq \mathrm{Aut}(\mathbf{M})$ is a finite set. Then there is a finite family $\mathcal{S}$ of orbital types in $\mathbf{M}$ so that, for all $g \in (V_{\overline{a}}F)^n$, there are $\overline{a}_0, \ldots, \overline{a}_n \in \mathcal{O}(\overline{a})$ with $\overline{a}_0 = \overline{a}$ and $\overline{a}_n = g\overline{a}$ satisfying $\mathcal{O}(\overline{a}_i, \overline{a}_{i+1}) \in \mathcal{S}$ for all i.*

Proof. We let $\mathcal{S}$ be the collection of $\mathcal{O}(\overline{a}, f\overline{a})$ with $f \in F$. Now suppose that $g \in (V_{\overline{a}}F)^n$ and write $g = h_1f_1 \cdots h_nf_n$ for $h_i \in V_{\overline{a}}$ and $f_i \in F$. Setting $\overline{a}_i = h_1f_1 \cdots h_if_i \cdot \overline{a}$, we see that

$$(\overline{a}_i, \overline{a}_{i+1}) = h_1f_1 \cdots h_if_ih_{i+1} \cdot (h_{i+1}^{-1}\overline{a}, f_{i+1}\overline{a}) = h_1f_1 \cdots h_if_ih_{i+1} \cdot (\overline{a}, f_{i+1}\overline{a}),$$

i.e., $\mathcal{O}(\overline{a}_i, \overline{a}_{i+1}) \in \mathcal{S}$ as required. $\square$

In order to simplify calculations of quasi-isometry types of various automorphism groups and provide a better visualisation of the group structure, we introduce the following graph.

DEFINITION 6.3. *Suppose $\overline{a}$ is a tuple in $\mathbf{M}$ and $\mathcal{S}$ is a finite family of orbital types in $\mathbf{M}$. We let $\mathbb{X}_{\overline{a},\mathcal{S}}$ denote the graph whose vertex set is the orbital type $\mathcal{O}(\overline{a})$ and whose edge relation is given by*

$$(\overline{b}, \overline{c}) \in \text{Edge}\, \mathbb{X}_{\overline{a},\mathcal{S}} \;\Leftrightarrow\; \overline{b} \neq \overline{c} \;\&\; \Big(\mathcal{O}(\overline{b}, \overline{c}) \in \mathcal{S} \;\; or \;\; \mathcal{O}(\overline{c}, \overline{b}) \in \mathcal{S}\Big).$$

Let also $\rho_{\overline{a},\mathcal{S}}$ denote the corresponding shortest path metric on $\mathbb{X}_{\overline{a},\mathcal{S}}$, where we stipulate that $\rho_{\overline{a},\mathcal{S}}(\overline{b}, \overline{c}) = \infty$ whenever $\overline{b}$ and $\overline{c}$ belong to distinct connected components.

We remark that, as the vertex set of $\mathbb{X}_{\overline{a},\mathcal{S}}$ is just the orbital type of $\overline{a}$, the automorphism group $\text{Aut}(\mathbf{M})$ acts transitively on the vertices of $\mathbb{X}_{\overline{a},\mathcal{S}}$. Moreover, the edge relation is clearly invariant, meaning that $\text{Aut}(\mathbf{M})$ acts vertex transitively by automorphisms on $\mathbb{X}_{\overline{a},\mathcal{S}}$. In particular, $\text{Aut}(\mathbf{M})$ preserves $\rho_{\overline{a},\mathcal{S}}$.

LEMMA 6.4. *Suppose $\overline{a}$ is a finite tuple in $\mathbf{M}$. Then the following are equivalent.*

(1) *The pointwise stabiliser $V_{\overline{a}}$ is coarsely bounded in* $\text{Aut}(\mathbf{M})$.
(2) *For every tuple $\overline{b}$ in $\mathbf{M}$, there is a finite family $\mathcal{S}$ of orbital types so that the set*

$$\{(\overline{a}, \overline{c}) \mid (\overline{a}, \overline{c}) \in \mathcal{O}(\overline{a}, \overline{b})\} = \{(\overline{a}, g\overline{b}) \mid g \in V_{\overline{a}}\}$$

has finite $\mathbb{X}_{(\overline{a},\overline{b}),\mathcal{S}}$-diameter.

PROOF. (1)⇒(2): Suppose that $V_{\overline{a}}$ is coarsely bounded in $\text{Aut}(\mathbf{M})$ and that $\overline{b}$ is a tuple in $\mathbf{M}$. This means that, for every neigbourhood $U \ni 1$, there is a finite set $F \subseteq \text{Aut}(\mathbf{M})$ and an $n \geqslant 1$ so that $V_{\overline{a}} \subseteq (UF)^n$. In particular, this holds for $U = V_{(\overline{a},\overline{b})}$. Let now $\mathcal{S}$ be the finite family of orbital types associated to F and the tuple $(\overline{a}, \overline{b})$ as given in Lemma 6.2. Thus, for any $g \in V_{\overline{a}} \subseteq (V_{(\overline{a},\overline{b})}F)^n$ there is a path in $\mathbb{X}_{(\overline{a},\overline{b}),\mathcal{S}}$ of length $n+1$ beginning at $(\overline{a}, \overline{b})$ and ending at $g(\overline{a}, \overline{b}) = (\overline{a}, g\overline{b})$. In other words, the set

$$\{(\overline{a}, g\overline{b}) \mid g \in V_{\overline{a}}\}$$

has diameter at most $2n+1$ in $\mathbb{X}_{(\overline{a},\overline{b}),\mathcal{S}}$.

(2)⇒(1): Assume that (2) holds. To see that $V_{\overline{a}}$ is coarsely bounded in $\text{Aut}(\mathbf{M})$, it is enough to verify that, for all tuples $\overline{b}$ in $\mathbf{M}$, there is a finite set $F \subseteq \text{Aut}(\mathbf{M})$ and an $n \geqslant 1$ so that $V_{\overline{a}} \subseteq (V_{(\overline{a},\overline{b})}FV_{(\overline{a},\overline{b})})^n$.

So suppose $\overline{b}$ is given and let $\mathcal{S}$ be a finite set of orbital types so that

$$\{(\overline{a}, g\overline{b}) \mid g \in V_{\overline{a}}\}$$

has finite $\mathbb{X}_{(\overline{a},\overline{b}),\mathcal{S}}$-diameter. Pick then a finite set $F \subseteq \text{Aut}(\mathbf{M})$ associated to $\mathcal{S}$ and the tuple $(\overline{a}, \overline{b})$ as provided by Lemma 6.1. This means that, for some n and all $g \in V_{\overline{a}}$, we have

$$g(\overline{a}, \overline{b}) = (\overline{a}, g\overline{b}) \in \big(V_{(\overline{a},\overline{b})}F\big)^n \cdot (\overline{a}, \overline{b}),$$

whence $g \in (V_{(\overline{a},\overline{b})}F)^n \cdot V_{(\overline{a},\overline{b})} = (V_{(\overline{a},\overline{b})}FV_{(\overline{a},\overline{b})})^n$ as required. $\square$

Using that the $V_{\overline{a}}$ form a neighbourhood basis at the identity in $\mathrm{Aut}(\mathbf{M})$, we obtain the following criterion for local boundedness and hence the existence of a coarsely proper metric.

THEOREM 6.5. *The following are equivalent for the automorphism group* $\mathrm{Aut}(\mathbf{M})$ *of a countable structure* $\mathbf{M}$.

(1) $\mathrm{Aut}(\mathbf{M})$ *admits a coarsely proper metric,*
(2) $\mathrm{Aut}(\mathbf{M})$ *is locally bounded,*
(3) *there is a tuple* $\overline{a}$ *so that, for every* $\overline{b}$, *there is a finite family* $\mathcal{S}$ *of orbital types for which the set*

$$\{(\overline{a},\overline{c}) \mid (\overline{a},\overline{c}) \in \mathcal{O}(\overline{a},\overline{b})\}$$

has finite $\mathbb{X}_{(\overline{a},\overline{b}),\mathcal{S}}$*-diameter.*

Note that $\rho_{\overline{a},\mathcal{S}}$ is an actual metric exactly when $\mathbb{X}_{\overline{a},\mathcal{S}}$ is a connected graph. Our next task is to decide when this happens.

LEMMA 6.6. *Suppose* $\overline{a}$ *is a tuple in* $\mathbf{M}$. *Then the following are equivalent.*

(1) $\mathrm{Aut}(\mathbf{M})$ *is finitely generated over* $V_{\overline{a}}$,
(2) *there is a finite family* $\mathcal{S}$ *of orbital types so that* $\mathbb{X}_{\overline{a},\mathcal{S}}$ *is connected.*

PROOF. (1)$\Rightarrow$(2): Suppose that $\mathrm{Aut}(\mathbf{M})$ is finitely generated over $V_{\overline{a}}$ and pick a finite set $F \subseteq \mathrm{Aut}(\mathbf{M})$ containing 1 so that $\mathrm{Aut}(\mathbf{M}) = \langle V_{\overline{a}} \cup F\rangle$. Let also $\mathcal{S}$ be the finite family of orbital types associated to F and $\overline{a}$ as given by Lemma 6.2. To see that $\mathbb{X}_{\overline{a},\mathcal{S}}$ is connected, let $\overline{b} \in \mathcal{O}(\overline{a})$ be any vertex and write $\overline{b} = g\overline{a}$ for some $g \in \mathrm{Aut}(\mathbf{M})$. Find also $n \geqslant 1$ so that $g \in (V_{\overline{a}}F)^n$. By the choice of $\mathcal{S}$, it follows that there are $\overline{c}_0,\ldots,\overline{c}_n \in \mathcal{O}(\overline{a})$ with $\overline{c}_0 = \overline{a}$, $\overline{c}_n = g\overline{a}$ and $\mathcal{O}(\overline{c}_i,\overline{c}_{i+1}) \in \mathcal{S}$ for all i. Thus, $\overline{c}_0,\ldots,\overline{c}_n$ is a path from $\overline{a}$ to $\overline{b}$ in $\mathbb{X}_{\overline{a},\mathcal{S}}$. Since every vertex is connected to $\overline{a}$, $\mathbb{X}_{\overline{a},\mathcal{S}}$ is a connected graph.

(2)$\Rightarrow$(1): Assume $\mathcal{S}$ is a finite family of orbital types so that $\mathbb{X}_{\overline{a},\mathcal{S}}$ is connected. We let $\mathcal{T}$ consist of all orbital types $\mathcal{O}(\overline{b},\overline{c})$ so that either $\mathcal{O}(\overline{b},\overline{c}) \in \mathcal{S}$ or $\mathcal{O}(\overline{c},\overline{b}) \in \mathcal{S}$ and note that $\mathcal{T}$ is also finite. Let also $F \subseteq \mathrm{Aut}(\mathbf{M})$ be the finite set associated to $\overline{a}$ and $\mathcal{T}$ as given by Lemma 6.2. Then, if $g \in \mathrm{Aut}(\mathbf{M})$, there is a path $\overline{c}_0,\ldots,\overline{c}_n$ in $\mathbb{X}_{\overline{a},\mathcal{S}}$ from $\overline{c}_0 = \overline{a}$ to $\overline{c}_n = g\overline{a}$, whence $\mathcal{O}(\overline{c}_i,\overline{c}_{i+1}) \in \mathcal{T}$ for all i. By the choice of F, it follows that $g\overline{a} = \overline{c}_n \in (V_{\overline{a}}F)^n \cdot \overline{a}$ and hence that $g \in (V_{\overline{a}}F)^n \cdot V_{\overline{a}}$. Thus, $\mathrm{Aut}(\mathbf{M}) = \langle V_{\overline{a}} \cup F\rangle$. $\square$

LEMMA 6.7. *Suppose* $\overline{a}$ *is a tuple in* $\mathbf{M}$ *so that the pointwise stabiliser* $V_{\overline{a}}$ *is coarsely bounded in* $\mathrm{Aut}(\mathbf{M})$ *and assume that* $\mathcal{S}$ *is a finite family of orbital types. Then, for all natural numbers* n, *the set*

$$\{g \in \mathrm{Aut}(\mathbf{M}) \bigm| \rho_{\overline{a},\mathcal{S}}(\overline{a},g\overline{a}) \leqslant n\}$$

is coarsely bounded in $\mathrm{Aut}(\mathbf{M})$.

In particular, if the graph $\mathbb{X}_{\overline{a},\mathcal{S}}$ *is connected, then the continuous isometric action*

$$\mathrm{Aut}(\mathbf{M}) \curvearrowright (\mathbb{X}_{\overline{a},\mathcal{S}},\rho_{\overline{a},\mathcal{S}})$$

is coarsely proper.

PROOF. Let $F \subseteq \mathrm{Aut}(\mathbf{M})$ be the finite set associated to $\overline{a}$ and $\mathcal{S}$ as given by Lemma 6.1. Then, if $g \in \mathrm{Aut}(\mathbf{M})$ is such that $\rho_{\overline{a},\mathcal{S}}(\overline{a},g\overline{a}) = m \leqslant n$, there is a path

$\overline{c}_0, \ldots, \overline{c}_m$ in $\mathbb{X}_{\overline{a},\mathcal{S}}$ with $\overline{c}_0 = \overline{a}$ and $\overline{c}_m = g\overline{a}$. Thus, by the choice of F, we have that $g\overline{a} = \overline{c}_m \in (V_{\overline{a}}F)^m \cdot \overline{a}$, i.e., $g \in (V_{\overline{a}}F)^m \cdot V_{\overline{a}} = (V_{\overline{a}}FV_{\overline{a}})^m$. In other words,

$$\{g \in \mathrm{Aut}(\mathbf{M}) \mid \rho_{\overline{a},\mathcal{S}}(\overline{a}, g\overline{a}) \leqslant n\} \subseteq \bigcup_{m \leqslant n} (V_{\overline{a}}FV_{\overline{a}})^m$$

and the latter set is coarsely bounded in $\mathrm{Aut}(\mathbf{M})$. □

With these preliminary results at hand, we can now give a full characterisation of when an automorphism group $\mathrm{Aut}(\mathbf{M})$ carries a well-defined quasi-isometry type and, moreover, provide a direct computation of this.

Theorem 6.8. *Let $\mathbf{M}$ be a countable structure. Then $\mathrm{Aut}(\mathbf{M})$ admits a maximal metric if and only if there is a tuple $\overline{a}$ in $\mathbf{M}$ satisfying the following two requirements.*

(1) *For every $\overline{b}$, there is a finite family $\mathcal{S}$ of orbital types for which the set*
$$\{(\overline{a}, \overline{c}) \mid (\overline{a}, \overline{c}) \in \mathcal{O}(\overline{a}, \overline{b})\}$$
has finite $\mathbb{X}_{(\overline{a},\overline{b}),\mathcal{S}}$-diameter,

(2) *there is a finite family $\mathcal{R}$ of orbital types so that $\mathbb{X}_{\overline{a},\mathcal{R}}$ is connected.*

Moreover, if $\overline{a}$ and $\mathcal{R}$ are as in (2), then the mapping

$$g \in \mathrm{Aut}(\mathbf{M}) \mapsto g \cdot \overline{a} \in \mathbb{X}_{\overline{a},\mathcal{R}}$$

is a quasi-isometry between $\mathrm{Aut}(\mathbf{M})$ and $(\mathbb{X}_{\overline{a},\mathcal{R}}, \rho_{\overline{a},\mathcal{R}})$.

Proof. Note that (1) is simply a restatement of $V_{\overline{a}}$ being coarsely bounded in $\mathrm{Aut}(\mathbf{M})$, while (2) states that $\mathrm{Aut}(\mathbf{M})$ is finitely generated over $V_{\overline{a}}$. By Theorem 2.53, these two properties together are equivalent to the existence of a maximal metric.

