A TOPOLOGICAL VERSION OF THE BERGMAN PROPERTY

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ABSTRACT. A topological group G is defined to have property (OB) if any G-action by isometries on a metric space, which is separately continuous, has bounded orbits. We study this topological analogue of strong uncountable cofinality in the context of Polish groups, where we show it to have several interesting reformulations and consequences. We subsequently apply the results obtained in order to verify property (OB) for a number of groups of isometries and homeomorphism groups of compact metric spaces. We also give a proof that the isometry group of the rational Urysohn metric space of diameter 1 has strong uncountable cofinality.

1. INTRODUCTION

We study in this paper a topological version of strong uncountable cofinality. This latter property has been the object of intense scrutiny by a number of people, since it was first discovered to hold for the infinite symmetric group, $\mathcal{S}_\infty$, by George M. Bergman [6].

Definition 1.1. *A group G is said to have strong uncountable cofinality if whenever $W_0 \subseteq W_1 \subseteq \ldots \subseteq G = \bigcup_n W_n$, there are n and k such that $G = W_n^k$.*

We should mention that the above property is sometimes called the *Bergman property* in the literature, but since this terminology is also used for a weaker property, we stick to "strong uncountable cofinality" throughout the paper.

We now have a large number of interesting results concerning this property, but most surprising is perhaps the fact that many large permutation groups indeed have strong uncountable cofinality. One can pick some of these results from [9, 11, 12, 21, 26, 34, 35]. For example, it holds for automorphism groups of 2-transitive linear orders (Droste and Holland [12]), the group of measure preserving automorphisms of the unit interval (Miller [26]) and oligomorphic permutation groups with ample generics, e.g., many automorphism groups of ω-stable, ω-categorical structures (Kechris and Rosendal [21]).

To see what strong uncountable cofinality is really worth, it is useful to consider some of its consequences and reformulations. First of all, it is clear that no group with strong uncountable cofinality can be written as a union of a countable chain of proper subgroups, or, in other words, such groups have *uncountable cofinality*. Similarly, if $1 \in E = E^{-1}$ is a generating set for a group with strong uncountable cofinality, then there is some finite power n such that every element of the group can be written as a word of length n in E. We express this by saying that the group is *Cayley bounded*, since it corresponds to every Cayley graph being of bounded diameter with respect to the word metric. Both uncountable cofinality and Cayley boundedness have been studied in the literature, though apparently mostly independently of each other. Uncountable cofinality grew out of J.P. Serre's work [30] on property (FA), which is a fixed point property for actions on trees, in which it proved to be one of the three conditions in his equivalent formulation of property

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(FA) for uncountable groups. It was first proved to hold for certain profinite groups by S. Koppelberg and J. Tits [23] and has subsequently been verified for a large number of primarily subgroups of the infinite symmetric group. S. Shelah [31], on the other hand, has constructed a group having width 240 with respect to any generating set, so the group is Cayley bounded. As a matter of fact, as was noticed by Droste and Holland [12], these two properties taken together are equivalent to having strong uncountable cofinality.

However, perhaps more useful for the geometric theory of groups with strong uncountable cofinality is the following basic characterisation, of which I learned the equivalence of (2) and (3) from B.D. Miller [26] and where the equivalence of (1) and (3) was independently noticed by Y. de Cornulier [9] and V. Pestov.

Theorem 1.2. *The following conditions are equivalent for a group G.*
(1) Whenever G acts by isometries on a metric space (X, d) every orbit is bounded.
(2) Any left-invariant metric on G is bounded.
(3) G has strong uncountable cofinality.
(4) Whenever G acts on a metric space (X, d) by mappings, which are Lipschitz for large distances, every orbit is bounded.
(5) Whenever G acts by uniformly continuous homeomorphisms on a geodesic space (X, d) every orbit is bounded.

Now, of course, (1),(4) and (5) are really properties one would tend to study in connection with topological groups modulo some continuity condition, and these are indeed the main topics of the present paper. We therefore propose the following definition.

Definition 1.3. *A topological group G is said to have* property (OB) *if whenever G acts by isometries on a metric space (X, d), such that for every $x \in X$ the function $g \in G \mapsto gx \in X$ is continuous, then every orbit is bounded.*

Similar properties have previously been considered by Jan Hejcman [16] in 1959 (see also his recent paper [17]) and later by Christopher Atkin [1] under the name of *boundedness* in the context of uniform spaces. However, let me point of the differences between their notions and property (OB). A topological group G is *bounded* if for any non-empty open subset $U \subseteq G$ there is a finite set $A \subseteq G$ and a number n such that $G = U^n A$. This, however, is easily seen to fail for our prime example of a group with strong uncountable cofinality, namely $\mathcal{S}_\infty$. For if we choose U to be an open subgroup of denumerable index, as for example the isotropy subgroup of $0 \in \mathbb{N}$, then clearly $U^n A = UA \neq \mathcal{S}_\infty$ for all n and finite A. Nevertheless, the two notions do turn out to be equivalent in the context of abelian groups, but most of the groups considered here are very much non-abelian.

One of the reasons for our interest in the property comes from the fact that it can be seen as an addition to a well-known spectrum of properties studied in geometric group theory, namely properties (FA), (FH), (T), amenability, etc. One easily sees that property (OB) implies property (FH) and actually, we shall see that it provides a fairly comprehensive class of new examples of non-locally compact groups with property (FH).

Much of the work on these properties has been restricted to the locally compact setting, where the strongest tools are available (e.g., Haar measure). But over the years, a number of very interesting results concerning the dynamics of non-locally compact Polish groups have surfaced, for example on the unitary group of ℓ_2, where Gromov and Milman proved that it is extremely amenable [15] and Bekka proved that it has property (T) [4]. Moreover, in logic, where, e.g., automorphism groups of countable structures tend to be non-locally compact, there is a multitude of results on permutation groups, e.g., Truss [36] and Hodges, Hodkinson, Lascar, and Shelah [19], and also for more inclusive classes of Polish groups, e.g., Becker and Kechris [3] and Hjorth [18].

Thus, for Polish groups it is natural to look for a similar characterisation of property (OB), and indeed we have the following result.

Theorem 1.4. *The following are equivalent for a Polish group G:*
(i) G has property (OB).
(ii) Whenever $W_0 \subseteq W_1 \subseteq \ldots \subseteq G$ is an increasing exhaustive sequence of sets with the Baire property, there are n and k such that $G = W_n^k$.
(iii) Any compatible left-invariant metric on G is bounded.
(iv) G is finitely generated of bounded width over any non-empty open subset.

For example, a locally compact Polish group has property (OB) if and only if it is compact.

This gives an indication of how to think of these properties. Namely, one should think of strong uncountable cofinality as a strong generalisation of finiteness and of property (OB) as a strong generalisation of compactness. Surprisingly though, this "compactness" can, apart from compact groups, only be found in large, very much non-locally compact groups.

We study first the dynamics of property (OB) groups acting continuously by Hölder mappings, showing that in this case the closure of the orbits gives a decomposition of the phase space into pieces on which the group acts minimally (often denoted by *semi-simplicity*). And secondly we consider the closure properties of the class of property (OB) groups, for example, it is quite easily seen that it is closed under infinite products, group extensions over a property (OB) group, and behaves well with respect to short exact sequences. I.e., if $\pi : G \to H$ is a continuous homomorphism with dense image where G has property (OB), then so does H. Most interesting in this connection is the fact that it passes to subgroups of finite index, for which we give a geometric proof. In his paper [6], Bergman originally asked whether strong uncountable cofinality was preserved between a group and its subgroups of finite index, and A. Khélif, in an announcement [22], stated that this is indeed the case. However, the mentioned geometric proof, which works also for strong uncountable cofinality, shows the usefulness of the reformulation of strong uncountable cofinality in terms of isometric actions, where one has the added advantage of geometric intuition.

However, a large chunk of the paper is concerned with the verification and construction of groups which have either strong uncountable cofinality or property (OB). Our first examples are isometry groups of sufficiently homogeneous metric spaces. We turn Theorem 1.4 on its head and ask for when the isometry group of a bounded complete metric space has property (OB). We provide one sufficient condition that is also of independent interest and use this to show that the isometry group of the Urysohn metric space of diameter 1 has property (OB). However, in the case of the rational Urysohn metric space of diameter 1, one can take advantage of the results of S. Solecki from [32], and use this to show outright that its isometry group have strong uncountable cofinality.

Theorem 1.5. *Let $\mathbb{U}_1$ be the Urysohn metric space of diameter 1 and Ω the rational Urysohn metric space of diameter 1. Then $\mathrm{Iso}(\mathbb{U}_1)$ has property (OB) and $\mathrm{Iso}(\Omega)$ strong uncountable cofinality.*

We then study a model theoretic version of the unitary group of ℓ_2 in some depth. This is a subgroup $\mathcal{U}(\mathbb{V})$ that sits as a dense subgroup in $\mathcal{U}(\ell_2)$, which we prove to have ample generics, the main tool used in proving strong uncountable cofinality for automorphism groups of ω-stable, ω-categorical structures in [21] and previously introduced by Hodges, Hodkinson, Lascar, and Shelah [19]. From ample generics, we prove that also $\mathcal{U}(\mathbb{V})$ has strong uncountable cofinality and has a number of other properties, e.g., the small index property and satisfies automatic continuity of homomorphisms.

Our final collection of examples comes from topology, where we prove property (OB) for homeomorphism groups of spheres and of the Hilbert cube.

Theorem 1.6. $\mathrm{Homeo}(S^m)$ *and* $\mathrm{Homeo}(Q)$ *have property (OB) and, by consequence, any Polish group is topologically isomorphic to a subgroup of a Polish group with property (OB).*

In the final section of the paper we consider property (FA), and provide a simple proof of a result of Dugald Macpherson and Simon Thomas stating that if a Polish group with a comeagre conjugacy class acts on a tree, there every element of the group fixes a vertex or an edge. Actually, we extend their theorem to all actions on Λ-trees, though of course, the case of simplicial trees is ultimately the most interesting, due to the structure theory of Serre for groups with property (FA) [30].

Though we shall from time to time use a little bit of descriptive set theory, the article should be comprehensible to the general analyst. Really, all one needs to know is the definition of a *Polish space* (a completely metrisable separable topological space), a *Polish group* (a topological group whose topology is Polish), *Borel sets* (the sets belonging to the σ-algebra generated by the open sets), *analytic set* (a subset in a Polish space which is the continuous image of a Borel set) and sets having the *Baire property* (i.e., sets A such that for some open set U and some meagre set M, $A = U \triangle M$). A basic result of Lusin and Sierpiński says that analytic sets have the Baire property.

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2. STRONG UNCOUNTABLE COFINALITY

For the following, recall that a *geodesic space* is a metric space such that between any two points x and y there is a path of length $d(x, y)$. For example, Banach spaces and $\mathbb{R}$-trees are geodesic spaces. Recall also that a mapping $\phi : X \to X$, where X is a metric space, is called *Lipschitz for large distances* if there are constants c, K such that for all $x, y \in X$, $d(\phi x, \phi y) \leq c \cdot d(x, y) + K$.

The following result states the basic equivalent formulations of strong uncountable cofinality.

Theorem 2.1. *The following conditions are equivalent for a group G.*
(1) Whenever G acts by isometries on a metric space (X, d) every orbit is bounded.
(2) Any left-invariant metric on G is bounded.
(3) G has strong uncountable cofinality.
(4) Whenever G acts on a metric space (X, d) by mappings which are Lipschitz for large distances, every orbit is bounded.
(5) Whenever G acts by uniformly continuous homeomorphisms on a geodesic space (X, d) every orbit is bounded.

Proof. Clearly, $1 \Rightarrow 2$ is trivial.

$2 \Rightarrow 3$: Suppose that $W_0 \subseteq W_1 \subseteq W_2 \subseteq \ldots \subseteq G$ is an exhaustive sequence of subsets of G. Notice that then $W_0 \cap W_0^{-1} \subseteq W_1 \cap W_1^{-1} \subseteq \ldots \subseteq G$ is also exhaustive. So we can suppose that the W_n are symmetric, and by renumbering the sequence, we can also suppose that $W_0 = \{1\}$. Notice now that the following left-invariant metric on G is bounded if and only if $G = W_n^k$ for some n and k:

$$d(f,g) = \min(k_1 + k_2 + \ldots + k_n \mid \exists h_i \in W_{k_i}\ fh_1 \ldots h_n = g).$$

$3 \Rightarrow 4$: Assume now that 3 holds and that G acts on a metric space (X,d) by mappings which are Lipschitz for large distances and find for each $g \in G$ constants c_g and K_g witnessing this. Then, if $g_1, \ldots, g_k \in G$ and $c_{g_1}, \ldots, c_{g_k}, K_{g_1}, \ldots, K_{g_k} \leq M$,

$$\begin{aligned} d(g_1 \ldots g_k \cdot x, g_1 \ldots g_k \cdot y) &\leq M \cdot d(g_2 \ldots g_k \cdot x, g_2 \ldots g_k \cdot y) + M \\ &\leq M^2 \cdot d(g_3 \ldots g_k \cdot x, g_3 \ldots g_k \cdot y) + M^2 + M \\ &\leq \ldots \\ &\leq M^k \cdot d(x,y) + M^k + M^{k-1} + \ldots + M \\ &\leq M^k \cdot (d(x,y) + k) \end{aligned} \tag{1}$$

Now, fix an $x_0 \in X$ and let for $n \geq 1$

$$W_n = \{g \in G \mid c_g, K_g \leq n \ \&\ d(x_0, g \cdot x_0) \leq n\}$$

This is clearly an increasing exhaustive sequence of subsets of G, so for some $M \geq 1$ and k, $G = W_M^k$. We claim that x_0's orbit is bounded in diameter by $2k^2M^k$. For if $g = g_1 \ldots g_k \in G$, with $g_1, \ldots, g_k \in W_M$,

$$\begin{aligned} d(x_0, g \cdot x_0) &= d(x_0, g_1 \ldots g_k \cdot x_0) \\ &\leq d(x_0, g_1 \cdot x_0) + d(g_1 \cdot x_0, g_1 g_2 \cdot x_0) + \\ &\quad \ldots + d(g_1 \ldots g_{k-1} \cdot x_0, g_1 \ldots g_{k-1} g_k \cdot x_0) \\ &\leq d(x_0, g_1 \cdot x_0) + M(d(x_0, g_2 \cdot x_0) + 1) + M^2(d(x_0, g_3 \cdot x_0) + 2) + \\ &\quad \ldots + M^{k-1}(d(x_0, g_k \cdot x_0) + k - 1) \\ &\leq M + M(M+1) + M^2(M+2) + \ldots + M^{k-1}(M + k - 1) \\ &\leq 2k^2 M^k \end{aligned} \tag{2}$$

So if x is any other point of X, then

$$\begin{aligned} d(g \cdot x, x) &\leq d(g \cdot x, g \cdot x_0) + d(g \cdot x_0, x_0) + d(x_0, x) \\ &\leq M^k(d(x, x_0) + k) + 2k^2 M^k + d(x, x_0) \end{aligned} \tag{3}$$

Whence x's orbit is bounded and thus showing 4.