For the moreover part, note that, as $\mathbb{X}_{\overline{a},\mathcal{R}}$ is a connected graph, the metric space $(\mathbb{X}_{\overline{a},\mathcal{R}}, \rho_{\overline{a},\mathcal{R}})$ is large scale geodesic. Thus, as the continuous isometric action $\mathrm{Aut}(\mathbf{M}) \curvearrowright (\mathbb{X}_{\overline{a},\mathcal{S}}, \rho_{\overline{a},\mathcal{S}})$ is transitive (hence cobounded) and coarsely proper, it follows from the Milnor–Schwarz lemma (Theorem 2.57) that

$$g \in \mathrm{Aut}(\mathbf{M}) \mapsto g \cdot \overline{a} \in \mathbb{X}_{\overline{a},\mathcal{R}}$$

is a quasi-isometry between $\mathrm{Aut}(\mathbf{M})$ and $(\mathbb{X}_{\overline{a},\mathcal{R}}, \rho_{\overline{a},\mathcal{R}})$. □

In cases where $\mathrm{Aut}(\mathbf{M})$ may not admit a maximal metric, but only a coarsely proper metric, it is still useful to have an explicit calculation of this. For this, the following lemma will be useful.

Lemma 6.9. *Suppose $\overline{a}$ is finite tuple in a countable structure $\mathbf{M}$. Let also $\mathcal{R}_1 \subseteq \mathcal{R}_2 \subseteq \mathcal{R}_3 \subseteq \ldots$ be an exhaustive sequence of finite sets of orbital types on $\mathbf{M}$ and define a metric $\rho_{\overline{a},(\mathcal{R}_n)}$ on $\mathcal{O}(\overline{a})$ by*

$$\rho_{\overline{a},(\mathcal{R}_n)}(\overline{b}, \overline{c}) = \min\Big(\sum_{i=1}^{k} n_i \cdot \rho_{\overline{a},\mathcal{R}_{n_i}}(\overline{d}_{i-1}, \overline{d}_i) \;\Big|\; n_i \in \mathbb{N} \;\;\&\;\; \overline{d}_i \in \mathcal{O}(\overline{a}) \;\;\&\;\; \overline{d}_0 = \overline{b} \;\;\&\;\; \overline{d}_k = \overline{c}\Big).$$

Assuming that $V_{\overline{a}}$ is coarsely bounded in $\mathrm{Aut}(\mathbf{M})$, then the isometric action

$$\mathrm{Aut}(\mathbf{M}) \curvearrowright \big(\mathcal{O}(\overline{a}), \rho_{\overline{a},(\mathcal{R}_n)}\big)$$

is coarsely proper.

Proof. Let us first note that, since the sequence $(\mathcal{R}_n)$ is exhaustive, every orbital type $\mathcal{O}(\bar{b},\bar{c})$ eventually belongs to some $\mathcal{R}_n$, whereby $\rho_{\bar{a},(\mathcal{R}_n)}(\bar{b},\bar{c})$ is finite. Also, $\rho_{\bar{a},(\mathcal{R}_n)}$ satisfies the triangle inequality by definition and hence is a metric.

Note now that, since the $\mathcal{R}_n$ are increasing with n, we have, for all $m\in\mathbb{N}$,

$$\rho_{\bar{a},(\mathcal{R}_n)}(\bar{b},\bar{c})\leqslant m \quad\Rightarrow\quad \rho_{\bar{a},\mathcal{R}_m}(\bar{b},\bar{c})\leqslant m$$

and thus

$$\{g\in\mathrm{Aut}(\mathbf{M})\mid \rho_{\bar{a},(\mathcal{R}_n)}(\bar{a},g\bar{a})\leqslant m\}\subseteq\{g\in\mathrm{Aut}(\mathbf{M})\mid \rho_{\bar{a},\mathcal{R}_m}(\bar{a},g\bar{a})\leqslant m\}.$$

By Lemma 6.7, the latter set is coarsely bounded in $\mathrm{Aut}(\mathbf{M})$, so the action $\mathrm{Aut}(\mathbf{M})\curvearrowright\big(\mathcal{O}(\bar{a}),\rho_{\bar{a},(\mathcal{R}_n)}\big)$ is coarsely proper. $\square$

3. Homogeneous and atomic models

3.1. Definability of metrics. Whereas the preceding sections have largely concentrated on the automorphism group $\mathrm{Aut}(\mathbf{M})$ of a countable structure $\mathbf{M}$ without much regard to the actual structure $\mathbf{M}$, its language $\mathcal{L}$ or its theory $T=\mathrm{Th}(\mathbf{M})$, in the present section, we shall study how the theory T may directly influence the large scale geometry of $\mathrm{Aut}(\mathbf{M})$.

We recall that a structure $\mathbf{M}$ is *ω-homogeneous* if, for all finite tuples $\bar{a}$ and $\bar{b}$ in $\mathbf{M}$ with the same type $\mathrm{tp}^{\mathbf{M}}(\bar{a})=\mathrm{tp}^{\mathbf{M}}(\bar{b})$ and all c in $\mathbf{M}$, there is some d in $\mathbf{M}$ so that $\mathrm{tp}^{\mathbf{M}}(\bar{a},c)=\mathrm{tp}^{\mathbf{M}}(\bar{b},d)$. By a back and forth construction, one sees that, in case $\mathbf{M}$ is countable, ω-homogeneity is equivalent to the condition

$$\mathrm{tp}^{\mathbf{M}}(\bar{a})=\mathrm{tp}^{\mathbf{M}}(\bar{b}) \quad\Leftrightarrow\quad \mathcal{O}(\bar{a})=\mathcal{O}(\bar{b}).$$

In other words, every orbital type $\mathcal{O}(\bar{a})$ is *type $\emptyset$-definable*, i.e., type definable without parameters.

For a stronger notion, we say that $\mathbf{M}$ is *ultrahomogeneous* if it satisfies

$$\mathrm{qftp}^{\mathbf{M}}(\bar{a})=\mathrm{qftp}^{\mathbf{M}}(\bar{b}) \quad\Leftrightarrow\quad \mathcal{O}(\bar{a})=\mathcal{O}(\bar{b}),$$

where $\mathrm{qftp}^{\mathbf{M}}(\bar{a})$ denotes the quantifier-free type of $\bar{a}$. In other words, every orbital type $\mathcal{O}(\bar{a})$ is defined by the quantifier-free type $\mathrm{qftp}^{\mathbf{M}}(\bar{a})$.

Another requirement is to demand that each individual orbital type $\mathcal{O}(\bar{a})$ is *$\emptyset$-definable* in $\mathbf{M}$, i.e., definable by a single formula $\phi(\bar{x})$ without parameters, that is, so that $\bar{b}\in\mathcal{O}(\bar{a})$ if and only if $\mathbf{M}\models\phi(\bar{b})$. We note that such a ϕ necessarily isolates the type $\mathrm{tp}^{\mathbf{M}}(\bar{a})$. Indeed, suppose $\psi\in\mathrm{tp}^{\mathbf{M}}(\bar{a})$. Then, if $\mathbf{M}\models\phi(\bar{b})$, we have $\bar{b}\in\mathcal{O}(\bar{a})$ and thus also $\mathbf{M}\models\psi(\bar{b})$, showing that $\mathbf{M}\models\forall\bar{x}\ (\phi\to\psi)$. Conversely, suppose $\mathbf{M}$ is a countable ω-homogeneous structure and $\phi(\bar{x})$ is a formula without parameters isolating some type $\mathrm{tp}^{\mathbf{M}}(\bar{a})$. Then, if $\mathbf{M}\models\phi(\bar{b})$, we have $\mathrm{tp}^{\mathbf{M}}(\bar{a})=\mathrm{tp}^{\mathbf{M}}(\bar{b})$ and thus, by ω-homogeneity, $\bar{b}\in\mathcal{O}(\bar{a})$.

We recall that a model $\mathbf{M}$ is *atomic* if every type realised in $\mathbf{M}$ is isolated. As is easy to verify (see Lemma 4.2.14 [**47**]), countable atomic models are ω-homogeneous. So, by the discussion above, we see that a countable model $\mathbf{M}$ is atomic if and only if every orbital type $\mathcal{O}(\bar{a})$ is $\emptyset$-definable.

For example, if $\mathbf{M}$ is a locally finite ultrahomogeneous structure in a finite language $\mathcal{L}$, then $\mathbf{M}$ is atomic. This follows from the fact that if $\mathbf{A}$ is a finite structure in a finite language, then its isomorphism type is described by a single quantifier-free formula.

LEMMA 6.10. *Suppose $\overline{a}$ is a finite tuple in a countable atomic model $\mathbf{M}$. Let $\mathcal{S}$ be a finite collection of orbital types in $\mathbf{M}$ and $\rho_{\overline{a},\mathcal{S}}$ denote the corresponding shortest path metric on $\mathbb{X}_{\overline{a},\mathcal{S}}$. Then, for every $n \in \mathbb{N}$, the relation $\rho_{\overline{a},\mathcal{S}}(\overline{b},\overline{c}) \leqslant n$ is $\emptyset$-definable in $\mathbf{M}$.*

Now, suppose instead that $\mathcal{S}_1 \subseteq \mathcal{S}_2 \subseteq \ldots$ is an exhaustive sequence of finite sets of orbital types on $\mathbf{M}$. Then, for every $n \in \mathbb{N}$, the relation $\rho_{\overline{a},(\mathcal{S}_m)}(\overline{b},\overline{c}) \leqslant n$ is similarly $\emptyset$-definable in $\mathbf{M}$.

PROOF. Without loss of generality, every orbital type in $\mathcal{S}$ is of the form $\mathcal{O}(\overline{b},\overline{c})$, where $\overline{b},\overline{c} \in \mathcal{O}(\overline{a})$. Moreover, for such $\mathcal{O}(\overline{b},\overline{c}) \in \mathcal{S}$, we may suppose that also $\mathcal{O}(\overline{c},\overline{b}) \in \mathcal{S}$. Let now $\phi_1(\overline{x},\overline{y}),\ldots,\phi_k(\overline{x},\overline{y})$ be formulas without parameters defining the orbital types in $\mathcal{S}$. Then

$$\rho_{\overline{a},\mathcal{S}}(\overline{b},\overline{c}) \leqslant n \quad\Leftrightarrow$$
$$\mathbf{M} \models \bigvee_{m=0}^{n} \exists \overline{y}_0,\ldots,\overline{y}_m \; \Big(\bigwedge_{j=0}^{m-1} \bigvee_{i=1}^{k} \phi_i(\overline{y}_j,\overline{y}_{j+1}) \;\&\; \overline{b} = \overline{y}_0 \;\&\; \overline{c} = \overline{y}_m \Big),$$

showing that $\rho_{\overline{a},\mathcal{S}}(\overline{b},\overline{c}) \leqslant n$ is $\emptyset$-definable in $\mathbf{M}$.

For the second case, pick formulas $\phi_{m,n}(\overline{x},\overline{y})$ without parameters defining the relations $\rho_{\overline{a},\mathcal{S}_m}(\overline{b},\overline{c}) \leqslant n$ in $\mathbf{M}$. Then

$$\rho_{\overline{a},(\mathcal{S}_m)}(\overline{b},\overline{c}) \leqslant n \quad\Leftrightarrow$$
$$\mathbf{M} \models \bigvee_{k=0}^{n} \exists \overline{x}_0,\ldots,\overline{x}_k \Big(\overline{x}_0 = \overline{b} \;\&\; \overline{x}_k = \overline{c} \;\&\; \bigvee \Big\{ \bigwedge_{i=1}^{k} \phi_{m_i,n_i}(\overline{x}_{i-1},\overline{x}_i) \;\Big|\; \sum_{i=1}^{k} m_i \cdot n_i \leqslant n \Big\} \Big),$$

showing that $\rho_{\overline{a},(\mathcal{S}_m)}(\overline{b},\overline{c}) \leqslant n$ is $\emptyset$-definable in $\mathbf{M}$. □

3.2. Stable metrics and theories. We recall the following notion originating in the work of J.-L. Krivine and B. Maurey on stable Banach spaces [**44**].

DEFINITION 6.11. *A metric d on a set X is said to be* stable *if, for all d-bounded sequences (x_n) and (y_m) in X, we have*

$$\lim_{n\to\infty} \lim_{m\to\infty} d(x_n,y_m) = \lim_{m\to\infty} \lim_{n\to\infty} d(x_n,y_m),$$

whenever both limits exist.

We mention that stability of the metric is equivalent to requiring that the limit operations $\lim_{n\to\mathcal{U}}$ and $\lim_{m\to\mathcal{V}}$ commute over d for all ultrafilters $\mathcal{U}$ and $\mathcal{V}$ on $\mathbb{N}$.

Now stability of metrics is tightly related to model theoretical stability of which we recall the definition.

DEFINITION 6.12. *Let T be a complete theory of a countable language $\mathcal{L}$ and let κ be an infinite cardinal number. We say that T is κ-*stable *if, for all models $\mathbf{M} \models T$ and subsets $A \subseteq \mathbf{M}$ with $|A| \leqslant \kappa$, we have $|S_n^{\mathbf{M}}(A)| \leqslant \kappa$. Also, T is* stable *if it is κ-stable for some infinite cardinal κ.*

In the following discussion, we shall always assume that T is a complete theory with infinite models in a countable language $\mathcal{L}$. Of the various consequences of stability of T, the one most closely related to stability of metrics is the fact that, if T is stable and $\mathbf{M}$ is a model of T, then there are no formula $\phi(\overline{x},\overline{y})$ and tuples $\overline{a}_n$, $\overline{b}_m$, $n,m \in \mathbb{N}$, so that

$$\mathbf{M} \models \phi(\overline{a}_n,\overline{b}_m) \;\Leftrightarrow\; n < m.$$

Knowing this, the following lemma is straightforward.

LEMMA 6.13. *Suppose* $\mathbf{M}$ *is a countable atomic model of a stable theory* T *and that* $\overline{a}$ *is a finite tuple in* $\mathbf{M}$. *Let also* ρ *be a metric on* $\mathcal{O}(\overline{a})$ *so that, for every* $n \in \mathbb{N}$, *the relation* $\rho(\overline{b}, \overline{c}) \leqslant n$ *is* $\emptyset$*-definable in* $\mathbf{M}$. *Then* ρ *is a stable metric.*

PROOF. Suppose towards a contradiction that $\overline{a}_n, \overline{b}_m \in \mathcal{O}(\overline{a})$ are bounded sequences in $\mathcal{O}(\overline{a})$ so that

$$r = \lim_{n\to\infty} \lim_{m\to\infty} \rho(\overline{a}_n, \overline{b}_m) \neq \lim_{m\to\infty} \lim_{n\to\infty} \rho(\overline{a}_n, \overline{b}_m)$$

and pick a formula $\phi(\overline{x}, \overline{y})$ so that

$$\rho(\overline{b}, \overline{c}) = r \quad \Leftrightarrow \quad \mathbf{M} \models \phi(\overline{b}, \overline{c}).$$

Then, using $\forall^\infty$ to denote "for all, but finitely many", we have

$$\forall^\infty n\, \forall^\infty m \quad \mathbf{M} \models \phi(\overline{a}_n, \overline{b}_m),$$

while

$$\forall^\infty m\, \forall^\infty n \quad \mathbf{M} \models \neg\phi(\overline{a}_n, \overline{b}_m).$$

So, upon passing to subsequences of $(\overline{a}_n)$ and $(\overline{b}_m)$, we may suppose that

$$\mathbf{M} \models \phi(\overline{a}_n, \overline{b}_m) \quad \Leftrightarrow \quad n < m.$$

However, the existence of such a formula ϕ and sequences $(\overline{a}_n)$ and $(\overline{b}_m)$ contradicts the stability of T. $\square$

THEOREM 6.14. *Suppose* $\mathbf{M}$ *is a countable atomic model of a stable theory* T *so that* $\mathrm{Aut}(\mathbf{M})$ *admits a maximal metric. Then* $\mathrm{Aut}(\mathbf{M})$ *admits a stable maximal metric.*

PROOF. By Theorem 6.8, there is a finite tuple $\overline{a}$ and a finite family $\mathcal{S}$ of orbital types so that the mapping

$$g \in \mathrm{Aut}(\mathbf{M}) \mapsto g\overline{a} \in \mathbb{X}_{\overline{a},\mathcal{S}}$$

is a quasi-isometry of $\mathrm{Aut}(\mathbf{M})$ with $(\mathbb{X}_{\overline{a},\mathcal{S}}, \rho_{\overline{a},\mathcal{S}})$. Also, by Lemma 6.13, $\rho_{\overline{a},\mathcal{S}}$ is a stable metric on $\mathbb{X}_{\overline{a},\mathcal{S}}$. Define also a compatible left-invariant stable metric $D \leqslant 1$ on $\mathrm{Aut}(\mathbf{M})$ by

$$D(g, f) = \sum_{n=1}^{\infty} \frac{\chi_{\neq}(g(b_n), f(b_n))}{2^n},$$

where (b_n) is an enumeration of $\mathbf{M}$ and $\chi_{\neq}$ is the characteristic funtion of inequality. The stability of D follows easily from it being an absolutely summable series of the functions $(g, f) \mapsto \frac{\chi_{\neq}(g(b_n), f(b_n))}{2^n}$.