$4 \Rightarrow 5$: This implication follows from the general fact that any uniformly continuous mapping ϕ on a geodesic space (X,d) is Lipschitz for large distances. To see this, notice that for some $\epsilon > 0$ and all $x, y \in X$,

$$d(x,y) \leq \epsilon \rightarrow d(\phi x, \phi y) \leq 1.$$

So if $d(x,y) \leq N \cdot \epsilon$, there are, since (X,d) is geodesic, $x_0 = x, x_1, \ldots, x_N = y \in X$ with $d(x_i, x_{i+1}) \leq \epsilon$, whence

$$d(\phi x, \phi y) \leq d(\phi x_0, \phi x_1) + d(\phi x_1, \phi x_2) + \ldots + d(\phi x_{N-1}, \phi x_N) \leq N.$$

In other words, for all $x, y \in X$,

$$d(\phi x, \phi y) \leq \left\lceil \frac{d(x,y)}{\epsilon} \right\rceil < \frac{1}{\epsilon} d(x,y) + 1.$$

$5 \Rightarrow 1$: Suppose that G acts by isometries on a metric space (X,d). Now, (X,d) might not be geodesic, but can be extended to a geodesic space as follows:

For any $x, y \in X$, let $\alpha(x,y)$ be a distinct isometric copy of $[0, d(x,y)]$. We let $\tilde{X}$ be the quotient space of $\mathsf{X} = \bigcup_{x,y\in X} \alpha(x,y)$ obtained by for all $x, y, z \in X$ identifying the left endpoints of $\alpha(x,y)$ and $\alpha(x,z)$, identifying the right endpoints of $\alpha(y,x)$ and $\alpha(z,x)$, and identifying the right endpoint of $\alpha(y,x)$ with the left endpoint of $\alpha(x,z)$. We define a metric $\tilde{d}$ on $\tilde{X}$ as follows: For $a \in \alpha(x,y)$ and $b \in \alpha(z,u)$, put

$$\tilde{d}(a,b) = \min \begin{cases} |a-0| + d(x,z) + |0-b| \\ |a-0| + d(x,u) + |d(z,u) - b| \\ |a - d(x,y)| + d(y,z) + |0-b| \\ |a - d(y,u)| + d(y,u) + |d(z,u) - b| \end{cases}$$

Then $(\tilde{X}, \tilde{d})$ contains (X,d) isometrically (sending $x \in X$ to the equivalence class of the unique (end)point of $\alpha(x,x)$). Moreover, the isometric action of G on X extends to $\tilde{X}$ by letting $g \in G$ send $\alpha(x,y)$ isometrically and order-preservingly to $\alpha(gx, gy)$. Thus, G acts by isometries on the geodesic space $\tilde{X}$ and hence, if 5 holds, then every orbit of $\tilde{X}$ and hence every orbit of X is bounded. $\square$

This allows us to give the following nice proof that strong uncountable cofinality is preserved under short exact sequences: Clearly, if $H \trianglelefteq G$ and any action by isometries of G has bounded orbits, then any action by isometries of G/H has bounded orbits. Conversely, assume any action by isometries of G/H and of H has bounded orbits and that G acts by isometries on (X,d). Let $\mathcal{O}$ be the closure of an H orbit in X and let $A = \{g \cdot \mathcal{O} \mid g \in G\}$ be equipped with the Hausdorff metric d_H. Then G acts transitively by isometries on (A, d_H) and the action factors through G/H. Therefore, A is bounded and so any orbit is bounded in X.

3. Polish groups with property (OB)

The preceding section is exclusively concerned with discrete groups, but we shall see that in the case of Polish groups there are again nice equivalent formulations of property (OB). We first notice that property (OB) can be slightly reformulated for Polish groups.

Lemma 3.1. *Let G be a Polish group acting by homeomorphisms on a metrisable space X, such that the mapping $g \in G \mapsto gx \in X$ is continuous for every $x \in X$. Then the action is actually jointly continuous, i.e., $(g,x) \mapsto g \cdot x$ is continuous from $G \times X$ to X.*

Proof. Assume that $g_n \to g$ and $x_n \to x$. Since the mapping $h \in G \mapsto hy \in X$ is continuous for every $y \in X$, we see that $G \cdot y$ is a continuous image of a separable space and thus separable for every y. Hence $Y = G \cdot x \cup \bigcup_n G \cdot x_n$ is an invariant separable subspace of X. Moreover, the action of G on Y is separately continuous, so, as Y is metrisable, the action of G on Y is jointly continuous (Kechris [20] (9.16)). Therefore, $g_n x_n \to gx$ in Y and thus also in X. $\square$

In particular, a Polish group has property (OB) if and only if all of its continuous actions by isometries on separable metric spaces have bounded orbits.

Property (OB) and strong uncountable cofinality fit quite nicely into the well-known hierarchy of group theoretical fixed-point properties such as property (T), (FH), (FA) etc. As first sight they do not appear to be a fixed-point properties, but it all depends on the perspective, as, for example, strong uncountable cofinality is equivalent to a fixed point property for its induced actions on the hyperspace of bounded subsets of any metric space it acts upon.

Definition 3.2. *A group G is said to have* property (FA) *if whenever it acts by automorphisms on a combinatorial tree (i.e., a uniquely path connected graph) it either fixes a vertex or an edge.*

A topological group G has property (topFA) *if whenever it acts by automorphisms on a combinatorial tree, such that the stabilisers of vertices are open, then it fixes either a vertex or an edge.*

A group G is said to have property (algFH) *if whenever it acts by isometries on a real Hilbert space $\mathcal{H}$, then it fixes a vector.*

A topological group G is said to have property (FH) *if whenever it acts by isometries on a real Hilbert space $\mathcal{H}$, such that for all $\xi \in \mathcal{H}$ the mapping $g \in G \mapsto g \cdot \xi \in \mathcal{H}$ is continuous, then it fixes a vector.*

Admittedly, the fixed point property on trees is mainly interesting in its algebraic version, property (FA). Indeed, it is the main object of Serre's book [30] in which he shows that it is equivalent to the conjunction of (i) the group has no infinite cyclic quotients, (ii) the group is not a non-trivial free product with amalgamation and (iii) the group is not the union of a countable chain of proper subgroups. The fixed point property on Hilbert spaces has correspondingly mostly been studied for countable discrete groups (in which case properties (algFH) and (FH) coincide) and for locally compact groups, where one is interested in property (FH). It is also well-known that (algFH) is stronger than (FA) and similarly, (FH) is stronger than (topFA). Moreover, for a group of isometries of Hilbert space to fix a point it is enough that there should be a bounded orbit. This follows from the lemma of the centre (see Bekka, de la Harpe and Valette [5]). So the following proposition sums up the connections between our properties.

Proposition 3.3. *The following diagram of implications holds for topological and abstract groups.*

$$\begin{array}{ccc}
\textit{Strong uncountable cofinality} & \Rightarrow & \textit{Property } (OB) \\
\Downarrow & & \Downarrow \\
\textit{Property } (algFH) & \Rightarrow & \textit{Property } (FH) \\
\Downarrow & & \Downarrow \\
\textit{Property } (FA) & \Rightarrow & \textit{Property } (topFA)
\end{array}$$

Now in turn, we will show the basic equivalences of the different formulations of property (OB) for Polish groups. The following extracts the basic properties of the usual proof of the Birkhoff-Kakutani metrisation theorem, see, e.g., Hjorth [18], Theorem 7.2.

Lemma 3.4. *Let G be a topological group and $(V_n)_{n\in\mathbb{Z}}$ a neighbourhood basis at the identity consisting of open sets such that*
(I) $V_n = V_n^{-1}$,
(II) $G = \bigcup_{n\in\mathbb{Z}} V_n$,
(III) $V_n^3 \subseteq V_{n+1}$.

Let $\delta(g_1, g_2) = \inf(2^n \mid g_1^{-1} g_2 \in V_n)$ *and put*

$$d(g_1, g_2) = \inf(\sum_{i=0}^{k} \delta(h_i, h_{i+1}) \mid h_0 = g_1, h_k = g_2).$$

Then

$$\delta(g_1, g_2) \le 2d(g_1, g_2) \le 2\delta(g_1, g_2) \tag{4}$$

and d *is a left-invariant compatible metric on* G.

Definition 3.5. *Let* G *be a Polish group. We say that* G *is* topologically Bergman *if whenever*

$$B_0 \subseteq B_1 \subseteq B_2 \subseteq \ldots \subseteq G$$

is an exhaustive sequence of subsets with the Baire property, then $G = B_n^k$ *for some* n *and* k. *If there is a* k *which works for all sequences* (B_n), *then we say that* G *is topologically* k*-Bergman.*

Theorem 3.6. *The following are equivalent for a Polish group* G.
(i) G *has property (OB).*
(ii) G *is topologically Bergman.*
(iii) Any compatible left-invariant metric on G *is bounded.*
(iv) G *is finitely generated of bounded width over any non-empty open subset.*

Proof. (iii)⇒(ii): Assume that G is not topologically Bergman as witnessed by some exhaustive sequence of subsets with the Baire property

$$B_0 \subseteq B_1 \subseteq B_2 \subseteq \ldots \subseteq G.$$

By considering $B_0 \cap B_0^{-1} \subseteq B_1 \cap B_1^{-1} \subseteq \ldots$ we can assume that the B_n are symmetric. Then, as G is Polish, some B_n must be non-meagre and contain 1_G, whence $V = \mathrm{int}(B_n^2) \neq \emptyset$ by Pettis' Lemma (see Kechris [20] (9.9)). Thus,

$$VB_n \subseteq VB_{n+1} \subseteq \ldots \subseteq G$$

is an exhaustive sequence of open sets and $(VB_m)^k \subseteq (B_m^3)^k \neq G$ for all $m \geq n$. Put now $V_m = (VB_{n+m})^{3^m}$ and notice that $(V_m)_{m\in\mathbb{N}}$ is an increasing and exhaustive sequence of open neighbourhoods of the identity satisfying $V_m^3 \subseteq V_{m+1}$. Supplementing this sequence with suitable V_m for $m < 0$ we get a neighbourhood basis $(V_m)_{m\in\mathbb{Z}}$ satisfying the conditions (II) and (III) of Lemma 3.4. Also, by replacing V_m by $V_m \cap V_m^{-1}$, we ensure condition (I). Moreover, as $V_m \neq G$ for all G, the resulting metric d is left-invariant, compatible, but unbounded, since in this case δ is unbounded.

The proof of the implication (ii)⇒(i) can be done as in the proof of $3 \Rightarrow 4$ in Theorem 2.1, and that (i) implies (iii) is trivial.

(ii)⇒(iv): If $V \subseteq G$ is non-empty open and $\{g_n\}_{n\in\mathbb{N}}$ is dense in G, then the sequence $B_n = g_0V \cup \ldots \cup g_nV$ is increasing and exhaustive. So, if G is topologically Bergman, then $G = B_n^k$ for some n and k. But then $G = \big(V \cup \{g_0, \ldots, g_n\}\big)^{2k}$, showing that G is finitely generated of bounded width over V.

(iv)⇒(ii): Suppose G is finitely generated of bounded width over any non-empty open set and $B_0 \subseteq B_1 \subseteq \ldots \subseteq G$ is an increasing exhaustive sequence of sets with the Baire property. Then some B_n is non-meagre and $V = \mathrm{int} B_n^2 \neq \emptyset$. Find some $g_0, \ldots, g_m \in G$ and k such that $G = \big(V \cup \{g_0, \ldots, g_m\}\big)^k$. Then $G = \big(B_n^2 \cup \{g_0, \ldots, g_m\}\big)^k \subseteq B_l^{2k}$ for $l \geq n$ large enough such that $1, g_0, \ldots, g_m \in B_l$. □

It was shown in Droste and Holland [12] that a group G has strong uncountable cofinality if and only if G satisfies the conjunction of the following two properties:

- (Uncountable cofinality) Whenever $H_0 \leq H_1 \leq \ldots \leq G = \bigcup_n H_n$, then $G = H_n$ for some n.
- (Cayley boundedness) Whenever $1 \in E = E^{-1}$ generates G then $G = E^n$ for some n.

In the same manner, we can define these concepts for Polish groups.

Definition 3.7. *Let G be a Polish group. We say that G has* uncountable topological cofinality *if G is not the union of a chain of proper open subgroups (or equivalently, a countable chain of subgroups with the Baire property). G is* topologically Cayley bounded *if it has finite width with respect to any analytic generating set.*

Proposition 3.8. *A Polish group G has property (OB) if and only if it has uncountable topological cofinality and is topologically Cayley bounded.*

Proof. Suppose G has property (OB), $H_0 \leq H_1 \leq \ldots, \leq G = \bigcup_n H_n$ are open subgroups and $1 \in E = E^{-1}$ is an analytic set generating G. Then by property (OB) applied to the sequences $H_0 \subseteq H_1 \subseteq \ldots \subseteq G$ and $E \subseteq E^2 \subseteq \ldots \subseteq G$, we see that $G = H_n^k = H_n = (E^n)^k = E^{nk}$ for some n and k. Thus, G has both uncountable topological cofinality and is topologically Cayley bounded.

Conversely, suppose G has uncountable topological cofinality, is topologically Cayley bounded and $W_0 \subseteq W_1 \subseteq \ldots \subseteq G = \bigcup_n W_n$ are sets with the Baire property. By considering instead a tail subsequence of the exhaustive sequence

$$W_0 \cap W_0^{-1} \subseteq W_1 \cap W_1^{-1} \subseteq \ldots \subseteq G$$

we can suppose each W_n is non-meagre, symmetric and contains 1. Thus, the sequence $\langle W_0 \rangle \leq \langle W_1 \rangle \leq \ldots \leq G$ consists of open subgroups and hence one of the W_n generates G. As W_n is symmetric and non-meagre, $\mathrm{int} W_n^2$ is symmetric and non-empty, so $W_n \cdot \mathrm{int} W_n^2 \cdot W_n \subseteq W_n^4$ is a symmetric generating open subset of G containing 1. So $G = W_n^k$ for some k. □

In the case of locally compact groups, uncountable topological cofinality is clearly equivalent to compact generation. Moreover, if a locally compact, compactly generated group is also topologically Cayley bounded, then its compact generating set generates by a finite power and hence the group is compact. Conversely, compact groups trivially have property (OB). So property (OB) for locally compact Polish groups is just equivalent with compactness, just as strong uncountable cofinality for countable groups is equivalent with finiteness. However, we can actually provide a bit more information.

Proposition 3.9. *A compact Polish group is topologically 2-Bergman.*

Proof. Assume that G is compact and that

$$B_0 \subseteq B_1 \subseteq B_2 \subseteq \ldots \subseteq G$$

is an exhaustive sequence of subsets with the Baire property. Then there is some B_{n_0} which is non-meagre and hence comeagre in some open set Vf, where V is an open neighbourhood of the identity. Pick some symmetric open set $U \subseteq V$ such that $U^2 \subseteq V$ and $g_1, \ldots, g_m \in G$ such that $G = g_1 U \cup \ldots \cup g_m U$. Then if $h_i \in g_i U$, we have $g_i U = h_i(h_i^{-1} g_i) U \subseteq h_i U^2 \subseteq h_i V$ and thus $G = h_1 U^2 \cup \ldots \cup h_m U^2 = h_1 V \cup \ldots \cup h_m V$. Considering now the sequences $(B_j \cap g_i U)_j$ for each $i = 1, \ldots, m$, we find $n_1 \geq n_0$ such that B_{n_1} is non-meagre in each of the $g_i U$. Hence, we can find open sets $W_i \subseteq g_i U$ such that B_{n_1} is comeagre in W_i for each i. Pick now $h_i \in W_i \cap B_{n_1} \subseteq g_i U$ and notice that $G = Gf = h_1 V f \cup \ldots \cup h_m V f$. But as B_{n_1} is comeagre in both W_i and Vf, we have by Pettis' Lemma $h_i V f \subseteq W_i V f \subseteq B_{n_1}^2$. Thus, $G = B_{n_1}^2$, showing that G is topologically 2-Bergman. □

As a locally compact, non-compact Polish group cannot have property (OB), we know that it must have a compatible left-invariant unbounded metric. But actually we can see that this metric can be chosen to be *proper*, i.e., such that any bounded closed set is compact.