Finally, let

$$d(g, f) = D(g, f) + \rho_{\overline{a},\mathcal{S}}(g\overline{a}, f\overline{a}).$$

Then d is a maximal and stable metric on $\mathrm{Aut}(\mathbf{M})$. $\square$

Similarly, when $\mathrm{Aut}(\mathbf{M})$ is only assumed to have a coarsely proper metric, this can also be taken to be stable. This can be done by working with the metric $\rho_{\overline{a},(\mathcal{S}_n)}$, where $\mathcal{S}_1 \subseteq \mathcal{S}_2 \subseteq \ldots$ is an exhaustive sequence of finite sets of orbital types on $\mathbf{M}$, instead of $\rho_{\overline{a},\mathcal{S}}$.

THEOREM 6.15. *Suppose* $\mathbf{M}$ *is a countable atomic model of a stable theory* T *so that* $\mathrm{Aut}(\mathbf{M})$ *admits a coarsely proper metric. Then* $\mathrm{Aut}(\mathbf{M})$ *admits a stable, coarsely proper metric.*

Using the equivalence of the local boundedness and the existence of coarsely proper metrics and that the existence of a stable coarsely proper metric implies coarsely proper actions on reflexive spaces (see Theorem 56 [**65**]), we have the following corollary.

Corollary 6.16. *Suppose* $\mathbf{M}$ *is a countable atomic model of a stable theory* T *so that* $\mathrm{Aut}(\mathbf{M})$ *is locally bounded. Then* $\mathrm{Aut}(\mathbf{M})$ *admits a coarsely proper continuous affine isometric action on a reflexive Banach space.*

We should briefly review the hypotheses of the preceding theorem. So, in the following, let T be a complete theory with infinite models in a countable language $\mathcal{L}$. We recall that that $\mathbf{M} \models T$ is said to be a *prime model* of T if $\mathbf{M}$ admits an elementary embedding into every other model of T. By the Omitting Types Theorem, prime models are necessarily atomic. In fact, $\mathbf{M} \models T$ is a prime model of T if and only if $\mathbf{M}$ is both countable and atomic. Moreover, the theory T admits a countable atomic model if and only if, for every n, the set of isolated types is dense in the type space $S_n(T)$. In particular, this happens if $S_n(T)$ is countable for all n.

Now, by definition, T is ω-stable, if, for every model $\mathbf{M} \models T$, countable subset $A \subseteq \mathbf{M}$ and $n \geqslant 1$, the type space $S_n^{\mathbf{M}}(A)$ is countable. In particular, $S_n(T)$ is countable for every n and hence T has a countable atomic model $\mathbf{M}$. Thus, provided that $\mathrm{Aut}(\mathbf{M})$ is locally bounded, Corollary 6.16 gives a coarsely proper affine isometric action of this automorphism group.

3.3. Fraïssé classes. A useful tool in the study of ultrahomogeneous countable structures is the theory of R. Fraïssé that alllows us to view every such object as a so called limit of the family of its finitely generated substructures. In the following, we fix a countable language $\mathcal{L}$.

Definition 6.17. *A* Fraïssé class *is a class* $\mathcal{K}$ *of finitely generated* $\mathcal{L}$*-structures so that*

(1) κ *contains only countably many isomorphism types,*
(2) *(hereditary property) if* $\mathbf{A} \in \mathcal{K}$ *and* $\mathbf{B}$ *is a finitely generated* $\mathcal{L}$*-structure embeddable into* $\mathbf{A}$*, then* $\mathbf{B} \in \mathcal{K}$*,*
(3) *(joint embedding property) for all* $\mathbf{A}, \mathbf{B} \in \mathcal{K}$*, there some* $\mathbf{C} \in \mathcal{K}$ *into which both* $\mathbf{A}$ *and* $\mathbf{B}$ *embed,*
(4) *(amalgamation property) if* $\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2 \in \mathcal{K}$ *and* $\eta_i \colon \mathbf{A} \hookrightarrow \mathbf{B}_i$ *are embeddings, then there is some* $\mathbf{C} \in \mathcal{K}$ *and embeddings* $\zeta_i \colon \mathbf{B}_i \hookrightarrow \mathbf{C}$ *so that* $\zeta_1 \circ \eta_1 = \zeta_2 \circ \eta_2$.

Also, if $\mathbf{M}$ is a countable $\mathcal{L}$-structure, we let $\mathrm{Age}(\mathbf{M})$ denote the class of all finitely generated $\mathcal{L}$-structures embeddable into $\mathbf{M}$.

The fundamental theorem of Fraïssé [**27**] states that, for every Fraïssé class $\mathcal{K}$, there is a unique (up to isomorphism) countable ultrahomogeneous structure $\mathbf{K}$, called the *Fraïssé limit* of $\mathcal{K}$, so that $\mathrm{Age}(\mathbf{K}) = \mathcal{K}$ and, conversely, if $\mathbf{M}$ is a countable ultrahomogeneous structure, then $\mathrm{Age}(\mathbf{M})$ is a Fraïssé class.

Now, if $\mathbf{K}$ is the limit of a Fraïssé class $\mathcal{K}$, then $\mathbf{K}$ is ultrahomogeneous and hence its orbital types correspond to quantifier-free types realised in $\mathbf{K}$. Now, as $\mathrm{Age}(\mathbf{K}) = \mathcal{K}$, for every quantifier-free type p realised by some tuple $\overline{a}$ in $\mathbf{K}$, we see that the structure $\mathbf{A} = \langle \overline{a} \rangle$ generated by $\overline{a}$ belongs to $\mathcal{K}$ and that the expansion $\langle \mathbf{A}, \overline{a} \rangle$ of $\mathbf{A}$ with names for $\overline{a}$ codes p by

$$\phi(\overline{x}) \in p \;\Leftrightarrow\; \langle \mathbf{A}, \overline{a} \rangle \models \phi(\overline{a}).$$

Vice versa, since $\mathbf{A}$ is generated by $\overline{a}$, the quantifier free type $\mathrm{qftp}^{\mathbf{A}}(\overline{a})$ fully determines the expanded structure $\langle \mathbf{A}, \overline{a}\rangle$ up to isomorphism. To conclude, we see that orbital types $\mathcal{O}(\overline{a})$ in $\mathbf{K}$ correspond to isomorphism types of expanded structures $\langle \mathbf{A}, \overline{a}\rangle$, where $\overline{a}$ is a finite tuple generating some $\mathbf{A} \in \mathcal{K}$. This also means that Theorem 6.8 may be reformulated using these isomorphism types in place of orbital types. We leave the details to the reader and instead concentrate on a more restrictive setting.

Suppose now that $\mathbf{K}$ is the limit of a Fraïssé class $\mathcal{K}$ consisting of finite structures, that is, if $\mathbf{K}$ is *locally finite*, meaning that every finitely generated substructure is finite. Then we note that every $\mathbf{A} \in \mathcal{K}$ can simply be enumerated by some finite tuple $\overline{a}$. Moreover, if $\mathbf{A}$ is a finite substructure of $\mathbf{K}$, then the pointwise stabiliser $V_{\mathbf{A}}$ is a finite index subgroup of the *setwise stabiliser*

$$V_{\{\mathbf{A}\}} = \{g \in \mathrm{Aut}(\mathbf{K}) \mid g\mathbf{A} = \mathbf{A}\},$$

so, in particular, $V_{\{\mathbf{A}\}}$ is coarsely bounded in $\mathrm{Aut}(\mathbf{K})$ if and only if $V_{\mathbf{A}}$ is. Similarly, if $\mathbf{B}$ is another finite substructure, then $V_{\mathbf{A}}$ is finitely generated over $V_{\{\mathbf{B}\}}$ if and only if it is finitely generated over $V_{\mathbf{B}}$. Finally, if $(\overline{a}, \overline{c}) \in \mathcal{O}(\overline{a}, \overline{b})$ and $\mathbf{B}$ and $\mathbf{C}$ are the substructures of $\mathbf{K}$ generated by $(\overline{a}, \overline{b})$ and $(\overline{a}, \overline{c})$ respectively, then, for every automorphism $g \in \mathrm{Aut}(\mathbf{K})$ mapping $\mathbf{B}$ to $\mathbf{C}$, there is an $h \in V_{\{\mathbf{B}\}}$ so that $gh(\overline{a}, \overline{b}) = (\overline{a}, \overline{c})$.

Using these observations, one may substitute the orbital types of finite tuples $\overline{a}$ by isomorphism classes of finitely generated substructures of $\mathbf{K}$ to obtain a reformulation of Theorem 6.8.

THEOREM 6.18. *Suppose $\mathcal{K}$ is a Fraïssé class of finite structures with Fraïssé limit $\mathbf{K}$. Then $\mathrm{Aut}(\mathbf{K})$ admits a maximal metric if and only if there is $\mathbf{A} \in \mathcal{K}$ satisfying the following two conditions.*

(1) *For every $\mathbf{B} \in \mathcal{K}$ containing $\mathbf{A}$, there are $n \geqslant 1$ and an isomorphism invariant family $\mathcal{S} \subseteq \mathcal{K}$, containing only finitely many isomorphism types, so that, for all $\mathbf{C} \in \mathcal{K}$ and embeddings $\eta_1, \eta_2 \colon \mathbf{B} \hookrightarrow \mathbf{C}$ with $\eta_1|_{\mathbf{A}} = \eta_2|_{\mathbf{A}}$, one can find some $\mathbf{D} \in \mathcal{K}$ containing $\mathbf{C}$ and a path $\mathbf{B}_0 = \eta_1\mathbf{B}, \mathbf{B}_1, \dots, \mathbf{B}_n = \eta_2\mathbf{B}$ of isomorphic copies of $\mathbf{B}$ inside $\mathbf{D}$ with $\langle \mathbf{B}_i \cup \mathbf{B}_{i+1}\rangle \in \mathcal{S}$ for all i,*
(2) *there is an isomorphism invariant family $\mathcal{R} \subseteq \mathcal{K}$, containing only finitely many isomorphism types, so that, for all $\mathbf{B} \in \mathcal{K}$ containing $\mathbf{A}$ and isomorphic copies $\mathbf{A}' \subseteq \mathbf{B}$ of $\mathbf{A}$, there is some $\mathbf{C} \in \mathcal{K}$ containing $\mathbf{B}$ and a path $\mathbf{A}_0, \mathbf{A}_1, \dots, \mathbf{A}_n \subseteq \mathbf{C}$ consisting of isomorphic copies of $\mathbf{A}$, beginning at $\mathbf{A}_0 = \mathbf{A}$ and ending at $\mathbf{A}_n = \mathbf{A}'$, satisfying $\langle \mathbf{A}_i \cup \mathbf{A}_{i+1}\rangle \in \mathcal{R}$ for all i.*

4. Orbital independence relations

The formulation of Theorem 6.8 is rather abstract and it is therefore useful to have some more familiar criteria for being locally bounded or having a well-defined quasi-isometry type. The first such criterion is simply a reformulation of an observation of P. Cameron.

PROPOSITION 6.19 (P. Cameron). *Let $\mathbf{M}$ be an $\aleph_0$-categorical countable structure. Then, for every tuple $\overline{a}$ in $\mathbf{M}$, there is a finite set $F \subseteq \mathrm{Aut}(\mathbf{M})$ so that $\mathrm{Aut}(\mathbf{M}) = V_{\overline{a}} F V_{\overline{a}}$, i.e., $\mathrm{Aut}(\mathbf{M})$ is Roelcke precompact.*

PROOF. Since $\mathbf{M}$ is $\aleph_0$-categorical, the pointwise stabiliser $V_{\overline{a}}$ induces only finitely many orbits on $\mathbf{M}^n$, where n is the length of $\overline{a}$. So let $B \subseteq \mathbf{M}^n$ be a finite set of $V_{\overline{a}}$-orbit representatives. Also, for every $\overline{b} \in B$, pick if possible some $f \in \mathrm{Aut}(\mathbf{M})$ so that $\overline{b} = f\overline{a}$ and let F be the finite set of these f. Then, if $g \in \mathrm{Aut}(\mathbf{M})$, as $g\overline{a} \in \mathbf{M}^n = V_{\overline{a}}B$, there is some $h \in V_{\overline{a}}$ and $\overline{b} \in B$ so that $g\overline{a} = h\overline{b}$. In particular, there is $f \in F$ so that $\overline{b} = f\overline{a}$, whence $g\overline{a} = h\overline{b} = hf\overline{a}$ and thus $g \in hfV_{\overline{a}} \subseteq V_{\overline{a}}FV_{\overline{a}}$. □

Thus, for an automorphism group $\mathrm{Aut}(\mathbf{M})$ to have a non-trivial quasi-isometry type, the structure $\mathbf{M}$ should not be $\aleph_0$-categorical. In this connection, we recall that if $\mathcal{K}$ is a Fraïssé class in a finite language $\mathcal{L}$ and $\mathcal{K}$ is *uniformly locally finite*, that is, there is a function $f\colon \mathbb{N} \to \mathbb{N}$ so that every $\mathbf{A} \in \mathcal{K}$ generated by n elements has size $\leqslant f(n)$, then the Fraïssé limit $\mathbf{K}$ is $\aleph_0$-categorical. In particular, this applies to Fraïssé classes in finite relational languages.

However, our first concern is to identify locally bounded automorphism groups and for this we consider model theoretical independence relations.

Definition 6.20. *Let $\mathbf{M}$ be a countable structure and $A \subseteq \mathbf{M}$ a finite subset. An* orbital A-independence relation *on $\mathbf{M}$ is a binary relation $\mathop{\smile\!\!\!\!\!\mid}_A$ defined between finite subsets of $\mathbf{M}$ so that, for all finite $B, C, D \subseteq \mathbf{M}$,*

(i) *(symmetry) $B \mathop{\smile\!\!\!\!\!\mid}_A C \Leftrightarrow C \mathop{\smile\!\!\!\!\!\mid}_A B$,*
(ii) *(monotonicity) $B \mathop{\smile\!\!\!\!\!\mid}_A C \;\&\; D \subseteq C \;\Rightarrow\; B \mathop{\smile\!\!\!\!\!\mid}_A D$,*
(iii) *(existence) there is $f \in V_A$ so that $fB \mathop{\smile\!\!\!\!\!\mid}_A C$,*
(iv) *(stationarity) if $B \mathop{\smile\!\!\!\!\!\mid}_A C$ and $g \in V_A$ satisfies $gB \mathop{\smile\!\!\!\!\!\mid}_A C$, then $g \in V_C V_B$, i.e., there is some $f \in V_C$ agreeing pointwise with g on B.*

We read $B \mathop{\smile\!\!\!\!\!\mid}_A C$ as "B is independent from C over A." Occasionally, it is convenient to let $\mathop{\smile\!\!\!\!\!\mid}_A$ be defined between finite tuples rather than sets, which is done by simply viewing a tuple as a name for the set it enumerates. For example, if $\overline{b} = (b_1, \ldots, b_n)$, we let $\overline{b} \mathop{\smile\!\!\!\!\!\mid}_A C$ if and only if $\{b_1, \ldots, b_n\} \mathop{\smile\!\!\!\!\!\mid}_A C$.

With this convention, the stationarity condition on $\mathop{\smile\!\!\!\!\!\mid}_{\overline{a}}$ can be reformulated as follows: If $\overline{b}$ and $\overline{b}'$ have the same orbital type over $\overline{a}$, i.e., $\mathcal{O}(\overline{b}, \overline{a}) = \mathcal{O}(\overline{b}', \overline{a})$, and are both independent from $\overline{c}$ over $\overline{a}$, then they also have the same orbital type over $\overline{c}$.