Proposition 3.10 (Folklore)**.** *Let G be a locally compact Polish group. Then G admits a left-invariant compatible proper metric.*

Proof. We start by fixing an open neighbourhood basis at the identity $(U_n)_{n\in\mathbb{N}}$ such that $U_{n+1}^3 \subseteq U_n$, $U_n = U_n^{-1}$ and U_0 being relatively compact. Since G is σ-compact we can also find an increasing sequence of symmetric relatively compact sets $(V_n)_{n\in\mathbb{N}}$ such that $V_0 = U_0$, $V_n^3 \subseteq V_{n+1}$ and $G = \bigcup_{n\in\mathbb{N}} V_n$. Letting now $V_{-n} = U_n$, we see that the sequence $(V_n)_{n\in\mathbb{Z}}$ satisfies the conditions of Lemma 3.4. Let now d be the metric given by the lemma, we claim that d is proper. For any d-bounded set A is δ-bounded and thus there is some n such that for any $g, h \in A$, $g^{-1}h \in V_n$. In particular, A is contained in some translate of V_n and thus relatively compact. Hence if A is closed it is compact, showing that d is proper. □

We should mention that locally compact Polish groups have *complete* left-invariant metrics and hence every left-invariant metric is complete, see Becker [2], section 3.

Remark : There are examples of compact Polish groups not having strong uncountable cofinality. In fact, Koppelberg and Tits [23] prove that if F is a finite non-trivial group, then $F^{\mathbb{N}}$ has uncountable cofinality if and only if F is perfect. So we have examples of profinite groups without uncountable cofinality.

We also see that the two properties (FH) and (OB) do not coincide. For there are plenty of examples of locally compact, non-compact Polish groups with property (FH), but of course without property (OB).

Definition 3.11. *Recall that a mapping f between metric spaces (X, d) and (Y, δ) is called a Hölder(α) map for some $\alpha > 0$ if there is a constant $c \geq 1$ such that*

$$\delta(f(x), f(y)) \leq c \cdot d(x, y)^\alpha$$

for all $x, y \in X$. Hölder(1) mappings are thus simply Lipschitz mappings.

Proposition 3.12. *Let G be a group with strong uncountable cofinality acting by Hölder maps on a metric space (X, d). Then the action of G is semi-simple, i.e., $\{\overline{G \cdot x}\}_{x\in X}$ partitions X into (bounded) invariant pieces each on which G acts minimally. Moreover, there is an N such that any $g \in G$ is Hölder(α) with constant N for some $\alpha \in [1/N, N]$.*

The same holds for Polish groups with property (OB) acting continuously and by Hölder maps on a Polish metric space (X, d).

Proof. First assume that G has strong uncountable cofinality. For each $g \in G$ let $\alpha_g \geq 0$ and $c_g \geq 1$ be such that g is Hölder(α_g) with constant c_g. Thus,

$$\begin{aligned} d(g_1 \cdot \ldots \cdot g_n x, g_1 \cdot \ldots \cdot g_n y) &\leq c_{g_1} d(g_2 \cdot \ldots \cdot g_n x, g_2 \cdot \ldots \cdot g_n y)^{\alpha_{g_1}} \\ &\leq c_{g_1} c_{g_2}^{\alpha_{g_1}} d(g_3 \cdot \ldots \cdot g_n x, g_3 \cdot \ldots \cdot g_n y)^{\alpha_{g_1}\alpha_{g_2}} \\ &\leq \ldots \\ &\leq c_{g_1} c_{g_2}^{\alpha_{g_1}} \ldots c_{g_n}^{\alpha_{g_1}\ldots\alpha_{g_{n-1}}} d(x, y)^{\alpha_{g_1}\ldots\alpha_{g_n}}. \end{aligned} \tag{5}$$

Put now $W_n = \{g \in G \mid c_g \leq n \ \& \ \alpha_g \in [1/n, n]\}$ and notice that the sequence W_n is increasing and exhaustive. By strong uncountable cofinality, $G = W_n^k$ for some n and k. By the inequality 5, we see that there is a fixed N such that any $g \in G$ is Hölder(α) with constant N for some $\alpha \in [1/N, N]$. Thus, we have

$$\forall \epsilon > 0 \ \exists \delta > 0 \ \forall x, y \ \forall g \ \ \big(d(x, y) < \delta \to d(gx, gy) < \epsilon\big). \tag{6}$$

So suppose $x, y \in X$ and that $y \in \overline{G \cdot x}$. Then for any $x' \in G \cdot x$ and $\epsilon > 0$, we can find $\delta > 0$ as above and $x'' \in G \cdot x$ with $d(x'', y) < \delta$. Now, if $x' = gx''$, then $d(x', gy) = d(gx'', gy) < \epsilon$, so $x' \in \overline{G \cdot y}$, showing that $\overline{G \cdot x} = \overline{G \cdot y}$. Thus, in $\overline{G \cdot x}$ every orbit is dense, and hence if $u \in \overline{G \cdot x} \cap \overline{G \cdot z}$ for any u, z, then $\overline{G \cdot x} = \overline{G \cdot z} = \overline{G \cdot u}$. Therefore, $\{\overline{G \cdot x}\}_{x \in X}$ partitions X and G acts minimally on each piece of the partition. The usual argument, as in the proof of Theorem 2.1, will also show that every orbit is bounded.

Now we only have to indicate the proof in the case that G is a Polish group with property (OB). In this case we fix a countable dense set $\{x_m\}$ in X and define W_n by

$$W_n = \{g \in G \mid \exists \alpha \in [1/n, n] \, \forall m, l \;\; d(gx_m, gx_l) \leq n \cdot d(x_m, x_l)^{\alpha}\}.$$

Then W_n is analytic (in fact closed) and hence has the Baire property. Notice also that if $g \in W_n$ then g is indeed Hölder(α) with constant n for some $\alpha \in [1/n, n]$. We can thus proceed as before using that G is topologically Bergman. □

In order to see that the result is not void, we can exhibit an action of $\mathbb{Z}$ by Lipschitz isomorphisms of $\mathbb{R}$ such that $\{\overline{\mathbb{Z} \cdot x}\}_{x \in \mathbb{R}}$ does not partition $\mathbb{R}$. Namely, let $T(x) = 2x$, whence $T^n(x) = 2^n x$ for all $n \in \mathbb{Z}$. Then T is a simple dilation of $\mathbb{R}$ and $0 \in \overline{\mathbb{Z} \cdot x}$ for any $x \in \mathbb{R}$.

We thus see from statement 6 that a group with strong uncountable cofinality acting by Hölder maps actually acts equicontinuously. One might wonder if this also holds if the group acts by, e.g., uniformly continuous homeomorphisms, but this is false. For example, $\mathcal{S}_\infty$ acts continuously on $2^{\mathbb{N}}$, and thus by uniformly continuous homeomorphisms, but the action fails to be equicontinuous. Moreover, there is no decomposition of $2^{\mathbb{N}}$ into closed minimal pieces.

4. CLOSURE PROPERTIES

Proposition 4.1. *Let G be Polish and $H \trianglelefteq G$ a closed subgroup. If both G/H and H have property (OB), so does G. Conversely, if G has property (OB) and $\phi : G \to K$ is a continuous homomorphism into a Polish group K with dense image, then K has property (OB).*

Proof. Let $V \subseteq G$ be an open neighbourhood of the identity in G. Then $U = H \cap V$ is non-empty open in H, whence for some finite set $A \subseteq H$ and $n \in \mathbb{N}$, $(UA)^n = H$. Therefore, $(VA)^n \supseteq H$. Since the quotient mapping $\pi : G \to G/H$ is continuous and open, $\pi[V] \subseteq G/H$ is open, non-empty, so for some finite set $B \subseteq G$ and $m \in \mathbb{N}$, $(\pi[V]\pi[B])^m = G/H$. Thus,

$$\forall g \in G \, \exists v_1, \ldots, v_m \in V \, \exists b_1, \ldots, b_m \in B \;\; \big(\pi(g) = \pi(v_1 b_1 \cdots v_m b_m)\big) \tag{7}$$

and

$$\begin{aligned} &\forall g \in G \, \exists v_1, \ldots, v_m \in V \, \exists b_1, \ldots, b_m \in B \, \exists v_1', \ldots, v_n' \in V \\ &\exists a_1, \ldots, a_n \in A \;\; g = v_1 b_1 \cdots v_m b_m v_1' a_1 \cdots v_n' a_n. \end{aligned} \tag{8}$$

So $G = (VB)^m (VA)^n$, showing that G has property (OB).

Now, if G has property (OB) and K acts continuously by isometries on a metric space, then this induces an action by G. Thus, every G-orbit is bounded and as $\phi(G)$ is dense in K, every K-orbit is bounded. □

Proposition 4.2. *Suppose $\{G_n\}_{\mathbb{N}}$ are Polish groups. Then $G = \prod_{\mathbb{N}} G_n$ has property (OB) if and only if each G_n has property (OB).*

Proof. Suppose each G_n has property (OB) and assume $W_0 \leq W_1 \leq \ldots \leq G$ is an exhaustive sequence of subsets with the Baire property. As in the proof of Proposition 3.8 we can suppose that each W_n is a symmetric open neighbourhood of the identity in G.

Thus, there is $n \in \mathbb{N}$ such that

$$\{1\} \times \ldots \times \{1\} \times \prod_{i>n} G_i \subseteq W_0$$

and thus to prove that $G = W_n^k$ for some n and k, it is enough to prove that $G_0 \times \ldots \times G_n$ is contained in some V_m^l, where

$$V_i = \{(g_0, \ldots, g_n) \in G_0 \times \ldots \times G_n \bigm| (g_0, \ldots, g_n, 1, 1, \ldots) \in W_i\}.$$

But $V_0 \subseteq V_1 \subseteq \ldots \subseteq G_0 \times \ldots \times G_n$ is an increasing exhaustive sequence of open subsets, and as $G_0 \times \ldots \times G_n$ has property (OB) by Proposition 4.1, the result follows. The other direction follows by Proposition 4.1. □

In [6] Bergman poses the problem of whether strong uncountable cofinality passes from a group to a subgroup of finite index. In an announcement [22] A. Khélif states that this is indeed the case. We shall see that the concept of induced representations also leads to this result and, moreover, also solves the corresponding problem for Polish groups.

Proposition 4.3. *Let G be a Polish group and $H \leq G$ a finite index closed subgroup. Then G has property (OB) if and only if H has.*

Proof. First the easy direction. Assume H has property (OB). Then if G acts continuously by isometries on some space (X, d), so does H and this latter has bounded orbits. Letting $g_1 \ \ldots, g_n$ be representatives for the left cosets of H in G, we see that $G \cdot x = \bigcup_i g_i H \cdot x$, which is a finite union of bounded sets, and thus bounded.

For the other direction, consider first the abstract case of two groups G and H with H a finite index subgroup of G. Fix a transversal $1 \in T \subseteq G$ for the left cosets of H in G. Now assume that H acts by isometries on a metric space (X, d). We define

$$Y = \{\xi : G \to X \bigm| \forall g \in G\ \forall h \in H\ \xi(gh) = h^{-1}\xi(g)\}$$

For example, if $x_0 \in X$ is some fixed element, we can define $\xi_0 : G \to X$ by $\xi_0(ah) = h^{-1}x_0$ for all $h \in H$ and all $a \in T$. Then clearly $\xi_0 \in Y$. So Y is non-empty.

We can now define the following metric ∂ on Y: $\partial(\xi, \zeta) = \sup_{g \in G} d(\xi(g), \zeta(g))$. If we can show that the supremum is finite, then this is clearly a metric. But

$$\begin{aligned}
\partial(\xi, \zeta) &= \sup_{g \in G} d(\xi(g), \zeta(g)) \\
&= \sup_{a \in T, h \in H} d(\xi(ah), \zeta(ah)) \\
&= \sup_{a \in T, h \in H} d(h^{-1}\xi(a), h^{-1}\zeta(a)) \\
&= \sup_{a \in T, h \in H} d(\xi(a), \zeta(a)) \\
&= \sup_{a \in T} d(\xi(a), \zeta(a)) \\
&< \infty,
\end{aligned} \tag{9}$$

where the last inequality holds since T is finite. Now, let G act on Y by left translation

$$(g \cdot \xi)(f) = \xi(g^{-1}f)$$

This is an action by isometries

$$\begin{aligned}\partial(g\cdot\xi, g\cdot\zeta) &= \sup_{f\in G} d((g\cdot\xi)(f),(g\cdot\zeta)(f))\\ &= \sup_{f\in G} d(\xi(g^{-1}f),\zeta(g^{-1}f))\\ &= \sup_{f'\in G} d(\xi(f'),\zeta(f'))\\ &= \partial(\xi,\zeta).\end{aligned}\tag{10}$$

Now, if G has strong uncountable cofinality, there is a bounded orbit $G\cdot\xi$ in Y. We now only need to see how this gives rise to a bounded orbit for H in X, whereby, of course, all orbits will be bounded. So let $x_0=\xi(1)$ and notice that for $h\in H$

$$\begin{aligned}d(x_0, h\cdot x_0) &= d(\xi(1), h\cdot\xi(1))\\ &= d(\xi(1),\xi(h^{-1}))\\ &= d(\xi(1),(h\cdot\xi)(1))\\ &\leq \sup_{g\in G} d(\xi(g),(h\cdot\xi)(g))\\ &= \partial(\xi, h\cdot\xi)\\ &\leq \mathrm{diam}_\partial(G\cdot\xi).\end{aligned}\tag{11}$$

This shows that strong uncountable cofinality passes to subgroups of finite index.