Similarly, the existence condition on $\mathop{\smile\!\!\!\!\!\mid}_{\overline{a}}$ can be stated as: For all $\overline{b}, \overline{c}$, there is some $\overline{b}'$ independent from $\overline{c}$ over $\overline{a}$ and having the same orbital type over $\overline{a}$ as $\overline{b}$ does.

We should note that, as our interest is in the permutation group $\mathrm{Aut}(\mathbf{M})$ and not the particular structure $\mathbf{M}$, any two structures $\mathbf{M}$ and $\mathbf{M}'$, possibly of different languages, having the same universe and the exact same automorphism group $\mathrm{Aut}(\mathbf{M}) = \mathrm{Aut}(\mathbf{M}')$ will essentially be equivalent for our purposes. We also remark that the existence of an orbital A-independence relation does not depend on the exact structure $\mathbf{M}$, but only on its universe and its automorphism group. Thus, in Examples 6.21, 6.22 and 6.23 below, any manner of formalising the mathematical structures as bona fide first-order model theoretical structures of some language with the indicated automorphism group will lead to the same results and hence can safely be left to the reader.

Example 6.21 (Measured Boolean algebras). Let $\mathbf{M}$ denote the Boolean algebra of clopen subsets of Cantor space $\{0,1\}^{\mathbb{N}}$ equipped with the usual dyadic probability measure μ, i.e., the infinite product of the $\{\frac{1}{2}, \frac{1}{2}\}$-distribution on $\{0,1\}$.

We note that $\mathbf{M}$ is ultrahomogeneous, in the sense that, if $\sigma\colon A \to B$ is a measure preserving isomorphism between two subalgebras of $\mathbf{M}$, then σ extends to a measure preserving automorphism of $\mathbf{M}$.

For two finite subsets A, B, we let $A \mathop{\downfootline}_{\emptyset} B$ if the Boolean algebras they generate are measure theoretically independent, i.e., if, for all $a_1, \ldots, a_n \in A$ and $b_1, \ldots, b_m \in B$, we have

$$\mu(a_1 \cap \ldots \cap a_n \cap b_1 \cap \ldots \cap b_m) = \mu(a_1 \cap \ldots \cap a_n) \cdot \mu(b_1 \cap \ldots \cap b_m).$$

Remark that, if $\sigma\colon A_1 \to A_2$ and $\eta\colon B_1 \to B_2$ are measure preserving isomorphisms between subalgebras of $\mathbf{M}$ with $A_i \mathop{\downfootline}_{\emptyset} B_i$, then there is a measure preserving isomorphism $\xi\colon \langle A_1 \cup B_1 \rangle \to \langle A_2 \cup B_2 \rangle$ between the algebras generated extending both σ and η. Namely, $\xi(a \cap b) = \sigma(a) \cap \eta(b)$ for atoms $a \in A$ and $b \in B$.

Using this and the ultrahomogeneity of $\mathbf{M}$, the stationarity condition (iv) of $\mathop{\downfootline}_{\emptyset}$ is clear. Also, symmetry and monotonicity are obvious. Finally, for the existence condition (iii), suppose that A and B are given finite subsets of $\mathbf{M}$. Then there is some finite n so that all elements of A and B can be written as unions of basic open sets $N_s = \{x \in \{0,1\}^{\mathbb{N}} \mid s \text{ is an initial segment of } x\}$ for $s \in 2^n$. Pick a permutation α of $\mathbb{N}$ so that $\alpha(i) > n$ for all $i \leqslant n$ and note that α induces measure preserving automorphism σ of $\mathbf{M}$ so that $\sigma(A) \mathop{\downfootline}_{\emptyset} B$.

Thus, $\mathop{\downfootline}_{\emptyset}$ is an orbital $\emptyset$-independence relation on $\mathbf{M}$. We also note that, by Stone duality, the automorphism group of $\mathbf{M}$ is isomorphic to the group $\mathrm{Homeo}(\{0,1\}^{\mathbb{N}}, \mu)$ of measure-preserving homeomorphisms of Cantor space.

Example 6.22 (The ended $\aleph_0$-regular tree). Let $\mathbf{T}$ denote the $\aleph_0$-regular tree. I.e., $\mathbf{T}$ is a countable connected undirected graph without loops in which every vertex has infinite valence. Since $\mathbf{T}$ is a tree, there is a natural notion of *convex hull*, namely, for a subset $A \subseteq \mathbf{T}$ and a vertex $x \in \mathbf{T}$, we set $x \in \mathrm{conv}(A)$ if there are $a, b \in A$ so that x lies on the unique path from a to b. Now, pick a distinguished vertex $t \in \mathbf{T}$ and, for finite $A, B \subseteq \mathbf{T}$, set

$$A \mathop{\downfootline}_{\{t\}} B \;\Leftrightarrow\; \mathrm{conv}(A \cup \{t\}) \cap \mathrm{conv}(B \cup \{t\}) = \{t\}.$$

That $\mathop{\downfootline}_{\{t\}}$ is both symmetric and monotone is obvious. Also, if A and B are finite, then so are $\mathrm{conv}(A \cup \{t\})$ and $\mathrm{conv}(B \cup \{t\})$ and so it is easy to find a ellitic isometry g with fixed point t, i.e., a rotation of $\mathbf{T}$ around t, so that $g\big(\mathrm{conv}(A \cup \{t\})\big) \cap \mathrm{conv}(B \cup \{t\}) = \{t\}$. Since $g\big(\mathrm{conv}(A \cup \{t\})\big) = \mathrm{conv}(gA \cup \{t\})$, one sees that $gA \mathop{\downfootline}_{\{t\}} B$, verifying the existence condition (iii).

Finally, for the stationarity condition (iv), suppose $B, C \subseteq \mathbf{T}$ are given and g is an elliptic isometry fixing t so that $B \mathop{\downfootline}_{\{t\}} C$ and $gB \mathop{\downfootline}_{\{t\}} C$. Then, using again that $\mathbf{T}$ is $\aleph_0$-regular, it is easy to find another elliptic isometry fixing all of $\mathrm{conv}(C \cup \{t\})$ that agrees with g on B.

So $\mathop{\downfootline}_{\{t\}}$ is an orbital $\{t\}$-independence relation on $\mathbf{T}$.

Example 6.23 (Unitary groups). Fix a countable field $\mathbb{Q} \subseteq \mathfrak{F} \subseteq \mathbb{C}$ closed under complex conjugation and square roots and let $\mathbf{V}$ denote the countable dimensional $\mathfrak{F}$-vector space with basis $(e_i)_{i=1}^{\infty}$. We define the usual inner product on $\mathbf{V}$ by letting

$$\Big\langle \sum_{i=1}^{n} a_i e_i \;\Big|\; \sum_{j=1}^{m} b_j e_j \Big\rangle = \sum_i a_i \overline{b_i}$$

and let $\mathcal{U}(\mathbf{V})$ denote the corresponding unitary group, i.e., the group of all invertible linear transformations of $\mathbf{V}$ preserving $\langle\cdot \mid \cdot\rangle$.

For finite subsets $A, B \subseteq \mathbf{V}$, we let

$$A \mathop{\downarrow}_{\emptyset} B \;\Leftrightarrow\; \operatorname{span}(A) \perp \operatorname{span}(B).$$

I.e., A and B are independent whenever they span orthogonal subspaces. Symmetry and monotonicity is clear. Moreover, since we chose our field $\mathfrak{F}$ to be closed under complex conjugaction and square roots, the inner product of two vectors lies in $\mathfrak{F}$ and hence so does the norm $\|v\| = \sqrt{\langle v \mid v\rangle}$ of any vector. It follows that the Gram–Schmidt orthonormalisation procedure can be performed within $\mathbf{V}$ and hence every orthonormal set may be extended to an orthonormal basis for $\mathbf{V}$. Using this, one may immitate the details of Example 6.21 to show that $\downarrow_\emptyset$ satisfies conditions (iii) and (iv). (See also Section 6 of [**62**] for additional details.)

Theorem 6.24. *Suppose $\mathbf{M}$ is a countable structure, $A \subseteq \mathbf{M}$ a finite subset and $\downarrow_A$ an orbital A-independence relation. Then the pointwise stabiliser subgroup V_A is coarsely bounded in itself. Thus, if $A = \emptyset$, the automorphism group* $\operatorname{Aut}(\mathbf{M})$ *is coarsely bounded and, if $A \neq \emptyset$,* $\operatorname{Aut}(\mathbf{M})$ *is locally bounded.*

Proof. Suppose U is an open neighbourhood of 1 in V_A. We will find a finite subset $F \subseteq V_A$ so that $V_A = UFUFU$. By passing to a further subset, we may suppose that U is of the form V_B, where $B \subseteq \mathbf{M}$ is a finite set containing A. We begin by choosing, using property (iii) of the orbital A-independence relation, some $f \in V_A$ so that $fB \downarrow_A B$ and set $F = \{f, f^{-1}\}$.

Now, suppose that $g \in V_A$ is given and choose again by (iii) some $h \in V_A$ so that $hB \downarrow_A (B \cup gB)$. By (ii), it follows that $hB \downarrow_A B$ and $hB \downarrow_A gB$, whereby, using (i), we have $B \downarrow_A hB$ and $gB \downarrow_A hB$. Since $g \in V_A$, we can apply (iv) to $C = hB$, whence $g \in V_{hB}V_B = hV_Bh^{-1}V_B$.

However, as $fB \downarrow_A B$ and $hB \downarrow_A B$, i.e., $(hf^{-1} \cdot fB) \downarrow_A B$, and also $hf^{-1} \in V_A$, by (iv) it follows that $hf^{-1} \in V_BV_{fB} = V_BfV_Bf^{-1}$. So, finally, $h \in V_BfV_B$ and

$$g \in hV_Bh^{-1}V_B \subseteq V_BfV_B \cdot V_B \cdot (V_BfV_B)^{-1} \cdot V_B \subseteq V_BFV_BFV_B$$

as required. □

By the preceding examples, we see that both the automorphism group of the measured Boolean algebra and the unitary group $\mathcal{U}(\mathbf{V})$ are coarsely bounded, while the automorphism group $\operatorname{Aut}(\mathbf{T})$ is locally bounded (cf. Theorem 6.20 [**40**], Theorem 6.11 [**62**], respectively Theorem 6.31 [**40**]).

We note that, if $\mathbf{M}$ is an ω-homogeneous structure, $\bar{a}$, $\bar{b}$ are tuples in $\mathbf{M}$ and $A \subseteq \mathbf{M}$ is a finite subset, then, by definition, $\operatorname{tp}^{\mathbf{M}}(\bar{a}/A) = \operatorname{tp}^{\mathbf{M}}(\bar{b}/A)$ if and only if $\bar{b} \in V_A \cdot \bar{a}$. In this case, we can reformulate conditions (iii) and (iv) of the definition of orbital A-independence relations as follows.

- (iii) For all $\bar{a}$ and B, there is $\bar{b}$ with $\operatorname{tp}^{\mathbf{M}}(\bar{b}/A) = \operatorname{tp}^{\mathbf{M}}(\bar{a}/A)$ and $\bar{b} \downarrow_A B$.
- (iv) For all $\bar{a}, \bar{b}$ and B, if $\bar{a} \downarrow_A B$, $\bar{b} \downarrow_A B$ and $\operatorname{tp}^{\mathbf{M}}(\bar{a}/A) = \operatorname{tp}^{\mathbf{M}}(\bar{b}/A)$, then $\operatorname{tp}^{\mathbf{M}}(\bar{a}/B) = \operatorname{tp}^{\mathbf{M}}(\bar{b}/B)$.

Also, for the next result, we remark that, if T is a complete theory with infinite models in a countable language $\mathcal{L}$, then T has a countable saturated model if and only if $S_n(T)$ is countable for all n. In particular, this holds if T is ω-stable.

THEOREM 6.25. *Suppose that* $\mathbf{M}$ *is a saturated countable model of an* ω*-stable theory. Then* $\mathrm{Aut}(\mathbf{M})$ *is coarsely bounded.*

PROOF. We note first that, since $\mathbf{M}$ is saturated and countable, it is ω-homogeneous. Now, since $\mathbf{M}$ is the model of an ω-stable theory, there is a corresponding notion of *forking independence* $\overline{a} \mathop{\smile\!\!\!\!\!\mid}_A B$ defined by

$$\begin{aligned}\overline{a} \mathop{\smile\!\!\!\!\!\mid}_A B &\Leftrightarrow \mathrm{tp}^{\mathbf{M}}(\overline{a}/A\cup B) \text{ is a non-forking extension of } \mathrm{tp}^{\mathbf{M}}(\overline{a}/A)\\ &\Leftrightarrow \mathrm{RM}(\overline{a}/A\cup B)=\mathrm{RM}(\overline{a}/A),\end{aligned}$$

where *RM* denotes the *Morley rank*. In this case, forking independence $\mathop{\smile\!\!\!\!\!\mid}_A$ always satisfies symmetry and monotonicity, i.e., conditions (i) and (ii), for all finite $A \subseteq \mathbf{M}$.

Moreover, by the existence of non-forking extensions, every type $\mathrm{tp}^{\mathbf{M}}(\overline{a}/A)$ has a non-forking extension $q \in S_n(A\cup B)$. Also, as $\mathbf{M}$ is saturated, this extension q is realised by some tuple $\overline{b}$ in $\mathbf{M}$, i.e., $\mathrm{tp}^{\mathbf{M}}(\overline{b}/A\cup B)=q$. Thus, $\mathrm{tp}^{\mathbf{M}}(\overline{b}/A\cup B)$ is a non-forking extension of $\mathrm{tp}^{\mathbf{M}}(\overline{a}/A)=\mathrm{tp}^{\mathbf{M}}(\overline{b}/A)$, which implies that $\overline{b} \mathop{\smile\!\!\!\!\!\mid}_A B$. In other words, for all for all $\overline{a}$ and A, B, there is $\overline{b}$ with $\mathrm{tp}^{\mathbf{M}}(\overline{b}/A)=\mathrm{tp}^{\mathbf{M}}(\overline{a}/A)$ and $\overline{b} \mathop{\smile\!\!\!\!\!\mid}_A B$, which verifies the existence condition (iii) for $\mathop{\smile\!\!\!\!\!\mid}_A$.

However, forking independence over A, $\mathop{\smile\!\!\!\!\!\mid}_A$, may not satisfy the stationarity condition (iv) unless every type $S_n(A)$ is stationary, i.e., unless, for all $B \supseteq A$, every type $p \in S_n(A)$ has a unique non-forking extension in $S_n(B)$. Nevertheless, as we shall show, we can get by with sligthly less.

We let $\mathop{\smile\!\!\!\!\!\mid}_\emptyset$ denote forking independence over the empty set. Suppose also that $B \subseteq \mathbf{M}$ is a fixed finite subset and let $\overline{a} \in \mathbf{M}^n$ be an enumeration of B. Then there are at most $\deg_M\big(\mathrm{tp}^{\mathbf{M}}(\overline{a})\big)$ non-forking extensions of $\mathrm{tp}(\overline{a})$ in $S_n(B)$, where $\deg_M\big(\mathrm{tp}^{\mathbf{M}}(\overline{a})\big)$ denotes the *Morley degree* of $\mathrm{tp}^{\mathbf{M}}(\overline{a})$. Choose realisations $\overline{b}_1,\ldots,\overline{b}_k \in \mathbf{M}^n$ for each of these non-forking extensions realised in $\mathbf{M}$. Since $\mathrm{tp}^{\mathbf{M}}(\overline{b}_i)=\mathrm{tp}^{\mathbf{M}}(\overline{a})$, there are $f_1,\ldots,f_k \in \mathrm{Aut}(\mathbf{M})$ so that $\overline{b}_i = f_i\overline{a}$. Let F be the set of these f_i and their inverses. Thus, if $\overline{c} \in \mathbf{M}^n$ satisfies $\mathrm{tp}^{\mathbf{M}}(\overline{c})=\mathrm{tp}^{\mathbf{M}}(\overline{a})$ and $\overline{c} \mathop{\smile\!\!\!\!\!\mid}_\emptyset B$, then there is some i so that $\mathrm{tp}^{\mathbf{M}}(\overline{c}/B)=\mathrm{tp}^{\mathbf{M}}(\overline{b}_i/B)$ and so, for some $h \in V_B$, we have $\overline{c}=h\overline{b}_i = hf_i\overline{a} \in V_BF\cdot\overline{a}$.

Now assume $g \in \mathrm{Aut}(\mathbf{M})$ is given and pick, by condition (iii), some $h \in \mathrm{Aut}(\mathbf{M})$ so that $hB \mathop{\smile\!\!\!\!\!\mid}_\emptyset (B\cup gB)$. By monotonicity and symmetry, it follows that $B \mathop{\smile\!\!\!\!\!\mid}_\emptyset hB$ and $gB \mathop{\smile\!\!\!\!\!\mid}_\emptyset hB$. Also, since Morley rank and hence forking independence are invariant under automorphisms of $\mathbf{M}$, we see that $h^{-1}B \mathop{\smile\!\!\!\!\!\mid}_\emptyset B$ and $h^{-1}gB \mathop{\smile\!\!\!\!\!\mid}_\emptyset B$. So, as $\overline{a}$ enumerates B, we have $h^{-1}\overline{a} \mathop{\smile\!\!\!\!\!\mid}_\emptyset B$ and $h^{-1}g\overline{a} \mathop{\smile\!\!\!\!\!\mid}_\emptyset B$, where clearly $\mathrm{tp}^{\mathbf{M}}(h^{-1}\overline{a})=\mathrm{tp}^{\mathbf{M}}(\overline{a})$ and $\mathrm{tp}^{\mathbf{M}}(h^{-1}g\overline{a})=\mathrm{tp}^{\mathbf{M}}(\overline{a})$. By our observation above, we deduce that $h^{-1}\overline{a} \in V_BF\cdot\overline{a}$ and $h^{-1}g\overline{a} \in V_BF\cdot\overline{a}$, whence $h^{-1} \in V_BFV_{\overline{a}} = V_BFV_B$ and similarly $h^{-1}g \in V_BFV_B$. Therefore, we finally have that

$$g \in (V_BFV_B)^{-1}V_BFV_B = V_BFV_BFV_B.$$

We have thus shown that, for all finite $B \subseteq \mathbf{M}$, there is a finite subset $F \subseteq \mathrm{Aut}(\mathbf{M})$ so that $\mathrm{Aut}(\mathbf{M}) = V_BFV_BFV_B$, verifying that $\mathrm{Aut}(\mathbf{M})$ is coarsely bounded. □

EXAMPLE 6.26. Let us note that Theorem 6.25 fails without the assumption of $\mathbf{M}$ being saturated. To see this, let T be the complete theory of ∞-regular forests, i.e., of undirected graphs without loops in which every vertex has infinite valence, in the language of a single binary edge relation. In all countable models of T, every connected component is then a copy of the $\aleph_0$-regular tree $\mathbf{T}$ and hence the number of connected components is a complete isomorphism invariant for the countable models of T. Moreover, the countable theory T is ω-stable (in fact, $\mathbf{T}$ is also ω-homogeneous). Nevertheless, the automorphism group of the unsaturated structure $\mathbf{T}$ fails to be coarsely bounded, as witnessed by its tautological isometric action on $\mathbf{T}$.

In view of Theorem 6.25 it is natural to wonder if the automorphism group of a countable atomic model of an ω-stable theory is at least locally bounded and indeed this was mentioned as an open problem in an earlier preprint of this chapter. However, J. Zielinski [**82**] was able to construct an example of such a structure with automorphism group isomorphic to the direct product $\mathbb{Q}^{\mathbb{Q}}$, which fails to be locally bounded. It would be useful to isolate other purely model theoretical properties of a countable structure that would ensure local boundedness of its automorphism group.

Example 6.27. Suppose $\mathfrak{F}$ is a countable field and let $\mathcal{L} = \{+,-,0\} \cup \{\lambda_t \mid t \in \mathfrak{F}\}$ be the language of $\mathfrak{F}$-vector spaces, i.e., $+$ and $-$ are respectively binary and unary function symbols and 0 a constant symbol representing the underlying Abelian group and λ_t are unary function symbols representing multiplication by the scalar t. Let also T be the theory of infinite $\mathfrak{F}$-vector spaces.

Since $\mathfrak{F}$-vector spaces of size $\aleph_1$ have dimension $\aleph_1$, we see that T is $\aleph_1$-categorical and thus complete and ω-stable. Moreover, provided $\mathfrak{F}$ is infinite, T fails to be $\aleph_0$-categorical, since, for exampe, the 1 and 2-dimensional $\mathfrak{F}$-vector spaces are non-isomorphic. However, since T is ω-stable, it has a countable saturated model, which is easily seen to be the $\aleph_0$-dimensional $\mathfrak{F}$-vector space denoted $\mathbf{V}$. It thus follows from Theorem 6.25 that the general linear group $\mathrm{GL}(\mathbf{V}) = \mathrm{Aut}(\mathbf{V})$ is coarsely bounded.

Oftentimes, Fraïssé classes admit a canonical form of amalgamation that can be used to define a corresponding notion of independence. One rendering of this is given by K. Tent and M. Ziegler (Example 2.2 [**71**]). However, their notion is too weak to ensure that the corresponding independence notion is an orbital independence relation. For this, one needs a stronger form of functoriality, which nevertheless is satisfied in most cases.

Definition 6.28. *Suppose $\mathcal{K}$ is a Fraïssé class and that $\mathbf{A} \in \mathcal{K}$. We say that $\mathcal{K}$ admits a* functorial amalgamation *over $\mathbf{A}$ if there is map θ that to all pairs of embeddings $\eta_1 \colon \mathbf{A} \hookrightarrow \mathbf{B}_1$ and $\eta_2 \colon \mathbf{A} \hookrightarrow \mathbf{B}_2$, with $\mathbf{B}_1, \mathbf{B}_2 \in \mathcal{K}$, associates a pair of embeddings $\zeta_1 \colon \mathbf{B}_1 \hookrightarrow \mathbf{C}$ and $\zeta_2 \colon \mathbf{B}_2 \hookrightarrow \mathbf{C}$ into another structure $\mathbf{C} \in \mathcal{K}$ so that $\zeta_1 \circ \eta_1 = \zeta_2 \circ \eta_2$ and, moreover, satisfying the following conditions.*

(1) *(symmetry) The pair $\Theta\big(\eta_2 \colon \mathbf{A} \hookrightarrow \mathbf{B}_2, \eta_1 \colon \mathbf{A} \hookrightarrow \mathbf{B}_1\big)$ is the reverse of the pair $\Theta\big(\eta_1 \colon \mathbf{A} \hookrightarrow \mathbf{B}_1, \eta_2 \colon \mathbf{A} \hookrightarrow \mathbf{B}_2\big)$.*

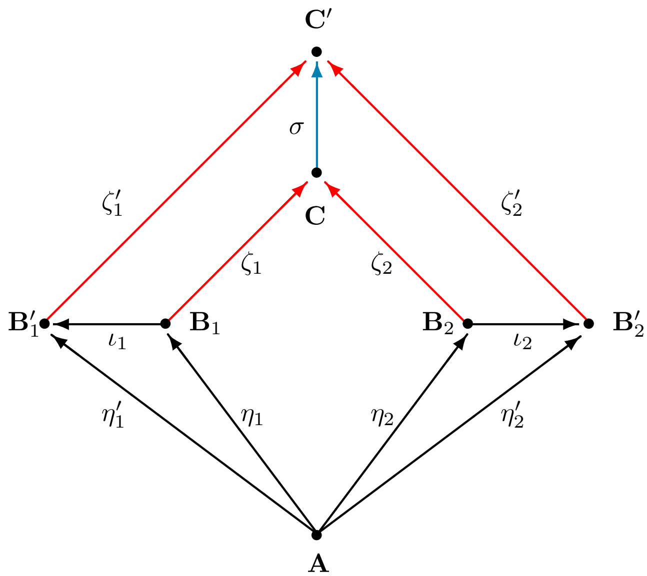
(2) *(functoriality) If $\eta_1 \colon \mathbf{A} \hookrightarrow \mathbf{B}_1$, $\eta_2 \colon \mathbf{A} \hookrightarrow \mathbf{B}_2$, $\eta_1' \colon \mathbf{A} \hookrightarrow \mathbf{B}_1'$ and $\eta_2' \colon \mathbf{A} \hookrightarrow \mathbf{B}_2'$ are embeddings with $\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_1', \mathbf{B}_2' \in \mathcal{K}$ and $\iota_1 \colon \mathbf{B}_1 \hookrightarrow \mathbf{B}_1'$ and $\iota_2 \colon \mathbf{B}_2 \hookrightarrow \mathbf{B}_2'$ are embeddings with $\iota_i \circ \eta_i = \eta_i'$, then, for*

$$\Theta\big(\eta_1 \colon \mathbf{A} \hookrightarrow \mathbf{B}_1, \eta_2 \colon \mathbf{A} \hookrightarrow \mathbf{B}_2\big) = \big(\zeta_1 \colon \mathbf{B}_1 \hookrightarrow \mathbf{C}, \zeta_2 \colon \mathbf{B}_2 \hookrightarrow \mathbf{C}\big)$$

and

$$\Theta\big(\eta_1 \colon \mathbf{A} \hookrightarrow \mathbf{B}_1, \eta_2 \colon \mathbf{A} \hookrightarrow \mathbf{B}_2\big) = \big(\zeta_1' \colon \mathbf{B}_1' \hookrightarrow \mathbf{C}', \zeta_2' \colon \mathbf{B}_2' \hookrightarrow \mathbf{C}'\big),$$

there is an embedding $\sigma \colon \mathbf{C} \hookrightarrow \mathbf{C}'$ so that $\sigma \circ \zeta_i = \zeta_i' \circ \iota_i$ for $i = 1, 2$.



We note that $\Theta\big(\eta_1\colon \mathbf{A}\hookrightarrow \mathbf{B}_1, \eta_2\colon \mathbf{A}\hookrightarrow \mathbf{B}_2\big)=\big(\zeta_1\colon \mathbf{B}_1\hookrightarrow \mathbf{C}, \zeta_2\colon \mathbf{B}_2\hookrightarrow \mathbf{C}\big)$ is simply the precise manner of decribing the amalgamation $\mathbf{C}$ of the two structures $\mathbf{B}_1$ and $\mathbf{B}_2$ over their common substructure $\mathbf{A}$ (with the additional diagram of embeddings). Thus, symmetry says that the amalgamation should not depend on the order of the structures $\mathbf{B}_1$ and $\mathbf{B}_2$, while functoriality states that the amalgamation should commute with embeddings of the $\mathbf{B}_i$ into larger structures $\mathbf{B}_i'$. With this concept at hand, for finite subsets A, B_1, B_2 of the Fraïssé limit $\mathbf{K}$, we may define B_1 and B_2 to be independent over A if $\mathbf{B}_1=\langle A\cup B_1\rangle$ and $\mathbf{B}_2=\langle A\cup B_2\rangle$ are amalgamated over $\mathbf{A}=\langle A\rangle$ in $\mathbf{K}$ as given by $\Theta(\mathrm{id}_\mathbf{A}\colon \mathbf{A}\hookrightarrow \mathbf{B}_1, \mathrm{id}_\mathbf{A}\colon \mathbf{A}\hookrightarrow \mathbf{B}_2)$. More precisely, we have the following definition.

Definition 6.29. *Suppose $\mathcal{K}$ is a Fraïssé class with limit $\mathbf{K}$, $A\subseteq \mathbf{K}$ is a finite subset and Θ is a functorial amalgamation on $\mathcal{K}$ over $\mathbf{A}=\langle A\rangle$. For finite subsets $B_1, B_2\subseteq \mathbf{K}$ with $\mathbf{B}_i=\langle A\cup B_i\rangle$, $\mathbf{D}=\langle A\cup B_1\cup B_2\rangle$ and*

$$\Theta\big(\mathrm{id}_\mathbf{A}\colon \mathbf{A}\hookrightarrow \mathbf{B}_1, \mathrm{id}_\mathbf{A}\colon \mathbf{A}\hookrightarrow \mathbf{B}_2\big)=\big(\zeta_1\colon \mathbf{B}_1\hookrightarrow \mathbf{C}, \zeta_2\colon \mathbf{B}_2\hookrightarrow \mathbf{C}\big),$$

we set

$$B_1 \mathop{\smile\hspace{-0.9em}\downarrow}_A B_2$$

if and only if there is an embedding $\pi\colon \mathbf{D}\hookrightarrow \mathbf{C}$ so that $\zeta_i=\pi\circ \mathrm{id}_{\mathbf{B}_i}$ for $i=1,2$.

With this setup, we readily obtain the following result.

Theorem 6.30. *Suppose $\mathcal{K}$ is a Fraïssé class with limit $\mathbf{K}$, $A\subseteq \mathbf{K}$ is a finite subset and Θ is a functorial amalgamation of $\mathcal{K}$ over $\mathbf{A}=\langle A\rangle$. Let also $\mathop{\smile\hspace{-0.9em}\downarrow}_A$ be the relation defined from Θ and A as in Definition 6.29. Then $\mathop{\smile\hspace{-0.9em}\downarrow}_A$ is an orbital A-independence relation on $\mathbf{K}$ and thus V_A is coarsely bounded. In particular,* $\mathrm{Aut}(\mathbf{K})$ *is locally bounded and hence admits a coarsely proper metric.*

Proof. Symmetry and monotonicity of $\mathop{\smile\hspace{-0.9em}\downarrow}_A$ follow easily from symmetry, respectively functoriality, of Θ. Also, the existence condition on $\mathop{\smile\hspace{-0.9em}\downarrow}_A$ follows from the ultrahomogeneity of $\mathbf{K}$ and the realisation of the amalgam Θ inside of $\mathbf{K}$.