For the case of Polish groups G and H, with H being a finite index closed and thus clopen subgroup, we of course restrict our attention to continuous ξ. Again we see that $\xi_0\in Y\neq\emptyset$. We claim that the action of G on Y is separately continuous. In the second variable this is trivial, as G acts by isometries. On the other hand, if we fix some $\xi\in Y$ and suppose that $g_n\to g$ in G, then by Equation 9

$$\partial(g_n\cdot\xi, g\cdot\xi) = \sup_{a\in T} d((g_n\cdot\xi)(a),(g\cdot\xi)(a)) = \sup_{a\in T} d(\xi(g_n^{-1}a),\xi(g^{-1}a)) \underset{n\to\infty}{\longrightarrow} 0.\tag{12}$$

Thus, by Lemma 3.1, the action of G on Y is continuous and we can finish the proof as in the discrete case. □

5. Groups of isometries

Definition 5.1. *Let (X,d) be a metric space and $G\leq\mathrm{Iso}(X,d)$. We let d_∞ denote the supremum metric on the spaces X^m induced by d. The group G is said to be* approximately oligomorphic *if for any $n\geq 1$ and $\epsilon>0$ there is a finite set $A\subseteq X^n$ such that $G\cdot A$ is ϵ-dense in X^n with respect to d_∞.*

Theorem 5.2. *Let (X,d) be a Polish metric space and G a closed subgroup of $\mathrm{Iso}(X,d)$ with the topology of pointwise convergence. If G is approximately oligomorphic, then G has property (OB).*

Proof. We need to show that G is finitely generated of bounded width over any non-empty open set $V\subseteq G$. So find $\overline{x}=(x_1,\dots,x_n)\in X^n$ and $\epsilon>0$ such that

$$U=\{g\in G \mid \forall i\leq n\ d(x_i,gx_i)<\epsilon\}\subseteq VV^{-1}.$$

We claim that there is a finite set $B\subseteq X^n$ such that $U\cdot B$ is $\frac{\epsilon}{2}$-dense in X^n. To see this, let $A\subseteq X^n\times X^n$ be a finite set such that $G\cdot A$ is $\frac{\epsilon}{2}$-dense in $X^n\times X^n$. Define $A'\subseteq A$

to be the set of $\overline{a} = (\overline{a}_1, \overline{a}_2) \in A$ such that for some $g_{\overline{a}} \in G$, $d_\infty(\overline{x}, g_{\overline{a}}\overline{a}_1) < \frac{\epsilon}{2}$. Finally, put $B = \{g_{\overline{a}}\overline{a}_2 \mid \overline{a} \in A'\}$.

Then, if $\overline{c} \in X^n$, there are $\overline{a} = (\overline{a}_1, \overline{a}_2) \in A$ and $g \in G$ such that

$$d_\infty\big((\overline{x}, \overline{c}), (g\overline{a}_1, g\overline{a}_2)\big) < \frac{\epsilon}{2}.$$

In particular, $\overline{a} \in A'$ and thus

$$\begin{aligned} d_\infty(gg_{\overline{a}}^{-1}\overline{x}, \overline{x}) &= d_\infty(g_{\overline{a}}^{-1}\overline{x}, g^{-1}\overline{x}) \\ &\leq d_\infty(g_{\overline{a}}^{-1}\overline{x}, \overline{a}_1) + d_\infty(\overline{a}_1, g^{-1}\overline{x}) \\ &= d_\infty(\overline{x}, g_{\overline{a}}\overline{a}_1) + d_\infty(g\overline{a}_1, \overline{x}) \\ &< \epsilon. \end{aligned} \tag{13}$$

So $gg_{\overline{a}}^{-1} \in U$ and

$$d_\infty(\overline{c}, gg_{\overline{a}}^{-1} \cdot g_{\overline{a}}\overline{a}_2) = d_\infty(\overline{c}, g\overline{a}_2) < \frac{\epsilon}{2}.$$

Thus, $U \cdot B$ is $\frac{\epsilon}{2}$-dense in X^n. Let now $B' \subseteq B$ be the set of $\overline{b} \in B$ such that

$$d_\infty(U \cdot \overline{b}, G \cdot \overline{x}) < \frac{\epsilon}{2}.$$

So for some $h_{\overline{b}} \in G$ and $g_{\overline{b}} \in U$,

$$d_\infty(g_{\overline{b}}\overline{b}, h_{\overline{b}}\overline{x}) < \frac{\epsilon}{2}.$$

Then, if $f \in G$, we can find $\overline{b} \in B$ and $g \in U$ such that $d_\infty(f\overline{x}, g\overline{b}) < \frac{\epsilon}{2}$. Thus, $\overline{b} \in B'$ and

$$\begin{aligned} d_\infty(\overline{x}, f^{-1}gg_{\overline{b}}^{-1}h_{\overline{b}}\overline{x}) &= d_\infty(g^{-1}f\overline{x}, g_{\overline{b}}^{-1}h_{\overline{b}}\overline{x}) \\ &\leq d_\infty(g^{-1}f\overline{x}, \overline{b}) + d_\infty(\overline{b}, g_{\overline{b}}^{-1}h_{\overline{b}}\overline{x}) \\ &= d_\infty(f\overline{x}, g\overline{b}) + d_\infty(g_{\overline{b}}\overline{b}, h_{\overline{b}}\overline{x}) \\ &< \epsilon. \end{aligned} \tag{14}$$

So $f^{-1}gg_{\overline{b}}^{-1}h_{\overline{b}} \in U$ and if $H = \{h_{\overline{b}} \mid \overline{b} \in B'\}$, we see that $G = UU^{-1}HU^{-1}$. $\square$

Corollary 5.3. *Let G be an oligomorphic closed subgroup of $\mathcal{S}_\infty$. Then G has property (OB).*

Proof. Notice that if we let $\mathbb{N}$ have the metric in which all points have distance 1, then G is approximately oligomorphic as a group of isometries exactly when it is oligomorphic. $\square$

The *Urysohn metric space* $\mathbb{U}$ is the unique separable complete metric space containing each finite metric space and such that any isometry between finite subsets extends to a full isometry of the space. This space, constructed by Urysohn [37] is also characterised by being separable, complete and satisfying the following *extension property*:

> If $\phi : X \to \mathbb{U}$ is an isometric embedding of a finite metric space X into $\mathbb{U}$ and $Y = X \cup \{y\}$ is a one point metric extension of X, then ϕ extends to an isometric embedding of Y.

In the same manner, there is a Urysohn metric space of diameter 1, designated by $\mathbb{U}_1$, which is the unique complete separable metric space whose diameter is at most 1 and satisfying the extension property, when Y varies over metric spaces of diameter at most 1.

Similarly, one can construct variants of the Urysohn metric space, where the metric takes values only in $\mathbb{Q} \cap [0, 1]$. Thus, the *rational Urysohn metric space of diameter* 1, denoted by Ω, is the unique countable metric space whose metric takes values in $\mathbb{Q} \cap [0,1]$ and satisfying the extension property for Y, whose metric also takes values in $\mathbb{Q} \cap [0,1]$.

Theorem 5.4. *Let $\mathbb{U}_1$ be the Urysohn metric space of diameter* 1. $\mathrm{Iso}(\mathbb{U}_1)$ *is approximately oligomorphic and hence has property (OB).*

For this proof we need some notions of metric theory. Let D_n be the set of $n \times n$ matrices $[a_{ij}]$ with entries in $[0,1]$ such that $d(i,j) = a_{ij}$ defines a pre-metric on $\{1,\dots,n\}$. Consider D_n as a subset of $[0,1]^{n^2}$ with the supremum metric d_∞. Clearly, the triangle inequality is a closed condition, so D_n is compact.

We define also the following distance d_1 on D_n à la Gromov and Hausdorff (see Chapter 3, Gromov [14]):

$$d_1(A,B) = \min\big(\mathrm{trace}(E) \mid \begin{bmatrix} A & E \\ E^t & B \end{bmatrix} \in D_{2n}\big)$$

Notice that the infimum is indeed attained, as we are minimising over a compact space. So if $A, B \in D_n$ are representing pre-metrics a and b on $\{1,\dots,n\}$ and $\{1',\dots,n'\}$ respectively (thus of diameter at most 1), d_1 is the minimum of $\sum_i c(i,i')$, where c varies over all pre-metrics on $\{1,\dots,n,1',\dots,n'\}$ of diameter at most 1 agreeing with a on $\{1,\dots,n\}$ and with b on $\{1',\dots,n'\}$. Therefore, d_1 measures how far the spaces have to be from each other, when they are both embedded into a metric space of diameter at most 1.

Lemma 5.5. $2d_1 \leq nd_\infty \leq nd_1$.

Proof. Let $A, B \in D_n$ and let a and b be the corresponding pre-metrics on $\{1,\dots,n\}$. Assume that

$$\delta = d_\infty(A,B) = \sup_{i,j} |a(i,j) - b(i,j)|$$

and let c be defined on $\{1,\dots,n,1',\dots,n'\}$ by

$$\begin{aligned} c(i,j) =& a(i,j), \\ c(i',j') =& b(i,j), \\ c(i,j') = c(j',i) =& \min_l \big(a(i,l) + \delta/2 + b(l,j)\big). \end{aligned}$$

We claim that c is a pre-metric and that $i \mapsto i$ and $i \mapsto i'$ are isometric embeddings of the spaces given by a and b respectively. Clearly, the triangle inequality is satisfied separately on $\{1,\dots,n\}$ and on $\{1',\dots,n'\}$, and

$$\begin{aligned} c(i,k') + c(k',j) &= \min_l \big(a(i,l) + \delta/2 + b(l,k)\big) + \min_p \big(b(k,p) + \delta/2 + a(p,j)\big) \\ &= \delta + \min_l \big(a(i,l) + b(l,k)\big) + \min_p \big(b(k,p) + a(p,j)\big) \\ &\geq \delta + \min_{l,p} \big(a(i,l) + b(l,p) + a(p,j)\big) \\ &\geq \min_{l,p} \big(a(i,l) + a(l,p) + a(p,j)\big) \\ &\geq a(i,j) \\ &= c(i,j). \end{aligned} \tag{15}$$

Similarly, $c(i',j') \leq c(i',l) + c(l,j')$. And

$$\begin{aligned} c(i,j') &\leq \min_k \big(a(i,k) + \delta/2 + b(k,j)\big) \\ &\leq a(i,l) + \min_k \big(a(l,k) + \delta/2 + b(k,j)\big) \qquad (16) \\ &= c(i,l) + c(l,j'). \end{aligned}$$

Similarly, $c(i,j') \leq c(i,l') + c(l',j')$, so the triangle inequality is verified. Unfortunately, c does not necessarily have diameter bounded by 1, but this can be remedied by letting $c'(x,y) = \min\{c(x,y),1\}$. Clearly, this does not affect the distances on $\{1,\dots,n\}$ and $\{1',\dots,n'\}$ separately, and only decreases other distances. So $c'(i,i') = \frac{\delta}{2}$. Let now

$$C' = \begin{bmatrix} A & E \\ E^t & B \end{bmatrix} \in D_{2n}$$

be the matrix corresponding to c', and notice that

$$d_1(A,B) \leq \mathrm{trace}(E) = n\frac{\delta}{2} = \frac{n}{2} d_\infty(A,B).$$

Thus, $d_1 \leq \frac{n}{2} d_\infty$.

On the other hand, if the two sets $\{1,\dots,n\}$ and $\{1',\dots,n'\}$ are very close to each other, pointwise, in some common metric space, then the distance between i and j cannot differ very much from the distance between i' and j'. And in fact, $d_\infty \leq d_1$. $\square$

Proof of Theorem 5.4: Fix some $n \geq 1$ and $\epsilon > 0$ and let ∂ be the metric on $\mathbb{U}_1$. As D_n is compact, we can find some finite $\mathcal{A} \subseteq D_n$, which is ϵ-dense in the metric d_1. By the universality property of the Urysohn metric space $\mathbb{U}_1$, this means that for any $\overline{x} = (x_1,\dots,x_n) \in \mathbb{U}_1^n$, there is $\overline{y} = (y_1,\dots,y_n) \in \mathbb{U}_1^n$ with distance matrix $A \in \mathcal{A}$, such that $\partial_\infty(\overline{x},\overline{y}) \leq \sum_i \partial(x_i,y_i) \leq \epsilon$. So pick for each $A \in \mathcal{A}$ some $\overline{z} \in \mathbb{U}_1^n$ with distance matrix A, and let $\mathbb{A}$ be the set of these. Then, if $\overline{x} = (x_1,\dots,x_n) \in \mathbb{U}_1^n$ there is $\overline{y} = (y_1,\dots,y_n)$ as above, and hence some $\overline{z} = (z_1,\dots,z_n) \in \mathbb{A}$ isometric to $\overline{y}$. But then as $\mathbb{U}_1$ is ultrahomogeneous, we see that $\overline{y}$ and $\overline{z}$ are in the same orbit of $\mathrm{Iso}(\mathbb{U}_1)$, showing that $\mathrm{Iso}(\mathbb{U}_1)$ is approximately oligomorphic. $\square$

We will now show that if we consider the isometry group of the rational Urysohn metric space of diameter 1, Ω, then we actually get strong uncountable cofinality outright. The results here were finally clear after a late night discussion with Stevo Todorčević.

Two tuples $\overline{x}$ and $\overline{y}$ in Ω are said to be *uniformly of distance 1 from each other* if $d(x_i,y_j) = 1$ for all i,j.

Lemma 5.6. *If $\overline{x}$ and $\overline{y}$ in Ω are uniformly of distance 1 and some $\overline{z}$ in Ω is given, then there are $\overline{x}'$ and $\overline{z}'$ such that $(\overline{x},\overline{z},\overline{y})$ and $(\overline{x}',\overline{z}',\overline{y})$ are isometric and $\overline{x}$ is uniformly of distance 1 from both $\overline{x}'$ and $\overline{z}'$.*

Proof. Notice that the distances between $\overline{x},\overline{y},\overline{z}',\overline{x}'$ are completely specified by the lemma, so we need only specify the distances between $\overline{z}$ and $(\overline{x}',\overline{z}')$. We let $\overline{z}$ be uniformly of distance 1 from $\overline{x}'$ and put

$$d(z_i,z_j') = \min\{1, \inf_{y_l} d(z_i,y_l) + d(y_l,z_j)\}.$$

The triangle inequality holds, which can be checked by hand, so let us just give a few representative cases.

- Clearly, $d(x_i', y_j) \leq d(x_i', v) + d(v, y_j)$ for all $v \in \overline{x}, \overline{z}, \overline{y}$, since $\overline{x}'$ is uniformly of distance 1 from all of $\overline{x}, \overline{z}, \overline{y}$. Moreover, it also holds for $v \in \overline{x}', \overline{z}'$, since $(\overline{x}', \overline{z}', \overline{y})$ is isometric to $(\overline{x}, \overline{z}, \overline{y})$ and thus is a metric space.

- Clearly, for all $v \in \overline{x}, \overline{y}, \overline{x}', \overline{z}'$, $d(x_i, z_j') \leq d(x_i, v) + d(v, z_j')$, since in this case one of the distances on the right hand side must equal 1. So for $v = z_j$ we have

$$\begin{aligned} d(x_i, z_k) + d(z_k, z_j') &\geq \min\{1, \inf_{y_l} d(x_i, z_k) + d(z_k, y_l) + d(y_l, z_j)\} \\ &\geq \min\{1, \inf_{y_l} d(x_i, y_l) + d(y_l, z_j)\} \\ &\geq 1 \\ &= d(x_i, z_j'). \end{aligned} \tag{17}$$

- Clearly, for all $v \in \overline{x}, \overline{x}', \overline{y}, \overline{z}$, $d(z_i, y_j) \leq d(z_i, v) + d(v, y_j)$, since in the first two cases one of the distances on the right hand side must equal 1 and in the last two cases it reduces to the triangle inequality on $(\overline{z}, \overline{y})$. And for $v = z_j'$ we have

$$\begin{aligned} d(z_i, z_k') + d(z_k', y_j) &\geq \min\{1, \inf_{y_l} d(z_i, y_l) + d(y_l, z_k) + d(z_k, y_j)\} \\ &\geq \min\{1, d(z_i, y_j)\} \\ &\geq d(z_i, y_j). \end{aligned} \tag{18}$$

□

Lemma 5.7. *Assume that $\overline{x}$ is a tuple in Ω. Then there is some $l \in \mathrm{Iso}(\Omega)$ such that*

$$\mathrm{Iso}(\Omega) = \big(l \cdot \mathrm{Iso}(\Omega, \overline{x})\big)^4$$

Proof. Find some $\overline{y}$, isometric to $\overline{x}$ and uniformly of distance 1 from it. Let $l(\overline{x}) = \overline{y}$ and $l(\overline{y}) = \overline{x}$. Then $l \cdot \mathrm{Iso}(\Omega, \overline{x}) \cdot l = \mathrm{Iso}(\Omega, \overline{y})$. Let $g \in \mathrm{Iso}(\Omega)$ be any element and put $\overline{z} = g(\overline{x})$.