For stationarity, we use the ultrahomogeneity of $\mathbf{K}$. So, suppose that finite $\overline{a}, \overline{b}$ and $B \subseteq \mathbf{K}$ are given so that $\overline{a} \mathop{\smile\!\!\!\!\!\mid}_A B$, $\overline{b} \mathop{\smile\!\!\!\!\!\mid}_A B$ and $\mathrm{tp}^{\mathbf{K}}(\overline{a}/A) = \mathrm{tp}^{\mathbf{K}}(\overline{b}/A)$. We set $\mathbf{B}_1 = \langle \overline{a} \cup A \rangle$, $\mathbf{B}_1' = \langle \overline{b} \cup A \rangle$, $\mathbf{B}_2 = \langle B \cup A \rangle$, $\mathbf{D} = \langle \overline{a} \cup B \cup A \rangle$ and $\mathbf{D}' = \langle \overline{b} \cup B \cup A \rangle$. Let also

$$\Theta\big(\mathrm{id}_{\mathbf{A}} \colon \mathbf{A} \hookrightarrow \mathbf{B}_1, \mathrm{id}_{\mathbf{A}} \colon \mathbf{A} \hookrightarrow \mathbf{B}_2\big) = \big(\zeta_1 \colon \mathbf{B}_1 \hookrightarrow \mathbf{C}, \zeta_2 \colon \mathbf{B}_2 \hookrightarrow \mathbf{C}\big),$$

$$\Theta\big(\mathrm{id}_{\mathbf{A}} \colon \mathbf{A} \hookrightarrow \mathbf{B}_1', \mathrm{id}_{\mathbf{A}} \colon \mathbf{A} \hookrightarrow \mathbf{B}_2\big) = \big(\zeta_1' \colon \mathbf{B}_1' \hookrightarrow \mathbf{C}', \zeta_2' \colon \mathbf{B}_2 \hookrightarrow \mathbf{C}'\big)$$

and note that there is an isomorphism $\iota \colon \mathbf{B}_1 \hookrightarrow \mathbf{B}_1'$ pointwise fixing A so that $\iota(\overline{a}) = \overline{b}$. By the definition of the independence relation, there are embeddings $\pi \colon \mathbf{D} \hookrightarrow \mathbf{C}$ and $\pi' \colon \mathbf{D}' \hookrightarrow \mathbf{C}'$ so that $\zeta_i = \pi \circ \mathrm{id}_{\mathbf{B}_i}$, $\zeta_1' = \pi' \circ \mathrm{id}_{\mathbf{B}_1'}$ and $\zeta_2' = \pi' \circ \mathrm{id}_{\mathbf{B}_2}$. On the other hand, by the functoriality of Θ, there is an embedding $\sigma \colon \mathbf{C} \hookrightarrow \mathbf{C}'$ so that $\sigma \circ \zeta_1 = \zeta_1' \circ \iota$ and $\sigma \circ \zeta_2 = \zeta_2' \circ \mathrm{id}_{\mathbf{B}_2}$. Thus, $\sigma \circ \pi \circ \mathrm{id}_{\mathbf{B}_1} = \pi' \circ \mathrm{id}_{\mathbf{B}_1'} \circ \iota$ and $\sigma \circ \pi \circ \mathrm{id}_{\mathbf{B}_2} = \pi' \circ \mathrm{id}_{\mathbf{B}_2} \circ \mathrm{id}_{\mathbf{B}_2}$, i.e., $\sigma \circ \pi|_{\mathbf{B}_1} = \pi' \circ \iota$ and $\sigma \circ \pi|_{\mathbf{B}_2} = \pi'|_{\mathbf{B}_2}$. Let now $\rho \colon \pi'[\mathbf{D}'] \hookrightarrow \mathbf{D}'$ be the isomorphism that is inverse to π'. Then $\rho\sigma\pi|_{\mathbf{B}_1} = \iota$ and $\rho\sigma\pi|_{\mathbf{B}_2} = \mathrm{id}_{\mathbf{B}_2}$. So, by ultrahomogeneity of $\mathbf{K}$, there is an automorphism $g \in \mathrm{Aut}(\mathbf{K})$ extending $\rho\sigma\pi$, whence, in particular, $g \in V_{\mathbf{B}_2} \subseteq V_B$, while $g(\overline{a}) = \overline{b}$. It follows that $\mathrm{tp}^{\mathbf{K}}(\overline{a}/B) = \mathrm{tp}^{\mathbf{K}}(\overline{b}/B)$, verifying stationarity. □

EXAMPLE 6.31 (Urysohn metric spaces, cf. Example 2.2 (c) [**71**]). Suppose $\mathcal{S}$ is a countable additive subsemigroup of the positive reals. Then the class of finite metric spaces with distances in $\mathcal{S}$ forms a Fraïssé class $\mathcal{K}$ with functorial amalgamation over the one-point metric space $\mathbf{P} = \{p\}$. Indeed, if $\mathbf{A}$ and $\mathbf{B}$ belong to $\mathcal{S}$ and intersect exactly in the point p, we can define a metric d on $\mathbf{A} \cup \mathbf{B}$ extending those of $\mathbf{A}$ and $\mathbf{B}$ by letting

$$d(a, b) = d_{\mathbf{A}}(a, p) + d_{\mathbf{B}}(p, b),$$

for $a \in \mathbf{A}$ and $b \in \mathbf{B}$. We thus take this to define the amalgamation of $\mathbf{A}$ and $\mathbf{B}$ over $\mathbf{P}$ and one easily verifies that this provides a functorial amalgamation over $\mathbf{P}$ on the class $\mathcal{K}$.

Two important particular cases are when $\mathcal{S} = \mathbb{Z}_+$, respectively $\mathcal{S} = \mathbb{Q}_+$, in which case the Fraïssé limits are the integer and rational Urysohn metric spaces $\mathbb{Z}\mathbb{U}$ and $\mathbb{Q}\mathbb{U}$. By Theorem 6.30, we see that their isometry groups $\mathrm{Isom}(\mathbb{Z}\mathbb{U})$ and $\mathrm{Isom}(\mathbb{Q}\mathbb{U})$ admit coarsely proper metrics.

We note also that it is vital that $\mathbf{P}$ is non-empty. Indeed, since $\mathrm{Isom}(\mathbb{Z}\mathbb{U})$ and $\mathrm{Isom}(\mathbb{Q}\mathbb{U})$ act transitively on metric spaces of infinite diameter, namely, on $\mathbb{Z}\mathbb{U}$ and $\mathbb{Q}\mathbb{U}$, they do not have property (OB) and hence the corresponding Fraïssé classes do not admit a functorial amalgamation over the empty space $\emptyset$.

Instead, if, for a given $\mathcal{S}$ and $r \in \mathcal{S}$, we let $\mathcal{K}$ denote the finite metric spaces with distances in $\mathcal{S} \cap [0, r]$, then $\mathcal{K}$ is still a Fraïssé class now admitting functorial amalgamation over the empty space. Namely, to join $\mathbf{A}$ and $\mathbf{B}$, one simply takes the disjoint union and stipulates that $d(a, b) = r$ for all $a \in \mathbf{A}$ and $b \in \mathbf{B}$.

As a particular example, we note that the isometry group $\mathrm{Isom}(\mathbb{Q}\mathbb{U}_1)$ of the rational Urysohn metric space of diameter 1 is coarsely bounded (see Theorem 5.8 [**62**]).

EXAMPLE 6.32 (The ended $\aleph_0$-regular tree). Let again $\mathbf{T}$ denote the $\aleph_0$-regular tree and fix an *end* $\mathfrak{e}$ of $\mathbf{T}$. That is, $\mathfrak{e}$ is an equivalence class of infinite paths $(v_0, v_1, v_2, \ldots)$ in $\mathbf{T}$ under the equivalence relation

$$(v_0, v_1, v_2, \ldots) \sim (w_0, w_1, w_2, \ldots) \;\Leftrightarrow\; \exists k, l\; \forall n\; v_{k+n} = w_{l+n}.$$

So, for every vertex $t \in \mathbf{T}$, there is a unique path $(v_0, v_1, v_2, \ldots) \in \mathfrak{e}$ beginning at $v_0 = t$. Thus, if r is another vertex in $\mathbf{T}$, we can set $t <_{\mathfrak{e}} r$ if and only if $r = v_n$ for some $n \geqslant 1$. Note that this defines a strict partial ordering $<_{\mathfrak{e}}$ on $\mathbf{T}$ so that every two vertices $t, s \in \mathbf{T}$ have a least upper bound and, moreover, this least upper bound lies on the geodesic from t to s. Furthermore, we define the function $\vartheta\colon \mathbf{T} \times \mathbf{T} \to \mathbf{T}$ by letting $\vartheta(t, s) = x_1$, where $(x_0, x_1, x_2, \ldots, x_k)$ is the geodesic from $x_0 = t$ to $x_k = s$, for $t \neq s$, and $\vartheta(t, t) = t$.

As is easy to see, the expanded structure $\big(\mathbf{T}, <_{\mathfrak{e}}, \vartheta\big)$ is ultrahomogeneous and locally finite and hence, by Fraïssé's Theorem, is the Fraïssé limit of its age $\mathcal{K} = \mathrm{Age}(\mathbf{T}, <_{\mathfrak{e}}, \vartheta)$. We also claim that $\mathcal{K}$ admits a functorial amalgamation over the structure on a single vertex t. Indeed, if $\mathbf{A}$ and $\mathbf{B}$ are finite subtructures of $\big(\mathbf{T}, <_{\mathfrak{e}}, \vartheta\big)$ and we pick a vertex in $t_{\mathbf{A}}$ and $t_{\mathbf{B}}$ in each, then there is a freest amalgamation of $\mathbf{A}$ and $\mathbf{B}$ identifying $t_{\mathbf{A}}$ and $t_{\mathbf{B}}$. Namely, let $t_{\mathbf{A}} = a_0 <_{\mathfrak{e}} a_1 <_{\mathfrak{e}} a_2 <_{\mathfrak{e}} \ldots <_{\mathfrak{e}} a_n$ and $t_{\mathbf{B}} = b_0 <_{\mathfrak{e}} b_1 <_{\mathfrak{e}} b_2 <_{\mathfrak{e}} \ldots <_{\mathfrak{e}} b_m$ be an enumeration of the successors of $t_{\mathbf{A}}$ and $t_{\mathbf{B}}$ in $\mathbf{A}$ and $\mathbf{B}$ respectively. We then take the disjoint union of $\mathbf{A}$ and $\mathbf{B}$ modulo the identifications $a_0 = b_0, \ \ldots \ , a_{\min(n,m)} = b_{\min(n,m)}$ and add only the edges from $\mathbf{A}$ and $\mathbf{B}$. There are then unique extensions of $<_{\mathfrak{e}}$ and ϑ to the amalgam making it a member of $\mathcal{K}$. Moreover, this amalgamation is functorial over the single vertex t.

It thus follows that $\mathrm{Aut}(\mathbf{T}, <_{\mathfrak{e}}, \vartheta)$ is locally bounded as witnessed by the pointwise stabiliser V_t of any fixed vertex $t \in \mathbf{T}$. Now, as ϑ commutes with automorphisms of $\mathbf{T}$ and $<_{\mathfrak{e}}$ and $\mathfrak{e}$ are interdefinable, we see that $\mathrm{Aut}(\mathbf{T}, <_{\mathfrak{e}}, \vartheta)$ is simply the group $\mathrm{Aut}(\mathbf{T}, \mathfrak{e})$ of all automorphisms of $\mathbf{T}$ fixing the end $\mathfrak{e}$.

5. Computing quasi-isometry types of automorphism groups

Thus far we have been able to show local boundedness of certain automorphism groups. The goal is now to identity their quasi-isometry type insofar as this is well-defined.

Example 6.33 (The $\aleph_0$-regular tree). Let $\mathbf{T}$ be the $\aleph_0$-regular tree and fix a vertex $t \in \mathbf{T}$. By Example 6.22 and Theorem 6.24, we know that V_t is coarsely bounded. Fix also a neighbour s of t in $\mathbf{T}$ and let $\mathcal{R} = \{\mathcal{O}(t, s)\}$. Now, since $\mathrm{Aut}(\mathbf{T})$ acts transitively on the set of oriented edges of $\mathbf{T}$, we see that, if $r \in \mathcal{O}(t) = \mathbf{T}$ is any vertex and $(v_0, v_1, \ldots, v_m)$ is the geodesic from $v_0 = t$ to $v_m = r$, then $\mathcal{O}(v_i, v_{i+1}) \in \mathcal{R}$ for all i. It thus follows from Theorem 6.8 that $\mathrm{Aut}(\mathbf{T})$ admits a maximal metric and, moreover, that

$$g \in \mathrm{Aut}(\mathbf{T}) \mapsto g(t) \in \mathbb{X}_{t,\mathcal{R}}$$

is a quasi-isometry. However, the graph $\mathbb{X}_{t,\mathcal{R}}$ is simply the tree $\mathbf{T}$ itself, which shows that

$$g \in \mathrm{Aut}(\mathbf{T}) \mapsto g(t) \in \mathbf{T}$$

is a quasi-isometry. In other words, the quasi-isometry type of $\mathrm{Aut}(\mathbf{T})$ is just the tree $\mathbf{T}$.

Example 6.34 (The ended $\aleph_0$-regular tree). Let $\big(\mathbf{T}, <_{\mathfrak{e}}, \vartheta\big)$ be as in Example 6.32. Again, if t is some fixed vertex, the vertex stabiliser V_t is coarsely bounded. Now, $\mathrm{Aut}(\mathbf{T}, <_{\mathfrak{e}}, \vartheta)$ acts transitively on the vertices and edges of $\mathbf{T}$, but no longer acts transitively on set of oriented edges. Namely, if (x, y) and (v, w) are edges of $\mathbf{T}$, then $(v, w) \in \mathcal{O}(x, y)$ if and only if $x <_{\mathfrak{e}} y \leftrightarrow v <_{\mathfrak{e}} w$. Therefore, let s be any neighbour of t in $\mathbf{T}$ and set $\mathcal{R} = \{\mathcal{O}(t,s), \mathcal{O}(s,t)\}$. Then, as in Example 6.33, we see that $\mathbb{X}_{t,\mathcal{R}} = \mathbf{T}$ and that

$$g \in \mathrm{Aut}(\mathbf{T}, \mathfrak{e}) \mapsto g(t) \in \mathbf{T}$$

is a quasi-isometry. So $\mathrm{Aut}(\mathbf{T}, \mathfrak{e})$ is quasi-isometric to $\mathbf{T}$ and thus also to $\mathrm{Aut}(\mathbf{T})$.

EXAMPLE 6.35 (Urysohn metric spaces). Let $\mathcal{S}$ be a countable additive subsemigroup of the positive reals and let $\mathcal{S}\mathbb{U}$ be the limit of the Fraïssé class $\mathcal{K}$ of finite metric spaces with distances in $\mathcal{S}$. As we have seen in Example 6.31, $\mathcal{K}$ admits a functorial amalgamation over the one point metric space $\mathbf{P} = \{p\}$ and thus the stabiliser V_{x_0} of any chosen point $x_0 \in \mathcal{S}\mathbb{U}$ is coarsely bounded.

We remark that $\mathcal{O}(x_0) = \mathcal{S}\mathbb{U}$ and fix some point $x_1 \in \mathcal{S}\mathbb{U} \setminus \{x_0\}$ and $s = d(x_0, x_1)$. Let also $\mathcal{R} = \{\mathcal{O}(x_0, x_1)\}$. By the ultrahomogeneity of $\mathcal{S}\mathbb{U}$, we see that $\mathcal{O}(y,z) = \mathcal{O}(x_0, x_1) \in \mathcal{R}$ for all $y, z \in \mathcal{S}\mathbb{U}$ with $d(y,z) = s$.

Now, for any two points $y, z \in \mathcal{S}\mathbb{U}$, let $n_{y,z} = \lceil \frac{d(y,z)}{s} \rceil + 1$. It is then easy to see that there is a finite metric space in $\mathcal{K}$ containing a sequence of points $v_0, v_1, \ldots, v_{n_{y,z}}$ so that $d(v_i, v_{i+1}) = s$, while $d(v_0, v_{n_{y,z}}) = d(y,z)$. By the ultrahomogeneity of $\mathcal{S}\mathbb{U}$, it follows that there is a sequence $w_0, w_1, \ldots, w_{n_{y,z}} \in \mathcal{S}\mathbb{U}$ with $w_0 = y$, $w_{n_{y,z}} = z$ and $d(w_i, w_{i+1}) = s$, i.e., $\mathcal{O}(w_i, w_{i+1}) \in \mathcal{R}$ for all i. In other words,

$$\rho_{x_o,\mathcal{R}}(y,z) \leqslant \frac{1}{s} d(y,z) + 2$$

and, in particular, the graph $\mathbb{X}_{x_0,\mathcal{R}}$ is connected. Conversely, if $\rho_{x_0,\mathcal{R}}(y,z) = m$, then there is a finite path $w_0, w_1, \ldots, w_m \in \mathcal{S}\mathbb{U}$ with $w_0 = y$, $w_m = z$ and $d(w_i, w_{i+1}) = s$, whereby $d(y,z) \leqslant ms$, showing that

$$\frac{1}{s} d(y,z) \leqslant \rho_{x_o,\mathcal{R}}(y,z) \leqslant \frac{1}{s} d(y,z) + 2.$$

Therefore, the identity map is a quasi-isometry between $\mathbb{X}_{x_0,\mathcal{R}}$ and $\mathcal{S}\mathbb{U}$. Since, by Theorem 6.8, the mapping

$$g \in \mathrm{Isom}(\mathcal{S}\mathbb{U}) \mapsto g(x_0) \in \mathbb{X}_{x_0,\mathcal{R}}$$

is a quasi-isometry, so is the mapping

$$g \in \mathrm{Isom}(\mathcal{S}\mathbb{U}) \mapsto g(x_0) \in \mathcal{S}\mathbb{U}.$$

By consequence, the isometry group $\mathrm{Isom}(\mathcal{S}\mathbb{U})$ is quasi-isometric to the Urysohn space $\mathcal{S}\mathbb{U}$.

APPENDIX A

Zappa–Szép products

1. The topological structure

A basic result in Banach space theory says that, if X is a Banach space with closed linear subspaces A and B so that $X = A + B$ and $A \cap B = \{0\}$, then X is naturally isomorphic to the direct sum $A \oplus B$ with the product topology. Moreover, the proof of this is a rather straightforward application of the closed graph theorem. Namely, suppose $P\colon X \to A$ is the linear operator defined by $P(x) = a$ if $x = a + b$ for some $b \in B$. Then P has closed graph and hence is bounded. For, if $x_n \xrightarrow[n]{} x$ and $P(x_n) \xrightarrow[n]{} a$, then $x - a = \lim_n x_n - P(x_n) \in B$ and so the decomposition $x = a + (x - a)$ shows that $P(x) = a$. Thus both projections $P\colon X \to A$ and $I - P\colon X \to B$ are bounded, whence $X \cong A \oplus B$.

Apart from being formulated specifically for linear spaces, the proof very much depends on the projection P being a morphism and thus gives little hint as to a generalisation to the non-commutative setting. Employing somewhat different ideas, we present this generalisation here.

In the following, we consider a Polish group G with two closed subgroups A and B so that $G = AB$ and $A \cap B = \{1\}$, i.e., so that each element $g \in G$ can be written in a unique manner as $g = ab$ with $a \in A$ and $b \in B$. This situation is expressed by saying that G is the *Zappa–Szép product* of A and B [**70, 81**] and, of course, includes the familiar cases of internal direct and semidirect products.

Let $A \times B$ denote the cartesian product with the product topology and define

$$\Phi\colon A \times B \to G$$

by $\Phi(a, b) = ab$. By unique decomposition, Φ is a bijection and, since A and B are homeomorphically embedded in G and multiplication in G is continuous, also Φ is continuous. It therefore follows that Φ is a homeomorphism between $A \times B$ and G if and only if the projection maps $\pi_A\colon G \to A$ and $\pi_B\colon G \to B$, defined by

$$g = \pi_A(g)\pi_B(g),$$

are continuous. We verify this by showing that Φ is an open mapping.

For a subset $D \subseteq G$, let $\mathcal{U}(D)$ denote the largest open subset of G in which D is comeagre. Then, by continuity of the group operations in G, we have $g\mathcal{U}(D)f = \mathcal{U}(gDf)$ for all $g, f \in G$.

Lemma A.1. *Suppose $V \subseteq A$ and $W \subseteq B$ are open. Then*

$$VW \subseteq \mathcal{U}(VW) \quad \textit{and} \quad \overline{V} \cdot \overline{W} \subseteq \overline{\mathcal{U}(VW)}.$$

Proof. Suppose that $a \in V$ and $b \in B$. We choose open sets $a \in V_0 \subseteq V$ and $b \in W_0 \subseteq W$ so that $V_0V_0^{-1}V_0 \subseteq V$ and $W_0W_0^{-1}W_0 \subseteq W$. Let also $P \subseteq A$ and

$Q \subseteq B$ be countable dense subsets, whereby $A = PV_0$, $B = W_0Q$ and

$$G = \bigcup_{p,q} pV_0W_0q.$$

It follows that some pV_0W_0q and thus also V_0W_0 is non-meagre in G. Since V_0W_0 is analytic and hence has the property of Baire, we have $\mathcal{U}(V_0W_0) \neq \emptyset$. So pick some $g \in V_0W_0 \cap \mathcal{U}(V_0W_0)$ and write $g = a_0b_0$ for $a_0 \in V_0$ and $b_0 \in W_0$. Then

$$\begin{aligned} ab &= aa_0^{-1} \cdot a_0b_0 \cdot b_0^{-1}b \\ &\in aa_0^{-1} \cdot \mathcal{U}(V_0W_0) \cdot b_0^{-1}b \\ &= \mathcal{U}(aa_0^{-1}V_0W_0b_0^{-1}b) \\ &\subseteq \mathcal{U}(VW), \end{aligned}$$

showing that $VW \subseteq \mathcal{U}(VW)$ and hence also $\overline{V} \cdot \overline{W} \subseteq \overline{VW} \subseteq \overline{\mathcal{U}(VW)}$. □

Lemma A.2. *Suppose $V \subseteq A$ and $W \subseteq B$ are regular open. Then $VW = \mathcal{U}(VW)$.*

Proof. Assume that $V \subseteq A$ and $W \subseteq B$ are regular open, i.e., that $\mathsf{int}\,\mathsf{cl}\, V = V$ and hence also $\mathsf{cl}\,\mathsf{int}\,(\sim V) = \sim V$, where $\sim V = A \setminus V$, and similarly for W.

Suppose toward a contradiction that $g \in \mathcal{U}(VW) \setminus VW$ and write $g = ab$ for $a \in A$ and $b \in B$. Then, either $a \notin V$ or $b \notin W$, say $a \notin V$, the other case being similar. Set $U = \mathsf{int}\,(\sim V)$, whence $\overline{U} = \mathsf{cl}\,\mathsf{int}\,(\sim V) = \sim V$. Then, by Lemma A.1, we have

$$g = ab \in (\sim V)B = \overline{U} \cdot \overline{B} \subseteq \overline{\mathcal{U}(UB)}.$$

As also $g \in \mathcal{U}(VW)$, it follows that $\mathcal{U}(VW) \cap \overline{\mathcal{U}(UB)} \neq \emptyset$ and thus also that $\mathcal{U}(VW) \cap \mathcal{U}(UB) \neq \emptyset$. Therefore, VW and UB are both comeagre in the non-empty open set $\mathcal{U}(VW) \cap \mathcal{U}(UB)$, so must intersect, $VW \cap UB \neq \emptyset$. However, as $V \cap U = \emptyset$, this contradicts unique decomposability. So $VW \subseteq \mathcal{U}(VW)$ and the reverse inclusion follows directly from Lemma A.1. □

Theorem A.3. *Let A and B be closed subgroups of a Polish group G so that $G = AB$ and $A \cap B = \{1\}$. Then the group multiplication is a homeomorphism from $A \times B$ to G.*

Proof. It suffices to note that the sets $V \times W$ with $V \subseteq A$ and $W \subseteq B$ regular open form a basis for the topology on $A \times B$, whence, by Lemma A.2, the multiplication map is a continuous and open bijection. □

We should point out that the above result fails entirely when G is no longer assumed to be Polish. Indeed, one could simply take two closed linear subspaces A and B of a separable Banach space X so that $A \cap B = \{0\}$, but not forming a direct sum. Then there are unit vectors $a_n \in A$ and $b_n \in B$ so that $a_n - b_n \xrightarrow[n]{} 0$ and so the topology on the linear subspace $A + B$ is not the product topology.

2. Examples

Apart from trivial examples such as direct products of Polish groups, there are common instances of the above setup.

Example A.4 (Internal semidirect products). Another particular case of the Zappa–Szép product is when a Polish group G is the internal semidirect product of closed subgroups H and N, that is, $G = HN$, $H \cap N = \{1\}$ and N normal in G.

Example A.5 (Homeomorphism groups of locally compact groups). Suppose H is a locally compact Polish group and consider the group $\mathrm{Homeo}(H)$ of homeomorphisms of H equipped with the compact-open topology on the one-point compactification. Then H can be identified with a closed subgroup of $\mathrm{Homeo}(H)$ via its left-regular representation $\lambda\colon H \to \mathrm{Homeo}(H)$ given by $\lambda_x(y) = xy$. Letting 1_H denote the identity in H and setting

$$K = \{g \in \mathrm{Homeo}(H) \mid g(1_H) = 1_H\},$$

we find that K is closed in $\mathrm{Homeo}(H)$, $K \cap H = \{\mathrm{id}\}$ and $KH = \mathrm{Homeo}(H)$. So $\mathrm{Homeo}(H)$ is the Zappa-Szép product of the pointwise stabiliser K of 1_H and the group H of translations.

Example A.6. For a concrete instance of Example A.5, consider $\mathrm{Homeo}(\mathbb{T}^2)$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Then K is the group of homeomorphisms of $\mathbb{T}^2$ fixing the point 0 and $\mathrm{Homeo}(\mathbb{T}^2)$ is the Zappa-Szép product of K and $\mathbb{T}^2$ itself.

Example A.7. For a more general class of examples, suppose H is a locally compact Polish group and G a subgroup of $\mathrm{Homeo}(H)$, which is Polish in some finer group topology and so that G contains the image of H via the left-regular representation, i.e., the group of left-multiplication by elements of H. For example, H could be a Lie group and $G = \mathrm{Diff}^\infty(H)$. Again, H and the pointwise stabiliser K of 1_H are both closed in G and thus G is the Zappa–Szép product of K and H.

Example A.8. Suppose $\mathbf{T}$ is the countably infinite regular tree, i.e., so that every vertex has denumerable valence. Then $\mathbf{T}$ is isomorphic to the Cayley graph of the free group $\mathbb{F}_\infty$ on a denumerable set of generators and hence the automorphism group $\mathrm{Aut}(\mathbf{T})$ can be viewed as a subgroup of the homeomorphism group of the countable discrete group $\mathbb{F}_\infty$. Moreover, under this identification, $\mathrm{Aut}(\mathbf{T})$ contains all left-translations by elements of $\mathbb{F}_\infty$ and thus $\mathrm{Aut}(\mathbf{T})$ is the Zappa–Szép product of $\mathbb{F}_\infty$ and the pointwise stabiliser

$$K = \{g \in \mathrm{Aut}(\mathbf{T}) \mid g(r) = r\},$$

where $r \in \mathbf{T}$ is the vertex corresponding to the identity $1 \in \mathbb{F}_\infty$.

In the preceding examples, the larger group G is decomposed via an action on one of the closed subgroups A and B appearing as a factor in the Zappa–Szép product. As we shall see, this is necessarily so.

Indeed, suppose G is a Polish group with closed subgroups A and B so that $G = AB$ and $A \cap B = \{1\}$. Let also $\pi_A\colon G \to A$ and $\pi_B\colon G \to B$ be the corresponding projection maps, i.e., so that $g = \pi_A(g) \cdot \pi_B(g)$ for all $g \in G$. Then, if $f, g \in G$ and $a \in A$, we have

$$\begin{aligned}\pi_A(fga)\pi_B(fga) &= fga\\ &= f \cdot \pi_A(ga)\pi_B(ga)\\ &= \pi_A(f \cdot \pi_A(ga))\pi_B(f \cdot \pi_A(ga))\pi_B(ga),\end{aligned}$$

that is,

$$\pi_A(f \cdot \pi_A(ga))^{-1}\pi_A(fga) = \pi_B(f \cdot \pi_A(ga))\pi_B(ga)\pi_B(fga)^{-1} \in A \cap B = \{1\}.$$

It follows, in particular, that

$$\pi_A(f \cdot \pi_A(ga)) = \pi_A(fga)$$

for all $f, g \in G$ and $a \in A$. As also $\pi_A(1a) = a$ and as π_A is continuous by Theorem A.3, we obtain a continuous action $\alpha\colon G \curvearrowright A$ on the topological space A by letting

$$\alpha_g(a) = \pi_A(ga).$$

Observe then that the α-action of A on itself is simply the left-regular representation $\lambda\colon A \curvearrowright A$,

$$\alpha_{a_1}(a_2) = \pi_A(a_1a_2) = a_1a_2 = \lambda_{a_1}(a_2),$$

while

$$B = \{g \in G \mid \alpha_g(1) = \pi_A(g) = 1\}.$$

So, in other words, the Zappa-Szép product $G = AB$ arrises from a continuous action of G on A, where B is the isotropy subgroup of 1 and A acts by left-multiplication on itself.

3. The coarse structure of Zappa–Szép products

In the following, we fix a Polish group G that is the Zappa–Szép product of two closed subgroups A and B. Define the identification $\phi\colon A \times B \to G$ by $\phi(a, b) = ab$ and let $\pi_A\colon G \to A$ and $\pi_B\colon G \to B$ be the associated projections, i.e., the maps defined by

$$g = \pi_A(g) \cdot \pi_B(g)$$

for all $g \in G$. For $g \in G$ and $X, Y \subseteq G$, we let $g^X = \{xgx^{-1} \mid x \in X\}$ and $Y^X = \{xyx^{-1} \mid x \in X \;\&\; y \in Y\}$.

Lemma A.9. *If $X \subseteq A$, $Y \subseteq B$ and $n \geqslant 1$, we have*

$$(XY)^n \subseteq \pi_A(X^B)^n \pi_B(X^B)^{n-1} Y^n.$$

Proof. It suffices to show that, for all $a_1, \ldots, a_n \in A$ and $b_1, \ldots, b_n \in B$, we have

$$\begin{aligned}
&a_1b_1a_2b_2\cdots a_nb_n\\
&\qquad \in a_1\pi_A(a_2^B)\pi_A(a_3^B)\cdots\pi_A(a_n^B)\cdot\pi_B(a_n^B)\pi_B(a_{n-1}^B)\cdots\pi_B(a_2^B)\cdot b_1b_2\cdots b_n.
\end{aligned}$$

We show this by induction on n with the case $n = 1$ being trivial. For the induction step, suppose that

$$a_1b_1a_2b_2\cdots a_nb_n = ab \cdot b_1b_2\cdots b_n$$

for some $a \in a_1\pi_A(a_2^B)\pi_A(a_3^B)\cdots\pi_A(a_n^B)$ and $b \in \pi_B(a_n^B)\pi_B(a_{n-1}^B)\cdots\pi_B(a_2^B)$. Then, for $a_{n+1} \in A$ and $b_{n+1} \in B$, we have

$$\begin{aligned}
&a_1b_1a_2b_2\cdots a_nb_n \cdot a_{n+1}b_{n+1}\\
&\quad = ab \cdot b_1b_2\cdots b_n \cdot a_{n+1}b_{n+1}\\
&\quad = a \cdot (bb_1b_2\cdots b_n)a_{n+1}(bb_1b_2\cdots b_n)^{-1}(bb_1b_2\cdots b_n)b_{n+1}\\
&\quad = a\pi_A\big((bb_1b_2\cdots b_n)a_{n+1}(bb_1b_2\cdots b_n)^{-1}\big)\\
&\qquad \cdot \pi_B\big((bb_1b_2\cdots b_n)a_{n+1}(bb_1b_2\cdots b_n)^{-1}\big) \cdot bb_1b_2\cdots b_nb_{n+1}\\
&\quad \in a\pi_A(a_{n+1}^B) \cdot \pi_B(a_{n+1}^B) \cdot bb_1b_2\cdots b_nb_{n+1}\\
&\quad \subseteq a_1\pi_A(a_2^B)\cdots\pi_A(a_n^B)\pi_A(a_{n+1}^B)\cdot\pi_B(a_{n+1}^B)\pi_B(a_n^B)\cdots\pi_B(a_2^B)\cdot b_1\cdots b_{n+1},
\end{aligned}$$

as claimed. ∎

LEMMA A.10. *The map $\phi\colon A \times B \to G$ is bornologous if and only if X^B is coarsely bounded in G for every coarsely bounded set $X \subseteq A$.*

PROOF. The coarse structure on $A \times B$ is simply the product of the coarse structures on A and on B. Thus, a basic entourage in $A \times B$ has the form $E_X \times E_Y$ where X and Y are coarsely bounded subsets of A and B respectively. We observe that

$$\begin{aligned}(\phi \times \phi)[E_X \times E_Y] &= \{\phi(a,b)^{-1}\phi(ax,by) \mid a \in A, x \in X, b \in B, y \in Y\}\\ &= \{b^{-1}a^{-1}axby \mid a \in A, x \in X, b \in B, y \in Y\}\\ &= \{b^{-1}xb \cdot y \mid a \in A, x \in X, b \in B, y \in Y\}\\ &= X^B \cdot Y.\end{aligned}$$

However, since Y is coarsely bounded in B and hence also in G, we find that $X^B \cdot Y$ is coarsely bounded in G if and only if X^B is coarsely bounded in G. It thus follows that ϕ is bornlogous if and only if X^B is coarsely bounded in G for every coarsely bounded subset X of A. □

LEMMA A.11. *Suppose that A is locally bounded. Then*

(1) *$\pi_A\colon G \to A$ is bornologous if and only if $\pi_A(X^B)$ is coarsely bounded in A for every coarsely bounded subset $X \subseteq A$,*
(2) *if for every coarsely bounded subset X of A the set $\pi_B(X^B)$ is coarsely bounded in B, then $\pi_B\colon G \to B$ is modest.*

PROOF. Fix a coarsely bounded identity neighbourhood $U \subseteq A$. By Theorem A.3, the sets of the form WV with W open in A and V open in B form a neighbourhood basis at the identity in G. Thus, if D is a coarsely bounded subset of G, there are finite sets $E \subseteq A$ and $F \subseteq B$ and an $n \geqslant 1$ so that

$$D \subseteq (EUFV)^n \subseteq \pi_A\big((EU)^B\big)^n \pi_B\big((EU)^B\big)^{n-1}(FV)^n.$$

In particular, for every coarsely bounded set $D \subseteq G$ and every identity neighbourhood $V \subseteq B$, there is a finite set $F \subseteq B$, a coarsely bounded set $X \subseteq A$ and an $n \geqslant 1$, so that

$$\pi_A(D) \subseteq \pi_A\big(X^B\big)^n$$

and

$$\pi_B(D) \subseteq \pi_B\big(X^B\big)^{n-1}(FV)^n.$$

(1) So, assume that $\pi_A(X^B)$ is coarsely bounded in A for all coarsely bounded subsets X of A. Then, by the above, $\pi_A\colon G \to A$ is modest. Suppose that $D \subseteq G$ is coarsely bounded, Then, for $g \in G$ and $d \in D$, write $g = a_1b_1$ and $d = a_2b_2$ for some $a_i \in A$ and $b_i \in B$, whence

$$\pi_A(g)^{-1}\pi_A(gd) = a_1^{-1}a_1\pi_A(b_1a_2b_2) = \pi_A(b_1a_2b_1^{-1}) \in \pi_A\big(\pi_A(D)^B\big).$$

Since, by modesty of π_A, the set $\pi_A(D)$ is coarsely bounded in A, also $\pi_A\big(\pi_A(D)^B\big)$ is coarsely bounded in A. So this shows that $\pi_A\colon G \to A$ is bornologous.