By Lemma 5.6, we can find $\overline{x}', \overline{z}'$ such that $(\overline{x}, \overline{z}, \overline{y})$ and $(\overline{x}', \overline{z}', \overline{y})$ are isometric and $\overline{x}$ is uniformly of distance 1 from both $\overline{x}'$ and $\overline{z}'$. Thus, there is some $h \in \mathrm{Iso}(\Omega, \overline{y})$ such that $h(\overline{x}) = \overline{x}'$ and $h(\overline{z}) = \overline{z}'$. Now, since $(\overline{x}, \overline{z}')$ and $(\overline{x}, \overline{x}')$ are isometric, there is some $f \in \mathrm{Iso}(\Omega, \overline{x})$ such that $f(\overline{z}') = \overline{x}'$. And finally, as $(\overline{y}, \overline{x}')$ and $(\overline{y}, \overline{x})$ are isometric, we can find $k \in \mathrm{Iso}(\Omega, \overline{y})$ such that $k(\overline{x}') = \overline{x}$.

Therefore, $kfhg(\overline{x}) = kfh(\overline{z}) = kf(\overline{z}') = k(\overline{x}') = \overline{x}$ and $kfhg \in \mathrm{Iso}(\Omega, \overline{x})$. So

$$g = h^{-1}f^{-1}k^{-1}(kfhg) \in \big(\mathrm{Iso}(\Omega, \overline{y}) \cdot \mathrm{Iso}(\Omega, \overline{x})\big)^2 = \big(l \cdot \mathrm{Iso}(\Omega, \overline{x}) \cdot l \cdot \mathrm{Iso}(\Omega, \overline{x})\big)^2.$$

□

Theorem 5.8. *The isometry group of the rational Urysohn metric space of diameter 1, $\mathrm{Iso}(\Omega)$, has strong uncountable cofinality.*

Proof. The proof relies on the result of S. Solecki [32], also independently announced by A.M. Vershik, that for any finite rational metric space X there is another finite rational metric space Y containing X and such that any partial isometry of X extends to a full isometry of Y.

First of all, we notice that this also implies the corresponding result for rational metric spaces of bounded diameter 1. For if X is of bounded diameter 1, then we find first some Y' (not necessarily of bounded diameter 1) extending Xsuch that every partial isometry of X extends to a full isometry of Y'. Now, if d' is the metric on Y', let d be the metric given by $d(y_0, y_1) = \min\{1, d'(y_0, y_1)\}$ and let Y be the metric space obtained. Then we see that the distances between points in X are preserved and thus X is still a subspace of Y, and if f is an isometry of Y' it is also an isometry of Y. Thus, every partial isometry of X extends to a full isometry of Y.

We now need the following concept, which will also be used in a later section.

Definition 5.9. *Suppose G is a Polish group and consider for each finite $m \geq 1$ the diagonal conjugacy action of G on G^m given by*

$$g \cdot (h_1, \ldots, h_m) = (gh_1g^{-1}, \ldots, gh_mg^{-1}).$$

G is said to have ample generics *if for each $m \geq 1$ there is a comeagre orbit in G^m for this action.*

Notice that since Ω is countable, $\mathrm{Iso}(\Omega)$ is a Polish group in the permutation group topology. In section 6 of Kechris and Rosendal [21] it is shown how the above extension property for finite rational metric spaces of bounded diameter 1 implies that $\mathrm{Iso}(\Omega)$ has ample generics and the Proposition 6.18 of [21] imply that a Polish group with ample generics has strong uncountable cofinality if and only if it has property (OB). Thus, it is enough to show that $\mathrm{Iso}(\Omega)$ is finitely generated of bounded width over any non-empty open subset. But this follows from Lemma 5.7. □

6. A DENSE SUBGROUP OF THE UNITARY GROUP WITH STRONG UNCOUNTABLE COFINALITY

In the following ℓ_2 will be the complex Hilbert space on the countable orthonormal basis $(e_i)_{i\in\mathbb{N}}$ and with usual inner product

$$\langle \sum a_i e_i \mid \sum b_i e_i \rangle = \sum a_i \overline{b_i}$$

We will also fix a countable algebraically closed field $\mathbb{Q} \subseteq \boldsymbol{Q} \subseteq \mathbb{C}$ closed under complex conjugation. In fact, it will only be essential that $\boldsymbol{Q}$ is closed under square root, and we could therefore work in some subfield of $\mathbb{R}$ too. This would give similar results for the orthogonal group of the real separable Hilbert space, but we shall be content with the above setting. Notice first that $\boldsymbol{Q}$ is dense in $\mathbb{C}$.

We let $\mathbb{V}$ be the $\boldsymbol{Q}$-vector space with basis (e_i) and notice that $\mathbb{V}$ is a dense subset of ℓ_2. The inner product restricts to an inner product on $\mathbb{V}$ taking values in $\boldsymbol{Q}$, as $\boldsymbol{Q}$ is a field closed under complex conjugation. Since $\boldsymbol{Q}$ is algebraically closed, the norm of an element of $\mathbb{V}$ also belongs to $\boldsymbol{Q}$. This will give us enough space to perform the usual tasks of Gram-Schmidt orthonormalisation etc.

Lemma 6.1. *Let T be a $\boldsymbol{Q}$-linear isometry of $\mathbb{V}$. Then T extends to a unique unitary operator on ℓ_2.*

Proof. Since $\mathbb{V}$ is dense in ℓ_2 and ℓ_2 is complete, any isometry of $\mathbb{V}$ extends to a unique isometry of ℓ_2. Hence T extends to an isometry of ℓ_2 preserving the origin. A simple argument shows that the extension is $\mathbb{C}$-linear. □

So the group $\mathcal{U}(\mathbb{V})$ of $\boldsymbol{Q}$-linear isometries of $\mathbb{V}$ can be seen as a subgroup of $\mathcal{U}(\ell_2)$. It will be useful to represent unitary operators as infinite matrices with respect to the canonical basis $(e_i)_{i\in\mathbb{N}}$. Since we are only considering finite $\boldsymbol{Q}$-linear combinations, this means that any row and any column is eventually zero. The following operators in $\mathcal{U}(\mathbb{V})$ are of particular interest.

Definition 6.2. *An operator $T \in \mathcal{U}(\mathbb{V})$ is* finitary *if it is the identity outside of a finite-dimensional subspace of $\mathbb{V}$.*

So the finitary operators are those that are supported on a finite-dimensional subspace and hence can be represented as

$$\begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix},$$

where I is the infinite identity matrix and A some finite unitary matrix.

Clearly, the finitary operators form a subgroup of $\mathcal{U}(\mathbb{V})$.

The unitary group $\mathcal{U}(\ell_2)$ naturally comes with the *strong topology*, which makes it a Polish group. The strong topology is the weakest topology that makes all the maps

$$T \mapsto T(x)$$

continuous, where x varies over the elements of ℓ_2.

Similarly, as $\mathbb{V}$ is countable, $\mathcal{U}(\mathbb{V})$ is naturally isomorphic to a subgroup of the group $\mathrm{Sym}(\mathbb{V})$ of all permutations of $\mathbb{V}$, with the Polish permutation group topology. Moreover, $\mathcal{U}(\mathbb{V})$ is easily seen to be closed in $\mathrm{Sym}(\mathbb{V})$ and hence is a Polish group itself. Notice that this topology is stronger than the topology induced by $\mathcal{U}(\ell_2)$.

We need that the Gram-Schmidt orthonormalisation procedure can be done in $\mathbb{V}$.

Lemma 6.3. *If $\mathbb{W}$ is any subspace of $\mathbb{V}$ and $w_1, \ldots, w_n$ is an orthonormal set of vectors in $\mathbb{W}$, then there is an orthonormal basis of $\mathbb{W}$ extending $\{w_1, \ldots, w_n\}$.*

Proof. Let $\{x_1, x_2, x_3, \ldots\}$ be a $\boldsymbol{Q}$-vector space basis of $\mathbb{W}$ such that $x_1 = w_1, \ldots, x_n = w_n$. Now, define inductively y_m, w_m by

$$y_{m+1} = x_m - \sum_{i=1}^{m} \langle x_{m+1} \mid w_i \rangle w_i$$

and notice that as $\langle x_{m+1} \mid w_i \rangle \in \boldsymbol{Q}$ also $y_{m+1} \in \mathbb{V}$. Now, put

$$w_{m+1} = \frac{y_{m+1}}{\|y_{m+1}\|}$$

and again, as $\|y_{m+1}\| \in \boldsymbol{Q}$ (using that $\boldsymbol{Q}$ is algebraically closed), $w_{m+1} \in \mathbb{V}$. So as usual, $\{w_1, w_2, \ldots\}$ is an orthonormal basis of $\mathbb{W}$. $\square$

Lemma 6.4. *Suppose S is a linear isometry between finite-dimensional spaces $\mathbb{W}_0$ and $\mathbb{W}_1$. Then S extends to a finitary operator $\tilde{S}$ in $\mathcal{U}(\mathbb{V})$.*

Proof. This is clear from Lemma 6.3. For choose an orthonormal basis $v_1, \ldots, v_n$ for $\mathbb{W}_0$ and find some sufficiently big i such that $\mathbb{W}_0, \mathbb{W}_1 \subseteq [e_1, \ldots, e_i]$. Then we can extend $v_1, \ldots, v_n$ and $S(v_1), \ldots, S(v_n)$ respectively to orthonormal bases $u_1, \ldots, u_i$ and $w_1, \ldots, w_i$ of $[e_1, \ldots, e_i]$. Letting $\tilde{S}(u_j) = w_j$ for $j \leq i$ and $\tilde{S}(e_j) = e_j$ for $j > i$, we have the result. $\square$

In particular, if $\{v_1, \ldots, v_n\}$ and $\{u_1, \ldots, u_n\}$ are orthonormal sets in $\mathbb{V}$, then there is a finitary operator F sending the ordered basis $\{v_1, \ldots, v_n\}$ to the ordered basis $\{u_1, \ldots, u_n\}$.

We recall the following fact (see, e.g., Proposition 2.2. in [21]).

Proposition 6.5. *Let G be a Polish group acting continuously on a Polish space X. Then the following are equivalent for any $x \in X$:*
(i) The orbit $G \cdot x$ is non-meager.
(ii) For every open neighbourhood $V \subset G$ of the identity, $V \cdot x$ is somewhere dense.

Proposition 6.6. *$\mathcal{U}(\mathbb{V})$ has ample generics.*

Proof. By abstract methods (see Truss [36] and Kechris and Rosendal [21]) it is enough to show that certain amalgamation properties are satisfied, but in our case an outright description of the comeagre orbits is not much longer, so we give this instead.

So fix an $m \geq 1$. We need to construct $K_1, \ldots, K_m \in \mathcal{U}(\mathbb{V})$ such that the conjugacy class of the m-tuple $(K_1, \ldots, K_m)$ is comeagre in $\mathcal{U}(\mathbb{V})$.

By Lemma 6.4 we see that for any $P_1, \ldots, P_m \in \mathcal{U}(\mathbb{V})$ and $v_1, \ldots, v_k \in \mathbb{V}$ there are finitary operators $H_1, \ldots, H_m \in \mathcal{U}(\mathbb{V})$ such that for all $t \leq m, s \leq k$, $P_t(v_s) = H_t(v_s)$.

So list all m-tuples $\mathsf{K} = (K_1, \ldots, K_m)$ of unitary operators on some common finite-dimensional subspace $\mathbb{V}_i = [e_1, \ldots, e_i]$ as

$$\mathsf{K}_1 = (K_1^1, \ldots, K_m^1), \mathsf{K}_2 = (K_1^2, \ldots, K_m^2), \ldots$$

and let a_n be the dimension of the space $[e_1, \ldots, e_i]$ on which the $K_1^n, \ldots, K_m^n$ act. Let also $b_n = \sum_{j=1}^n a_j$. We suppose furthermore that each K_i is repeated infinitely often.

We can now paste these operators together as

$$M_t = \begin{bmatrix} K_t^1 & & & \\ & K_t^2 & & \\ & & K_t^3 & \\ & & & \ddots \end{bmatrix}.$$

In other words, $M_1, \ldots, M_m$ are disjoint sums of unitary operators on finite-dimensional spaces of the form $[e_l, \ldots, e_k]$ such that each conjugacy type of m-tuples appears infinitely often.

To see that the conjugacy type of $(M_1, \ldots, M_m)$ is comeagre in $\mathcal{U}(\mathbb{V})^m$, we show first that it is dense and non-meagre. Thus, by Proposition 6.5, it is enough to show that it is dense and that for every $l \in \mathbb{N}$ the set

$$\mathcal{A} = \big\{(T^{-1}M_1T, \ldots, T^{-1}M_mT) \mid T = \begin{bmatrix} I_l & 0 \\ 0 & A \end{bmatrix} \text{ for some } A\big\}$$

is somewhere dense in $\mathcal{U}(\mathbb{V})^m$, where I_l is the $l \times l$ identity matrix. To see this latter, find first some $b_i \geq l$. We claim that $\mathcal{A}$ is dense in the open set consisting of all $(P_1, \ldots, P_m)$ such that for every $t \leq m$,

$$P_t = \begin{bmatrix} K_t^1 & & & \\ & \ddots & & \\ & & K_t^i & \\ & & & A_t \end{bmatrix} = \begin{bmatrix} M_t \restriction_{\mathbb{V}_{b_i}} & \\ & A_t \end{bmatrix}$$

for some A_t. For if $(P_1, \ldots, P_m)$ is above, then the tuple can be approximated arbitrarily well by a tuple of finitary operators. So we can suppose that $P_1, \ldots, P_m$ are finitary themselves. Assume that we want to approximate $P_1, \ldots, P_m$ on $\mathbb{V}_k = [e_1, \ldots, e_k]$, where $k > b_i$ is such that $P_t(e_p) = e_p$ for all $t \leq m$ and $p \geq k$. Then we can find $j > i$ such that $k = b_i + a_j$ and

$$P_t = \begin{bmatrix} M_t \restriction_{\mathbb{V}_{b_i}} & & \\ & K_t^j & \\ & & I \end{bmatrix}.$$

Find a unitary operator T such that $T \upharpoonright [e_1, \ldots, e_{b_i}] = I_{b_i}$ and T sends the ordered basis $\{e_{b_{j-1}+1}, \ldots, e_{b_j}\}$ to $\{e_{b_i+1}, \ldots, e_{b_i+a_j}\}$. Then

$$TM_tT^{-1} = T \begin{bmatrix} K_t^1 & & \\ & K_t^2 & \\ & & \ddots \end{bmatrix} T^{-1} = \begin{bmatrix} M_t \upharpoonright_{\mathbb{V}_{b_i}} & & \\ & K_t^j & \\ & & B_t \end{bmatrix}$$

for some B_t. Thus, TM_tT^{-1} agrees with P_t on $\mathbb{V}_k = [e_1, \ldots, e_{b_i+a_j}]$ for every $t = 1, \ldots, m$.