Conversely, assume $\pi_A\colon G \to A$ is bornologous and that X is a coarsely bounded subset of A. Then X is coarsely bounded in G and

$$\begin{aligned}(\pi_A \times \pi_A)[E_X] &= \{\pi_A(g)^{-1}\pi_A(gx) \mid g \in G, x \in X\}\\ &= \{\pi_A(ab)^{-1}\pi_A(abx) \mid a \in A, b \in B, x \in X\}\\ &= \{a^{-1}a\pi_A(bx) \mid a \in A, b \in B, x \in X\}\\ &= \{\pi_A(bxb^{-1}) \mid b \in B, x \in X\}\\ &= \pi_A(X^B),\end{aligned}$$

which is coarsely bounded in A.

(2) Assume now that $\pi_B(X^B)$ is coarsely bounded in B for all coarsely bounded subsets X of A. Suppose that $D \subseteq G$ is coarsely bounded and V is an identity neighbourhood in B. Find as above a coarsely bounded subset X of A, a finite set $F \subseteq B$ and an $n \geqslant 1$ so that $\pi_B(D) \subseteq \pi_B\big(X^B\big)^{n-1}(FV)^n$. Since $\pi_B\big(X^B\big)$ is coarsely bounded in B, this means that there is a finite set $E \subseteq B$ containing F and an $m \geqslant 1$ so that $\pi_B\big(X^B\big) \subseteq (EV)^m$, whence

$$\pi_B(D) \subseteq (EV)^{m(n-1)+n},$$

showing that $\pi_B\colon G \to B$ is modest. $\square$

Theorem A.12. *Suppose that A is locally bounded. The $\phi\colon A \times B \to G$ is a coarse equivalence if and only if $\pi_A(X^B)$ is coarsely bounded in A and $\pi_B(X^B)$ is coarsely bounded in B for every coarsely bounded subset $X \subseteq A$.*

Proof. Observe that the inverse of $\phi\colon A \times B \to G$ is the map $g \in G \mapsto \big(\pi_A(g), \pi_B(g)\big) \in A \times B$. Thus, as the coarse structure on $A \times B$ is the product of the coarse structures on A and on B, we see that ϕ^{-1} is bornologous if and only if both $\pi_A\colon G \to A$ and $\pi_B\colon G \to B$ are bornologous. It thus follows that ϕ is a coarse equivalence exactly when all of ϕ, π_A and π_B are bornologous.

Suppose first that $\phi\colon A \times B \to G$ is a coarse equivalence, whence ϕ, π_A and π_B are bornologous, and assume that X is a coarsely bounded subset of A. Then, by Lemma A.10, X^B is coarsely bounded in G and hence $\pi_A(X^B)$ and $\pi_B(X^B)$ are coarsely bounded in A and B respectively.

For the remainder of the proof, we assume conversely that, for all coarsely bounded subsets X of A, the images $\pi_A(X^B)$ and $\pi_B(X^B)$ are coarsely bounded in A and B respectively.

By Lemma A.11, π_A is bornologous. Also, if X is a coarsely bounded subset of A, then $\pi_A(X^B)$ and $\pi_B(X^B)$ are coarsely bounded in G and hence

$$X^B \subseteq \pi_A(X^B) \cdot \pi_B(X^B)$$

is coarsely bounded in G. By Lemma A.10, this implies that ϕ is bornologous.

Finally, to see that π_B is bornologous, assume $D \subseteq G$ is coarsely bounded. Then

$$\begin{aligned}(\pi_B \times \pi_B)E_D &= \{\pi_B(g)^{-1}\pi_B(gd) \mid g \in G, d \in D\}\\ &= \{\pi_B(g)^{-1}\pi_A(gd)^{-1}gd \mid g \in G, d \in D\}\\ &= \{\pi_B(g)^{-1} \cdot \pi_A(gd)^{-1}\pi_A(g) \cdot \pi_B(g)d \mid g \in G, d \in D\}\\ &\subseteq \big((\pi_A \times \pi_A)E_{D^{-1}}\big)^B \cdot D.\end{aligned}$$

As π_A is bornologous, $(\pi_A\times\pi_A)E_{D^{-1}}$ is coarsely bounded in A and hence the sets $\big((\pi_A\times\pi_A)E_{D^{-1}}\big)^B$ and $(\pi_B\times\pi_B)E_D$ are coarsely bounded in G. Since $(\pi_B\times\pi_B)E_D\subseteq B$ and π_B is modest by Lemma A.11, it follows that $(\pi_B\times\pi_B)E_D$ is coarsely bounded in B and thus that $\pi_B\colon G\to B$ is bornologous. $\square$

REMARK A.13. While the two closed subgroups A and B may initially appear to play symmetric rôles in the Zappa–Szép product $G=AB$, in light of Theorem A.12 this is note quite so. Of course, if $G=AB$, then also $G=BA$, but stating that

$$\phi\colon A\times B\to G,\quad \phi(a,b)=ab$$

is a coarse equivalence is not the same as stating that

$$\psi\colon B\times A\to G,\quad \psi(b,a)=ba$$

is a coarse equivalence. This is due to the fact that we work with the left coarse structure $\mathcal{E}_L$, which is not in general bi-invariant.

EXAMPLE A.14 (Internal semidirect products). Suppose a Polish group G is the internal semidirect product of two closed subgroups N and H with N locally bounded and normal in G. That is, $G=NH$ with $N\cap H=\{1\}$ and $N\trianglelefteq G$. Let π_N and π_H be the corresponding projections, i.e.,

$$g=\pi_N(g)\cdot\pi_H(g)$$

for $g\in G$. In this case, for any subset $X\subseteq N$, we have $X^H\subseteq N$, so $\pi_N(X^H)=X^H$ and $\pi_H(X^H)=\{1\}$.

It thus follows from Theorem A.12 that the map

$$\phi\colon N\times H\to G,\quad \phi(n,h)=nh$$

is a coarse equivalence if and only if $X^H=\{hxh^{-1}\mid x\in X, h\in H\}$ is coarsely bounded in N for every coarsely bounded set $X\subseteq N$.

EXAMPLE A.15 (External semidirect products). Suppose $\alpha\colon H\curvearrowright N$ is a continuous action of a Polish group H by continuous automorphisms on a locally bounded Polish group N and let $G=N\rtimes_\alpha H$ be the corresponding topological semidirect product. Thus, G is simply the topological space $N\times H$ equipped with the multiplication

$$(n_1,h_1)\cdot(n_2,h_2)=\big(n_1\alpha_{h_1}(n_2),h_1h_2\big).$$

Moreover, N and H can be identified with the subgroups $N\times\{1_H\}$ and $\{1_N\}\times H$ of G with $N\times\{1_H\}$ normal in G.

So G is the Zappa–Szép product of $N\times\{1_H\}$ and $\{1_N\}\times H$. Moreover, as $(n,1_H)\cdot(1_N,h)=(h,n)$, we see that the projections $\pi_{\{1_H\}\times N}$ and $\pi_{H\times\{1_N\}}$ defined by

$$(h,n)=\pi_{\{1_H\}\times N}(h,n)\cdot\pi_{H\times\{1_N\}}(h,n)$$

are the projection maps to $N\times\{1_H\}$ and $\{1_N\}\times H$ respectively. Therefore,

$$\phi\colon N\times H\to N\rtimes_\alpha H,\quad \phi(n,h)=(n,h)$$

is a coarse equivalence if and only if, for all coarsely bounded subsets X of N, the set

$$\alpha_H(X)=\{\alpha_h(x)\bigm| h\in H, x\in X\}$$

is coarsely bounded in N.

We sum this up in the following proposition.

Proposition A.16. *Let $\alpha\colon H \curvearrowright N$ be a continuous action of a Polish group H by continuous automorphisms on a locally bounded Polish group N and let $N \rtimes_\alpha H$ be the corresponding topological semidirect product. Then the formal identity*

$$\phi\colon N \times H \to N \rtimes_\alpha H$$

is a coarse equivalence if and only if, for all coarsely bounded subsets X of N, the set $\alpha_H(X)$ is coarsely bounded in N.

Example A.17 (Affine isometry groups). Suppose $(X, \|\cdot\|)$ is a separable Banach space. Then the group $\mathrm{Aff}(X)$ of affine isometries of X decomposes as a semidirect product

$$\mathrm{Aff}(X) = (X, +) \rtimes \mathrm{Isom}(X),$$

where each $A \in \mathrm{Aff}(X)$ is identified with the pair (x, T), so that $A(y) = T(y) + x$ for all $y \in Y$. That is, the projection $\pi_{\mathrm{Isom}(X)}$ associates to $A \in \mathrm{Aff}(X)$ its linear part, while π_X is simply the associated cocycle $b\colon \mathrm{Aff}(X) \to X$. Since, if $D \subseteq X$ is norm bounded, also

$$D^{\mathrm{Isom}(X)} = \{T(x) \mid T \in \mathrm{Isom}(X) \;\&\; x \in D\}$$

is norm bounded, we find that $X \times \mathrm{Isom}(X)$ is coarsely equivalent with $\mathrm{Aff}(X)$ via the map that takes (x, T) to the affine isometry $y \mapsto T(y) + x$.

In particular, the cocycle $b\colon \mathrm{Aff}(X) \mapsto X$ is a coarse equivalence (and hence a quasi-isometry) if and only if $\mathrm{Isom}(X)$ is a coarsely bounded group. This reproves Proposition 3.13.

Example A.18 (Homeomorphisms of locally compact groups). Suppose G is a subgroup of the group $\mathrm{Homeo}(H)$ of homeomorphisms of a locally compact Polish group H and that G is equipped with a finer Polish group topology. Assume also that G contains the group $\lambda_H \cong H$ of left-translations λ_h by elements $h \in H$ and let $K = \{g \in G \mid g(1) = 1\}$ be the pointwise stabiliser of the identity in H. As observed in Example A.5, G is then the Zappa–Szép product of λ_H and K. Moreover, if $\pi_{\lambda_H}\colon G \to \lambda_H$ and $\pi_K\colon G \to K$ are the projections associated with the decomposition $G = \lambda_H \cdot K$, we find that

$$\pi_{\lambda_H}(g) = \lambda_{g(1)} \quad \text{and} \quad \pi_K(g) = \lambda_{g(1)^{-1}} \circ g.$$

Indeed, it suffices to note that $\big(\lambda_{g(1)^{-1}} \circ g\big)(1) = \lambda_{g(1)^{-1}}\big(g(1)\big) = 1$ and therefore $\lambda_{g(1)^{-1}} \circ g \in K$.

In particular, for $k \in K$ and $h \in H$, we have

$$\pi_{\lambda_H}(k\lambda_h k^{-1}) = \lambda_{k(h)} \quad \text{and} \quad \pi_K(k\lambda_h k^{-1}) = \lambda_{k(h)^{-1}} k\lambda_h k^{-1}.$$

Applying Lemma A.11 and the fact that coarsely bounded sets in H are simply the relatively compact sets, we find that π_{λ_H} is bornologous if and only if, for every relatively compact open set $U \subseteq H$, the K-invariant open set $K(U) = \bigcup_{k \in K} k[U]$ is relatively compact. It thus follows that π_{λ_H} is bornologous if and only if H admits a covering by K-invariant relatively compact open subsets. Note that this is, in general, stronger than requiring the action $K \curvearrowright H$ to be modest.

Investigating when $\pi_K(X^K)$ is coarsely bounded in K for coarsely bounded subsets $X \subseteq \lambda_H$ ultimately depends on the coarse geometry of K. When K is coarsely bounded even as a subset of G, we are led to the following criterion.

PROPOSITION A.19. *Suppose H is a locally compact Polish groups and $G \leqslant \mathrm{Homeo}(H)$ is a subgroup equipped with a finer Polish group topology. Assume also that G contains the group of left-translations λ_h by elements $h \in H$ and that the pointwise stabiliser $K = \mathrm{stab}_G(1)$ is coarsely bounded. Then the inclusion*

$$h \in H \mapsto \lambda_h \in G$$

is a coarse equivalence if and only if H admits a covering by K-invariant relatively compact open subsets.

PROOF. Suppose first that H admits a covering $\{U_n\}_n$ by K-invariant relatively compact open subsets. Then π_{λ_H} is bornologous and hence the inclusion of H into G is bornologous with a bornologous inverse. In particular, H is coarsely embedded in G and is cobounded since $G = HK$ with K coarsely bounded in G. So the inclusion is a coarse equivalence between H and G.

Conversely, suppose that the inclusion $h \in H \mapsto \lambda_h \in G$ is a coarse equivalence. Then H is coarsely embedded in G. Also, since K is coarsely bounded in G, the map $\lambda_h k \in G \mapsto \lambda_h \in G$ is bornologous. By composition, it follows that $\pi_{\lambda_H} : G \to \lambda_H$ is bornologous and thus H is covered by K-invariant relatively compact open sets. $\square$

Instances of this include, for example, $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$, where $K = \mathrm{Stab}(0)$ is isomorphic to $\mathrm{Homeo}_+([0,1])$, and hence is coarsely bounded, and the action of K on $\mathbb{R}$ leaves every interval $]-n, n[$ invariant. So again we see that the inclusion of $\mathbb{R}$ into $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ as the group of translations is a coarse equivalence.

APPENDIX B

Open problems

PROBLEM B.1. *Let G be a Polish group of finite asymptotic dimension. Is G necessarily locally bounded?*

PROBLEM B.2. *Let G be a Polish group of bounded geometry. Must G be coarsely equivalent to a locally compact (second countable) group?*

PROBLEM B.3. *Find a non-locally compact, topologically simple, Polish group of bounded geometry that is not coarsely bounded.*

PROBLEM B.4. *Let $\mathbb{H}$ be the infinite-dimensional hyperbolic space and G its group of isometries. Is G quasi-isometric to $\mathbb{H}$?*

PROBLEM B.5. *Suppose that K is a closed subgroup of a Polish group G and that both K and G/K are locally bounded. Does it follow that also G is locally bounded?*

PROBLEM B.6. *Suppose M is a compact manifold. Is* $\mathrm{Homeo}_0(M)$ *ultralocally bounded?*

PROBLEM B.7. *Suppose H is a cocompact closed subgroup of a Polish group G, i.e., $G = HK$ for some compact set $K \subseteq G$. Is H coarsely embedded in G?*

PROBLEM B.8. *Suppose H is an open subgroup of a coarsely bounded Polish group G. Is also H coarsely bounded?*

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