A similar argument shows that the conjugacy class of $(K_1, \ldots, K_m)$ is dense. But in fact, this also follows from the next proposition. Thus, as there is a dense orbit, the diagonal conjugacy action of $\mathcal{U}(\mathbb{V})$ on $\mathcal{U}(\mathbb{V})^m$ is generically ergodic, i.e., any invariant Borel set is either meagre or comeagre. So, as the conjugacy class of $(K_1, \ldots, K_m)$ is non-meagre, it must be comeagre. $\square$

Definition 6.7. *A Polish group G is said to have a* cyclically dense conjugacy class *if there are elements $g, h \in G$ such that $\{g^n h g^{-n}\}_{n\in\mathbb{Z}}$ is dense in G. We say that G has an* ample cyclically dense conjugacy class *if there is a $g \in G$ and some infinite sequence $(h_k)_k \in G^{\mathbb{N}}$ such that the set $\{(g^n h_k g^{-n})_k\}_{n\in\mathbb{Z}}$ is dense in $G^{\mathbb{N}}$.*

Proposition 6.8. *$\mathcal{U}(\mathbb{V})$ has an ample cyclically dense conjugacy class.*

Proof. Notice first that if G is a Polish group and for some $g \in G$ there is some m-tuple $(h_1, \ldots, h_m) \in G^m$ such that the set

$$\{(g^n h_1 g^{-n}, \ldots, g^n h_m g^{-n})\}_{n\in\mathbb{Z}}$$

is dense in G^m, then set of such $(h_1, \ldots, h_m)$ is certainly dense in G^m. Moreover, since it is also G_δ, it is comeagre. Thus, if for each $m \in \mathbb{N}$ there is such $(h_1, \ldots, h_m) \in G^m$, then there is an infinite sequence $(h_k)_k \in G^{\mathbb{N}}$ such that $\{(g^n h_k g^{-n})_k\}_{n\in\mathbb{Z}}$ is dense in $G^{\mathbb{N}}$.

Therefore, we only need to find some unitary operator S that fills the rôle of g. For this we will consider instead a biinfinite orthonormal basis $(e_i)_{i\in\mathbb{Z}}$ of $\mathbb{V}$ and let S be the bilateral shift on this basis. Fix also some dimension m.

We can now take $H_t = I \oplus M_t$, where we let the identity I act on $\mathbb{V}_- = [\ldots, e_{-2}, e_{-1}, e_0]$ and let M_t be as in the proof of Proposition 6.6 defined on $\mathbb{V}_+ = [e_1, e_2, e_3, \ldots]$. One easily sees that $(S^{-n}H_1S^n, \ldots, S^{-n}H_mS^n)_{n\in\mathbb{N}}$ is dense in $\mathcal{U}(\mathbb{V})^m$. For suppose we wish to approximate some $(P_1, \ldots, P_m)$, which we can suppose are finitary, on some space $\mathbb{W} = [e_{-n}, \ldots, e_n]$. Since each P_t is finitary, we can find $k > n$ such that each P_t is on the form

$$P_t = \begin{bmatrix} I & & \\ & A_t & \\ & & I \end{bmatrix},$$

where A_t is a $(2k + 1) \times (2k + 1)$ matrix acting on $[e_{-k}, \ldots, e_k]$. Now find some j such that $K_t^j = A_t$ for each $t \leq m$. Then we see that $H_t \upharpoonright [e_{b_{j-1}+1}, \ldots, e_{b_j}]$ and thus

$$S^{-b_{j-1}-1-k} H_t S^{b_{j-1}+1+k} \upharpoonright [e_{-k}, \ldots, e_k] = A_t.$$

Therefore,

$$(S^{-b_{j-1}-1-k} H_1 S^{b_{j-1}+1+k}, \ldots, S^{-b_{j-1}-1-k} H_m S^{b_{j-1}+1+k})$$

agrees with $(P_1, \ldots, P_m)$ on $[e_{-k}, \ldots, e_k] \supseteq \mathbb{W}$. $\square$

It follows from Proposition 6.18 in [21] that a Polish group with ample generics has strong uncountable cofinality if and only if it has property (OB).

Proposition 6.9. *$\mathcal{U}(\mathbb{V})$ has strong uncountable cofinality.*

Proof. Since $\mathcal{U}(\mathbb{V})$ has ample generics, it is enough to show that it has property (OB). We show that $\mathcal{U}(\mathbb{V})$ is finitely generated of bounded width over any open neighbourhood U of the identity. So suppose k is given such that U contains all operators fixing $\mathbb{W}_0 = [e_1, \ldots, e_k]$ pointwise. Find an operator M such that $M[\mathbb{W}_0] = \mathbb{W}_1 = [e_{k+1}, \ldots, e_{2k}]$ and notice that $M\mathsf{U}M^{-1}$ contains all operators fixing $\mathbb{W}$ pointwise. Let now $T \in \mathcal{U}(\mathbb{V})$ and find a finite dimensional space $\mathbb{H}_0 \subseteq (\mathbb{W}_0 \oplus \mathbb{W}_1)^\perp$ such that $T[\mathbb{W}_0] \subseteq \mathbb{W}_0 \oplus \mathbb{W}_1 \oplus \mathbb{H}_0$. Let $R_0 \in \mathsf{U}$ send $\mathbb{W}_1$ into $(\mathbb{W}_0 \oplus \mathbb{W}_1 \oplus \mathbb{H}_0)^\perp$ and fix $\mathbb{W}_0 \oplus \mathbb{H}_0$ pointwise. Thus,

$$R_0T[\mathbb{W}_0] \subseteq R_0[\mathbb{W}_0 \oplus \mathbb{W}_1 \oplus \mathbb{H}_0] \subseteq \mathbb{W}_0 \oplus R_0[\mathbb{W}_1] \oplus \mathbb{H}_0 \subseteq \mathbb{W}_1^\perp.$$

We can therefore find some $R_1 \in M\mathsf{U}M^{-1}$ such that R_1R_0T is the identity on $\mathbb{W}_0$ and R_1 fixes $\mathbb{W}_1$ pointwise. Hence, $R_1R_0T \in \mathsf{U}$ and $T \in R_0^{-1}R_1^{-1}\mathsf{U} \subseteq \mathsf{U}^{-1}M\mathsf{U}^{-1}M^{-1}\mathsf{U}$. Thus, $\mathcal{U}(\mathbb{V}) = \mathsf{U}^{-1}M\mathsf{U}^{-1}M^{-1}\mathsf{U}$. □

Lemma 6.10. *$\mathcal{U}(\mathbb{V})$ is dense in $\mathcal{U}(\ell_2)$.*

Proof. It is enough to see that any unitary $T \in \mathcal{U}(\ell_2)$ can be approximated arbitrarily well on any finite set of orthonormal vectors. So suppose $x_1, \ldots, x_n$ is an orthonormal set and $\epsilon > 0$. By the continuity of the inner product, we can find $\delta > 0$ such that if $v_1, \ldots, v_n, u_1, \ldots, u_n$ are normalised vectors such that $\|x_i - v_i\| < \delta$ and $\|T(x_i) - u_i\| < \delta$ for every i, then if $\hat{v}_1, \ldots, \hat{v}_n$ and $\hat{u}_1, \ldots, \hat{u}_n$ are the orthonormal bases obtained by applying the Gram-Schmidt orthonormalisation process to $v_1, \ldots, v_n$ and $u_1, \ldots, u_n$ respectively, we still have $\|x_i - \hat{v}_i\| < \epsilon/2$ and $\|T(x_i) - \hat{u}_i\| < \epsilon/2$ for every i. Thus, choose $v_i, u_i \in \mathbb{V}$ as above and pick some $R \in \mathcal{U}(\mathbb{V})$ sending the ordered basis $\hat{v}_1, \ldots, \hat{v}_n$ to the ordered basis $\hat{u}_1, \ldots, \hat{u}_n$. Then

$$\|T(x_i) - R(x_i)\| \leq \|T(x_i) - R(\hat{v}_i)\| + \|R(\hat{v}_i) - R(x_i)\| \leq \epsilon$$

and hence approximating T on $x_1, \ldots, x_n$. □

So let us sum up the results so far.

Theorem 6.11. *$\mathcal{U}(\mathbb{V})$ has ample generics, an ample cyclically dense conjugacy class and strong uncountable cofinality. Thus, $\mathcal{U}(\ell_2)$ has property (OB) and an ample cyclically dense conjugacy class.*

Added in proof: It has now been been verified by Éric Ricard and the author [28] that the unitary group has strong uncountable cofinality.

We should mention that the existence of ample generics in a Polish group has quite remarkable consequences for the structure of the group, for example, it implies that any homomorphism from it into a separable group is automatically continuous, and the group cannot be covered by countably many non-open cosets (see Hodges, Hodkinson, Lascar and Shelah [19], Kechris and Rosendal [21]).

Theorem 6.12. *Let $\mathcal{U}(\ell_2)$ act continuously by Hölder maps on a complete metric space (X, d). If the bilateral shift S induces a relatively compact orbit on X, then $\mathcal{U}(\ell_2)$ fixes a point of X.*

Proof. First, we can evidently suppose that X is in fact separable and thus Polish. So $\mathcal{U}(\ell_2)$ acts continuously on $\mathcal{K}(X)$. Moreover, if $g \in G$ is Hölder(α) with constant c on (X, d), then g is Hölder(α) with constant c on $(\mathcal{K}(X), d_H)$, where $\mathcal{K}(X)$ is the space of all compact subsets of X equipped with the Hausdorff metric d_H. For

$$d_H(K, L) = \max\big(\sup_{x \in K} d(x, L), \sup_{x \in L} d(K, x)\big).$$

But

(19)
$$\sup_{x\in gK} d(x,gL) = \sup_{x\in K}\inf_{y\in L} d(gx,gy) \leq \sup_{x\in K}\inf_{y\in L} c\cdot d(x,y)^\alpha = c\cdot(\sup_{x\in K}\inf_{y\in L} d(x,y))^\alpha$$

and thus

$$\begin{aligned} d_H(gK,gL) &= \max\big(\sup_{x\in gK} d(x,gL), \sup_{y\in gL} d(gK,y)\big) \\ &\leq \max\big(c\cdot(\sup_{x\in K}\inf_{y\in L} d(x,y))^\alpha, c\cdot(\sup_{y\in L}\inf_{x\in K} d(x,y))^\alpha\big) \\ &= c\cdot\max\big(\sup_{x\in K}\inf_{y\in L} d(x,y), \sup_{y\in L}\inf_{x\in K} d(x,y)\big)^\alpha \\ &= c\cdot d_H(gK,gL)^\alpha. \end{aligned} \tag{20}$$

Let now $\mathcal{O} = \overline{\{S^n\cdot x\}_{n\in\mathbb{Z}}}$ be compact and find by the proof of Proposition 6.8 some $T\in\mathcal{U}(\ell_2)$ such that $\{S^nTS^{-n}\}_{n\in\mathbb{Z}}$ is dense in $\mathcal{U}(\ell_2)$. Then

$$\begin{aligned} d_H(T\cdot\mathcal{O},\mathcal{O}) &= d_H(TS^{-n}\mathcal{O}, S^{-n}\mathcal{O}) \\ &\leq c_n d_H(S^nTS^{-n}\mathcal{O}, S^nS^{-n}\mathcal{O})^{\alpha_n} \\ &\leq c_n d_H(S^nTS^{-n}\mathcal{O},\mathcal{O})^{\alpha_n}, \end{aligned} \tag{21}$$

where S^n is Hölder(α_n) with constant c_n. Using now that $\mathcal{U}(\ell_2)$ has property (OB), we find a universal N such that we can choose all the $\alpha_n\in[1/N,N]$ and $c_n\leq N$. Picking a subsequence n_i such that $S^{n_i}TS^{-n_i}\to I$, we see that

$$c_{n_i}d_H(S^{n_i}TS^{-n_i}\mathcal{O},\mathcal{O})^{\alpha_{n_i}}\to 0$$

and thus $d_H(T\cdot\mathcal{O},\mathcal{O}) = 0$. Hence, $\mathcal{O}$ is both S and T invariant and thus also $\mathcal{U}(\ell_2)$-invariant. Moreover, as $\mathcal{O}$ is compact and $\mathcal{U}(\ell_2)$ is extremely amenable (this is a result of Gromov and Milman [15]), $\mathcal{U}(\ell_2)$ fixes a point of $\mathcal{O}$. □

7. Groups of homeomorphisms

7.1. **Circle groups.** We shall first consider the homeomorphism group of the unit circle S^1 and its model-theoretic counterpart, the automorphism group of the countable dense circular order, $\mathrm{Aut}(\mathfrak{C})$.

Let first $\pi:\mathbb{R}\to\mathbb{R}/\mathbb{Z} = S^1$ and let d be the metric on S^1 induced by the metric on $\mathbb{R}$. I.e., $d(x,y) = dist(\pi^{-1}(x),\pi^{-1}(y))$. So d takes values in $[0,1/2]$.

Let $\mathfrak{C}\subseteq S^1$ be a countable dense set, for concreteness we can take $\mathfrak{C} = \pi[\mathbb{Q}]$, and $\mathrm{Aut}(\mathfrak{C})$ the set of all permutations of $\mathfrak{C}$ that preserve the relation $B\subseteq\mathfrak{C}^3$ defined as follows:

For $x,y,z\in\mathfrak{C}$ let $B(x,y,z)$ if and only if

- x,y and z are distinct, and
- going clockwise along the unit circle S^1 from x to z one passes through y.

In this case, we say that y is *between* x and z.

Theorem 7.1. $\mathrm{Aut}(\mathfrak{C})$ *and* $\mathrm{Homeo}(S^1)$ *have strong uncountable cofinality.*

Proof. Pick distinct $x_1,x_2,x_3\in\mathfrak{C}$. Notice that for any $y\in\mathfrak{C}$, the groups $\mathrm{Aut}(\mathfrak{C},y)$ and $\mathrm{Aut}(\mathbb{Q},<)$ are naturally isomorphic. Thus, if

$$W_0\subseteq W_1\subseteq\ldots\subseteq\mathrm{Aut}(\mathfrak{C}) = \bigcup_n W_n$$

then there are n and k such that $\mathrm{Aut}(\mathfrak{C}, x_i) \subseteq W_n^k$ for $i = 1, 2, 3$. This follows from the result of Droste and Holland [12] that $\mathrm{Aut}(\mathbb{Q}, <)$ has strong uncountable cofinality. Now,

$$\mathrm{Aut}(\mathfrak{C}) = \mathrm{Aut}(\mathfrak{C}, x_1) \cdot \mathrm{Aut}(\mathfrak{C}, x_2) \cup \mathrm{Aut}(\mathfrak{C}, x_1) \cdot \mathrm{Aut}(\mathfrak{C}, x_2).$$

For given $g \in \mathrm{Aut}(\mathfrak{C})$, either $g(x_1) \neq x_2$ or $g(x_1) \neq x_2$. Say $g(x_1) \neq x_2$. Find some $h \in \mathrm{Aut}(\mathfrak{C}, x_2)$ such that $h(x_1) = g(x_1)$. Then $gh^{-1} \in \mathrm{Aut}(\mathfrak{C}, x_1)$ and so $g \in \mathrm{Aut}(\mathfrak{C}, x_1) \cdot \mathrm{Aut}(\mathfrak{C}, x_2)$. Similarly, if $g(x_1) \neq x_3$, then $g \in \mathrm{Aut}(\mathfrak{C}, x_1) \cdot \mathrm{Aut}(\mathfrak{C}, x_3)$. Thus,

$$\mathrm{Aut}(\mathfrak{C}) = \mathrm{Aut}(\mathfrak{C}, x_1) \cdot \mathrm{Aut}(\mathfrak{C}, x_2) \cup \mathrm{Aut}(\mathfrak{C}, x_1) \cdot \mathrm{Aut}(\mathfrak{C}, x_3) \subseteq W_n^k W_n^k = W_n^{2k}.$$

The same argument applies to $\mathrm{Homeo}_+(S^1)$, using that for any $x \in S^1$ the groups

$$\mathrm{Homeo}_+(S^1, x) = \{g \in \mathrm{Homeo}_+(S^1) \bigm| g(x) = x\}$$

and $\mathrm{Homeo}_+(\mathbb{R})$ are isomorphic and that $\mathrm{Homeo}_+(\mathbb{R})$ has strong uncountable cofinality by the results of Droste and Holland. Now, strong uncountable cofinality for $\mathrm{Homeo}(S^1)$ follows, as $\mathrm{Homeo}_+(S^1)$ is a subgroup of index 2 in $\mathrm{Homeo}(S^1)$. $\square$

7.2. **Spheres.**

Theorem 7.2. *Let* $\mathrm{Homeo}(S^m)$ *be the group of homeomorphisms of the* m*-dimensional sphere with the topology of uniform convergence. Then* $\mathrm{Homeo}(S^m)$ *has property (OB).*

Proof. We let d be the standard Euclidean metric on $\mathbb{R}^{m+1}$ and d_∞ the supremum metric on $\mathrm{Homeo}(S^m)$, $d_\infty(g, f) = \sup_{x \in S^m} d(gx, fx)$. We show that $\mathrm{Homeo}(S^m)$ is finitely generated of bounded width over any non-empty open subset U. So pick an $\epsilon_0 > 0$ such that

$$V = \{g \in \mathrm{Homeo}(S^m) \bigm| d_\infty(g, id) < 3\epsilon_0\} \subseteq UU^{-1}.$$

Let also $x_0 = (1, 0, 0, \ldots, 0) \in S^m$. Then for any $\epsilon_0 > \delta > 0$ there is some homeomorphism ϕ_δ of S^m such that $\phi_\delta(B_{\epsilon_0}(x_0)) = B_\delta(x_0)$ and $d_\infty(\phi_\delta, id) < \epsilon_0$. Moreover, there is an involution homeomorphism ψ of S^m fixing $\partial B_{\epsilon_0}(x_0)$ pointwise, while switching $\mathrm{int} B_{\epsilon_0}(x_0)$ with $\mathrm{ext} B_{\epsilon_0}(x_0) = S^m \setminus B_{\epsilon_0}(x_0)$. Finally, let ι be the orientation inverting involution

$$\iota(x_0, x_1, x_2, \ldots, x_m) = (x_0, -x_1, x_2, \ldots, x_m).$$

We notice that $SO(m+1)$ is a compact subgroup of $\mathrm{Homeo}(S^m)$, so $\bigcup_n (SO(m+1) \cap VV^{-1})^n$ is an open subgroup of $SO(m+1)$. But this latter is connected and compact, so $SO(m+1) \subseteq (VV^{-1})^k$ for some k.

Claim 7.3. $\mathrm{Homeo}(S^m) \subseteq (VV^{-1})^k \{\iota, id\} V^2 \psi V \psi V^{-1}$.

To see this, let $g \in \mathrm{Homeo}(S^m)$ and find $f \in SO(m+1)$ such that $fg(x_0) = x_0$. Then put

$$\hat{f} = \begin{cases} f & \text{if } fg \text{ is orientation preserving,} \\ \iota f & \text{if } \iota fg \text{ is orientation preserving.} \end{cases}$$

It follows that $\hat{f}g$ preserves the orientation and fixes x_0. Therefore, by Lemma 3.1. of Glasner and Weiss [13], which itself relies on the proof of the annulus conjecture, there is some $\epsilon_0 > \delta > 0$ and a homeomorphism h of S^m such that

$$\begin{aligned} &\forall x \in S^m \ (d(x, x_0) > \epsilon_0 \to hx = \hat{f}gx), \\ &\forall x \in S^m \ (d(x, x_0) < \delta \to hx = x). \end{aligned} \tag{22}$$

In particular,

$$d_\infty(h, \hat{f}g) = \sup_{x \in S^m} d(hx, \hat{f}gx) = \sup_{x \in B_{\epsilon_0}(x_0)} d(hx, \hat{f}gx) < 3\epsilon_0.$$

So $\hat{f}gh^{-1} \in V$ and $g \in \hat{f}^{-1}Vh$. Thus,

$$\phi_\delta^{-1} h \phi_\delta \restriction B_{\epsilon_0}(x_0) = id$$

and

$$\psi\phi_\delta^{-1} h \phi_\delta \psi \restriction extB_{\epsilon_0}(x_0) = id.$$

In particular, $d_\infty(\psi\phi_\delta^{-1} h \phi_\delta \psi, id) < 3\epsilon_0$, so $\psi\phi_\delta^{-1} h \phi_\delta \psi \in V$. Therefore, $h \in \phi_\delta \psi V \psi \phi_\delta^{-1}$ and

$$\begin{aligned} g &\in \hat{f}^{-1}Vh \\ &\subseteq \hat{f}^{-1}V\phi_\delta\psi V\psi\phi_\delta^{-1} \\ &\subseteq (VV^{-1})^k \cdot \{id, \iota\} \cdot V^2\psi V\psi V^{-1}. \end{aligned} \tag{23}$$

□

Added in proof: Subsequent to the appearance of this paper in preprint, D. Calegari, M. Freedman, and Y. de Cornulier [7] were able to show that the groups $\mathrm{Homeo}(S^n)$ actually have strong uncountable cofinality, thus strengthening Theorem 7.2.

7.3. **The Hilbert cube.** Consider now the Hilbert cube $Q = [0,1]^{\mathbb{N}}$ and its homeomorphism group $\mathrm{Homeo}(Q)$ equipped with the topology of uniform convergence. We let d be the metric on Q given by

$$d((x_n), (y_n)) = \sum_{n \in \mathbb{N}} \frac{|x_n - y_n|}{2^{n+1}}$$

and d_∞ the supremum metric on $\mathrm{Homeo}(Q)$ given by

$$d_\infty(f, g) = \sup_{\vec{x} \in Q} d(f(\vec{x}), g(\vec{x})),$$

which is right invariant.

Theorem 7.4. $\mathrm{Homeo}(Q)$*, with the topology of uniform convergence, has property (OB).*

Proof. Fix some open neighbourhood V of the identity in $\mathrm{Homeo}(Q)$, which we can suppose is of the form

$$V = \{g \in \mathrm{Homeo}(Q) \bigm| d_\infty(g, id) < 2\epsilon\}$$

for some $\epsilon > 0$. Thus, if n is sufficiently large such that $2^{-n} < \epsilon$, then for any $f \in \mathrm{Homeo}(Q)$ that does not change the first n coordinates of any $\vec{x} \in Q$, i.e.,

$$f((x_0, x_1, \ldots, x_{n-1}, x_n, x_{n+1}, \ldots)) = (x_0, x_1, \ldots, x_{n-1}, y_n, y_{n+1}, \ldots)$$

for all $\vec{x} \in Q$, we have $f \in V$.

Claim 7.5. *If* $\frac{1}{k} < \epsilon$ *, then there is a finite set* $\mathbb{F} \subseteq \mathrm{Homeo}(Q)$ *such that for every* $\vec{x} \in Q$*,* $\vec{0} = (0, 0, 0, \ldots) \in \mathbb{F}V^{k+1} \cdot \vec{x}$*.*

Proof of claim: Let $\frac{1}{2^n} < \epsilon$. For each $s \in \{0, \frac{1}{2}, 1\}^n$ let $\vec{z}_s = (s_0, s_1, \ldots, s_{n-1}, 0, 0, \ldots)$. As Q is homogeneous, $\mathrm{Homeo}(Q)$ acts transitively on Q (see van Mill, Theorem 6.1.6. [25]), and we can therefore find some $h_s \in \mathrm{Homeo}(Q)$ such that $h_s(\vec{z}_s) = \vec{0}$ for each s. Let $\mathbb{F} = \{h_s \bigm| s \in \{0, \frac{1}{2}, 1\}^n\}$. So it is enough to show that $\exists s\ \vec{z}_s \in V^{k+1} \cdot \vec{x}$. So first use the homogeneity of Q to adjust the tail $(x_n, x_{n+1}, \ldots)$ of $\vec{x}$ by some element of V to become $(0, 0, \ldots)$. This can be done since a homeomorphism leaving the first n coordinates invariant belongs to V. Now we can subsequently adjust each of the first n coordinates (leaving the tail invariant) to be equal to either 0, $\frac{1}{2}$ or 1. For this operation it is enough to use a product of at most k elements of V. $\square$

Now, it follows from Brouwer's fixed point Theorem that any homeomorphism of Q fixes a point. Thus, up to a conjugate by an element of the set $\mathbb{F}V^{k+1}$ from Claim 7.5, any homeomorphism of Q fixes $\vec{0}$. As we wish to show that $\mathrm{Homeo}(Q)$ is finitely generated of bounded width over V, we can suppose that any homeomorphism fixes $\vec{0}$.

Claim 7.6. *(Glasner and Weiss* [13]*) If $f \in \mathrm{Homeo}(Q)$ fixes $\vec{0}$, then there is some $\delta > 0$ and $g \in \mathrm{Homeo}(Q)$ such that $d_\infty(g, f) < \epsilon$ and $g\restriction_{B_\delta(\vec{0})} = id$.*

Proof of claim: Pick $\delta > 0$ sufficiently small such that $\sup_{\vec{x} \in B_\delta(\vec{0})} d(f(\vec{x}), \vec{x}) < \epsilon$. As both $\partial B_\delta(\vec{0})$ and $\partial(f[B_\delta(\vec{0})])$ are Z-sets, we can extend the homeomorphism

$$f^{-1} \colon \partial(f[B_\delta(\vec{0})]) \to \partial B_\delta(\vec{0})$$

to a homeomorphism $h \in \mathrm{Homeo}(Q)$ satisfying $d_\infty(h, id) < \epsilon$ (see van Mill, Theorem 6.4.6. [25]). Thus, $d_\infty(hf, f) = d_\infty(h, id) < \epsilon$ and we can let

$$g(\vec{y}) = \begin{cases} \vec{y} & \text{if } \vec{y} \in B_\delta(\vec{0}) \\ hf(\vec{y}) & \text{otherwise} \end{cases}$$

$\square$

Claim 7.7. *For any $g \in \mathrm{Homeo}(Q)$ and $0 < \delta < \epsilon$ such that $g\restriction_{B_\delta(\vec{0})} = id$, there is $h \in V^2$ such that $h^{-1}gh\restriction_{[0,\epsilon[^{n+1} \times [0,1]^{\mathbb{N}}} = id$.*

Proof of claim: Notice that $[0, \delta[^{n+1+l} \times [0, 1]^{\mathbb{N}} \subseteq B_\delta(\vec{0})$ for some $l > 0$. Moreover, it is not hard to see that $[0, \epsilon[\times[0, 1]^l$ is homeomorphic to $[0, \delta[^{l+1}$ by some function a, which is a homeomorphism of $[0, 1]^{l+1}$. Thus,

$$h_0 = id_{[0,1]^n} \otimes a \otimes id_{[0,1]^{\mathbb{N}}} : Q \to Q$$

belongs to V and sends

$$[0, 1]^n \times \Big([0, \epsilon[\times[0, 1]^l\Big) \times [0, 1]^{\mathbb{N}}$$

to

$$[0, 1]^n \times [0, \delta[^{l+1} \times [0, 1]^{\mathbb{N}}.$$

Now, let $h_1 : Q \to Q$ be a homeomorphism that moves the set $[0, \epsilon[^n \times [0, 1]^{\mathbb{N}}$ to $[0, \delta[^n \times [0, 1]^{\mathbb{N}}$, preserves all coordinates $\geq n$ and $d_\infty(h_1, id) < \epsilon$. Then $h_0, h_1 \in V$ and $h = h_1 h_0$ moves $[0, \epsilon[^{n+1} \times [0, 1]^{\mathbb{N}}$ to $[0, \delta[^{n+1+l} \times [0, 1]^{\mathbb{N}}$. $\square$

Now, let $\iota \in \mathrm{Homeo}([0, 1])$ be an involution homeomorphism that fixes ϵ and switches 0 and 1. Define $i \in \mathrm{Homeo}(Q)$ by

$$i(x_0, x_1, \ldots, x_n, x_{n+1}, \ldots) = (\iota(x_0), \iota(x_1), \ldots, \iota(x_n), x_{n+1}, x_{n+2}, \ldots).$$

Then i interchanges $[0, \epsilon[^{n+1} \times [0, 1]^{\mathbb{N}}$ and $]\epsilon, 1]^{n+1} \times [0, 1]^{\mathbb{N}}$.

We can now conclude our result. For suppose $f \in \mathrm{Homeo}(Q)$. Then, up to a conjugate by an element of $\mathbb{F}V^{k+1}$ we can suppose that f fixes $\vec{0}$. By Claim 7.6, we can find $g \in Vf$ and $0 < \delta < \epsilon$ such that $g\restriction_{B_\delta(\vec{0})} = id$. So pick by Claim 7.7 some $h \in V^2$ such that $h^{-1}gh\restriction_{[0,\epsilon[^{n+1} \times [0,1]^{\mathbb{N}}} = id$. But then $ih^{-1}ghi\restriction_{]\epsilon,1]^{n+1} \times [0,1]^{\mathbb{N}}} = id$ and hence $ih^{-1}ghi \in V$. All in all, this shows that $f \in (\mathbb{F}V^{k+1})^{-1}V^{-1}V^2 iViV^{-2}\mathbb{F}V^{k+1}$. $\square$

Since any Polish group is a closed subgroup of $\mathrm{Homeo}(Q)$ (see Uspenskiĭ [38] or the exposition in Kechris [20]), we have

Corollary 7.8. *Any Polish group is topologically isomorphic to a closed subgroup of a Polish group with property (OB).*

8. ACTIONS ON TREES

Comeagre conjucagy classes. We give first a simple proof of a result of Dugald Macpherson and Simon Thomas. We will actually prove a result stronger than theirs, for which we need some basic computations by M. Culler and J.W. Morgan [10]. Note first that if a group G acts by isometries on an $\mathbb{R}$-tree T, then each $g \in G$ has associated a characteristic non-empty subtree T_g of T, which either is the set of points fixed by g (in which case g is called *elliptic*) or a line on which g acts by translation (in which case g is called *hyperbolic*). We let $\|g\| = \inf(r \in \mathbb{R}_+ \bigm| \exists x \in T\ d(x, g \cdot x) = r)$. This infimum is in fact attained as shown in [10]. Thus, g is elliptic if and only if $\|g\| = 0$.

The interested reader should consult the very readable article by Culler and Morgan for more information on the general theory of group actions on $\mathbb{R}$-trees.

Lemma 8.1. [10] *Suppose g and h are isometries of an $\mathbb{R}$-tree T . If $T_g \cap T_h$ is empty, then*

$$\|gh\| = \|g\| + \|h\| + 2dist(T_g, T_h)$$

Lemma 8.2. [10] *Let g and h be hyperbolic isometries of an $\mathbb{R}$-tree T such that $T_g \cap T_h \neq \emptyset$. Then*

$$\max(\|gh\|, \|gh^{-1}\|) = \|g\| + \|h\|$$

From Lemma 8.1 follows the following important special case.

Theorem 8.3. *(Serre's Lemma) Suppose g, h and gh are elliptic isometries of an $\mathbb{R}$-tree T. Then $T_g \cap T_h \neq \emptyset$.*

Theorem 8.4. *(D. Macpherson and S. Thomas for combinatorial trees* [24]*.) Suppose G is a Polish group with a comeagre conjugacy class C acting by isometries on an $\mathbb{R}$-tree T. Then every element of G is elliptic.*

Proof. We claim that $\|\cdot\|$ is constantly 0 on C. Assume towards a contradiction that this is not the case. Notice first that $\|\cdot\|$ is conjugacy invariant, so constant on C. Pick $g, h \in C$ such that also $gh, gh^{-1} \in C$. By Lemma 8.1, if $T_g \cap T_h = \emptyset$ then

$$\|gh\| = \|g\| + \|h\| + 2dist(T_g, T_h) > \|g\|$$

contradicting that $\|\cdot\|$ is constant on C. So $T_g \cap T_h \neq \emptyset$, whence by Lemma 8.2,

$$\max(\|gh\|, \|gh^{-1}\|) = \|g\| + \|h\| > \|g\|$$

again contradicting that $\|\cdot\|$ is constant on C and thus proving the claim.

Assume now that f is an arbitrary element of G and pick $g, h \in C$ such that $f = hg$. Then

$$C^{-1} \cap C \cap hC^{-1} \cap gC^{-1} \cap fC^{-1} \neq \emptyset$$

so we can find $k_0, k_1, k_2, k_3 \in C$ with $k_0 = hk_1^{-1} = gk_2^{-1} = fk_3^{-1}$, $k_0^{-1} \in C$, i.e., $k_0k_1 = h$, $k_0k_2 = g$ and $k_0k_3 = f = k_0k_1k_0k_2$.

Notice that $k_0, k_1, k_0k_1 \in C$, $k_1, k_0k_2, k_1k_0k_2 = k_3 \in C$ and $k_0^{-1}, k_0k_2, k_0^{-1}k_0k_2 = k_2 \in C$, so applying Serre's Lemma to each of these three situations, we have $T_{k_0} \cap T_{k_1} \neq \emptyset$, $T_{k_1} \cap T_{k_0k_2} \neq \emptyset$ and $T_{k_0} \cap T_{k_0k_2} \neq \emptyset$. The three trees T_{k_0}, T_{k_1} and $T_{k_0k_2}$ therefore intersect pairwise, and thus there is some x in their common intersection. But then clearly $f \cdot x = k_0 k_1 k_0 k_2 \cdot x = x$, whence f is elliptic. □

Notice that if a group G acts by automorphisms on a tree T, then the action extends to the tree T' obtained from T by adding a midpoint on every edge. Moreover, the action on T' is *without inversion*, i.e., there are no vertices $a \neq b$ in T' such that $\{a, b\}$ is an edge and $g \cdot a = b$, $g \cdot b = a$ for some $g \in G$. We also see that G fixes a vertex if T' if and only if G fixes either a vertex or an edge of T. Thus, to see that a group has property (FA) it is enough to show that any action without inversion on a tree has a fixed vertex.

The proof of Theorem 8.4 translates word for word into the corresponding proof for Λ-trees (a generalisation of $\mathbb{R}$-trees with a metric taking values in an arbitrary ordered abelian group). The only thing that has to be checked is that the appropriate lemmas are true also in this setting. Well, here they are. In the following, Λ is a fixed ordered abelian group and (X, d) a given Λ-tree. We define the norm of elements of G in the same manner as for actions on $\mathbb{R}$-trees.

Lemma 8.5. *(Lemma 2.1.11 in* [8]*) Suppose $X_1, \ldots, X_n$ are subtrees of X such that $X_i \cap X_j \neq \emptyset$ for all i, j. Then $X_1 \cap \ldots \cap X_n \neq \emptyset$.*

Lemma 8.6. *(Lemma 3.2.2 in* [8]*) Suppose g and h are isometries of (X, d), which are not inversions, such that $T_g \cap T_h = \emptyset$. Then*

$$\|gh\| = \|g\| + \|h\| + 2dist(T_g, T_h)$$

Lemma 8.7. *(Lemmas 3.2.3 and 3.3.1 in* [8]*) Suppose g and h are hyperbolic isometries of (X, d) such that $T_g \cap T_h \neq \emptyset$. Then*

$$\max(\|gh\|, \|gh^{-1}\|) = \|g\| + \|h\|$$

From Lemma 8.6 we have again a version of Serre's Lemma.

Lemma 8.8. *Suppose g, h and gh are elliptic isometries of a Λ-tree (X, d). Then $T_g \cap T_h \neq \emptyset$.*

Theorem 8.9. *Suppose G is a Polish group with a comeagre conjugacy class C acting by isometries and without inversion on a Λ-tree (X, d). Then every element of G is elliptic.*

Dense conjugacy classes.

Lemma 8.10. *Suppose a topological group G acts continuously and without inversion on a tree T, i.e., such that the stabilisers of vertices in T are open in G. Then $\|\cdot\| : G \to \mathbb{N}$ is continuous.*

Proof. Suppose first that $\|g\| = 0$. Then for some $a \in T$, $g \cdot a = a$, i.e. $g \in G_a$ and G_a is an open neighbourhood of g on which $\|\cdot\|$ is constantly 0.

Now, suppose $\|g\| = n > 0$. Then by a theorem of Tits (Proposition 24 in [30]) there is a line $\ell_g = (a_i \mid i \in \mathbb{Z})$ in T such that $g \cdot a_i = a_{i+n}$ for all i. Now, if $h \in G$ is elliptic, then for any $a \in T$, h fixes the midpoint of the geodesic from a to $h \cdot a$. So if $h \cdot a_0 = g \cdot a_0 = a_n$ then $n = 2m$, $m > 0$ and $h \cdot a_m = a_m \neq g \cdot a_m$, by uniqueness of the geodesic. Hence if

$$U = \{f \in G \mid f \cdot a_0 = g \cdot a_0, \ldots, f \cdot a_n = g \cdot a_n\}$$

then U is an open neighbourhood of g containing only hyperbolic points of norm $\leq n$.

Moreover, if h is hyperbolic of norm $k < n$, then ℓ_h would contain exactly the $k+1$ midpoints of the arc $a_0, a_1, \ldots, a_n$ from a_0 to $h \cdot a_0 = a_n$. So for some $0 < i < n$,

$$dist_T(a_i, h \cdot a_i) = k \neq n$$

which is a contradiction. So U only contains hyperbolic points of norm n. □

Proposition 8.11. *Suppose G is a Polish group with a dense conjugacy class, which is not the union of a countable sequence of proper open subgroups. Then whenever G acts continuously and without inversion on a tree T, it fixes a vertex of T. In other words, G has property (topFA).*

Proof. Notice first that $\|\cdot\|$ is conjugacy invariant and continuous, so must be constantly 0 on G. I.e. every element of G is elliptic. So if G does not fix a vertex, it fixes an end $\alpha = (a_0, a_1, \ldots) \subseteq T$ (Tits, Exercise 2, page 66 [30]). But then $G = \bigcup_n G_{(a_n, a_{n+1}, \ldots)}$, where $G_{(a_n, a_{n+1}, \ldots)}$ is the pointwise stabiliser of the set $\{a_n, a_{n+1}, \ldots\}$. Since these subgroups are closed, almost all of them must be open, as G satisfies Baire's category theorem. And as G is not the union of a countable chain of proper open subgroups, $G = G_{(a_n, a_{n+1}, \ldots)}$ for some N, contradicting that G did not fix a vertex. $\square$

S. Solecki [32] has shown that the isometry group of the rational Urysohn metric space, $\mathrm{Iso}(\mathbb{U}_\mathbb{Q})$, with the permutation group topology, has ample generics and a cyclically dense conjugacy class. Moreover, in Kechris and Rosendal [21] it is shown that Polish groups with ample generics and a cyclically dense conjugacy class cannot be written as the union of a countable chain of proper subgroups. So this means that $\mathrm{Iso}(\mathbb{U}_\mathbb{Q})$ has property (FA). Moreover, V.G. Pestov [27] shows that $\mathrm{Iso}(\mathbb{U})$ has no non-trivial continuous representations by isometries in a reflexive Banach space, so in particular it has property (FH). However, this does not solve the corresponding problem for $\mathrm{Iso}(\mathbb{U}_\mathbb{Q})$.

References

[1] ATKIN, C. J. *Boundedness in uniform spaces, topological groups, and homogeneous spaces.* Acta Math. Hungar. 57 (1991), no. 3-4, 213-232.

[2] BECKER, H. *Polish group actions: dichotomies and generalized elementary embeddings.* J. Amer. Math. Soc. 11 (1998), no. 2, 397–449.

[3] BECKER, H. & KECHRIS, A. S. *The descriptive set theory of Polish group actions.* London Mathematical Society Lecture Note Series, 232. Cambridge University Press, Cambridge, 1996.

[4] BEKKA, B. *Kazhdan's property (T) for the unitary group of a separable Hilbert space.* Geom. Funct. Anal. 13 (2003), no. 3, 509–520.

[5] BEKKA, B.; DE LA HARPE, P. & VALETTE, A. *Kazhdan's property (T)*, forthcoming book 2003.

[6] BERGMAN, G. M. *Generating infinite symmetric groups.* Bull. London Math. Soc. 38 (2006), no. 3, 429–440.

[7] CALEGARI, D. & FREEDMAN, M. H. *Distortion in transformation groups.* With an appendix by Yves de Cornulier. Geom. Topol. 10 (2006), 267–293.

[8] CHISWELL, I. *Introduction to Λ-trees.* World Scientific Publishing Co., Inc., River Edge, NJ, 2001. pp.

[9] DE CORNULIER, Y. *Strongly bounded groups and infinite powers of finite groups.* Comm. Algebra 34 (2006), no. 7, 2337–2345.

[10] CULLER, M. & MORGAN, J. W. *Group actions on $\mathbb{R}$-trees.* Proc. London Math. Soc. (3) 55 (1987), no. 3, 571–604.

[11] DROSTE, M. & GÖBEL, R. *Uncountable cofinalities of permutation groups.* J. London Math. Soc. (2) 71 (2005) 335–344.

[12] DROSTE, M. & HOLLAND, W. C. *Generating automorphism groups of chains.* Forum Math. 17 (2005) 699–710.

[13] GLASNER, E. & WEISS, B. *The topological Rohlin property and topological entropy.* Amer. J. Math. 123 (2001), no. 6, 1055–1070.

[14] GROMOV, M. *Metric structures for Riemannian and non-Riemannian spaces.* Based on the 1981 French original. With appendices by M. Katz, P. Pansu and S. Semmes. Translated from the French by Sean Michael Bates. Progress in Mathematics, 152. Birkhuser Boston, Inc., Boston, MA, 1999.

[15] GROMOV, M. & MILMAN, V. D. *A topological application of the isoperimetric inequality.* Amer. J. Math. 105 (1983), no. 4, 843–854.

[16] HEJCMAN, J. *Boundedness in uniform spaces and topological groups.* Czechoslovak Math. J. 9 (84) 1959 544–563.

[17] HEJCMAN, J. *On simple recognizing of bounded sets.* Comment. Math. Univ. Carolin. 38 (1997), no. 1, 149–156.

[18] HJORTH, G. *Classification and orbit equivalence relations.* Mathematical Surveys and Monographs, 75. American Mathematical Society, Providence, RI, 2000.

[19] HODGES, W.; HODKINSON, I.; LASCAR, D. & SHELAH, S. *The small index property for ω-stable ω-categorical structures and for the random graph.* J. London Math. Soc. (2) 48 (1993), no. 2, 204–218.

[20] KECHRIS, A. S. *Classical descriptive set theory.* Graduate Texts in Mathematics, 156. Springer-Verlag, New York, (1995).

[21] KECHRIS, A. S. & ROSENDAL, C. *Turbulence, amalgamation and generic automorphisms of homogeneous structures.* Proc. London Math. Soc., 94 (2007) no.2, 302-350.

[22] KHÉLIF, A. *À propos de la propriété de Bergman.* C. R. Math. Acad. Sci. Paris 342 (2006), no. 6, 377–380.

[23] KOPPELBERG, S. & TITS, J. *Une propriété des produits directs infinis de groupes finis isomorphes.* C. R. Acad. Sci. Paris Sr. A 279 (1974), 583–585.

[24] MACPHERSON, D. & THOMAS, S. *Comeagre conjugacy classes and free products with amalgamation.* Discrete Math. 291 (2005), no. 1-3, 135–142.

[25] VAN MILL, J. *Infinite-dimensional topology. Prerequisites and introduction.* North-Holland Mathematical Library, 43. North-Holland Publishing Co., Amsterdam, 1989.

[26] MILLER, B. D. *Full groups, classification, and equivalence relations.* Dissertation, UC Berkeley 2004.

[27] PESTOV, V.G. *The isometry group of the Urysohn space as a Lévy group.* To appear in: Proceedings of the 6-th Iberoamerican Conference on Topology and its Applications (Puebla, Mexico, 4-7 July 2005).

[28] RICARD, É. & ROSENDAL, C. *On the algebraic structure of the unitary group*, Collect. Math. 58,2 (2007), 181-192.

[29] SAXL, J.; SHELAH, S. & THOMAS, S. *Infinite products of finite simple groups.* Trans. Amer. Math. Soc. 348 (1996), no. 11, 4611–4641.

[30] SERRE, J.-P. *Trees.* Translated from the French original by John Stillwell. Corrected 2nd printing of the 1980 English translation. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003.

[31] SHELAH, S. *On a problem of Kurosh, Jónsson groups, and applications.* Word problems, II (Conf. on Decision Problems in Algebra, Oxford, 1976), pp. 373–394, Stud. Logic Foundations Math., 95, North-Holland, Amsterdam-New York, 1980.

[32] SOLECKI, S. *Extending partial isometries.* Israel J. Math. 150 (2005), 315–332.

[33] THOMAS, S. *Infinite products of finite simple groups. II.* J. Group Theory 2 (1999), no. 4, 401–434.

[34] TOLSTYKH, V. *Infinite dimensional general linear groups are groups of universally finite width.* Infinite-dimensional general linear groups are groups of finite width. (Russian) Sibirsk. Mat. Zh. 47 (2006), no. 5, 1160–1166.

[35] TOLSTYKH, V. *On the Bergman property for the automorphism groups of relatively free groups.* J. London Math. Soc. (2) 73 (2006), no. 3, 669–680.

[36] TRUSS, J. K. *Generic automorphisms of homogeneous structures.* Proc. London Math. Soc. (3) 65 (1992), no. 1, 121–141.

[37] URYSOHN, P. *Sur un espace métrique universel.* Bull. Sci. Math. 51 (1927), 43-64, 74-90.

[38] USPENSKIĬ, V. V. *A universal topological group with a countable basis.* (Russian) Funktsional. Anal. i Prilozhen. 20 (1986), no. 2, 86–87.

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