A PURELY COMBINATORIAL PROOF OF GOWERS' THEOREM

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Abstract. We prove an exact, i.e., formulated without Δ-expansions, Ramsey principle for infinite block sequences in vector spaces over countable fields, where the two sides of the dichotomic principle are represented by respectively winning strategies in Gowers' block sequence game and winning strategies in the infinite asymptotic game. This allows us to recover Gowers' dichotomy theorem for block sequences in normed vector spaces by a simple application of the basic determinacy theorem for infinite asymptotic games.

1. Introduction

The results presented here represent a new approach to the fundamental result of W.T. Gowers [7], whose uses in Banach space theory seem far from exhausted (for applications see, e.g., [7, 6]). Gowers' result is a Ramsey theoretic statement for Banach spaces that combines Ramsey theory and game theory to compensate for the fact that a true Ramsey theoretic result fails to hold in general. The proof of Gowers' theorem, however, involves approximation arguments, which at times are a bit delicate, as can be seen from the existing proofs [7, 3, 2, 8, 1], and also hitherto seemed to require tricks not previously used in infinite-dimensional Ramsey theory. Perhaps more importantly, the notion of *weakly Ramsey sets* extracted from the proof incorporates approximations, which makes it hard to induct over and extend beyond the class of analytic sets. For example, it was unknown whether $\mathbf{\Sigma}^1_2$ sets are weakly Ramsey assuming Martin's axiom, though it was shown to hold under a strengthening of MA not equiconsistent with ZF by J. Bagaria and J. López-Abad [3].

The novelty of our approach lies in the replacement of both sides of the dichotomy with game theoretical statements, which completely eschew approximations and allow for a very simple inductive proof. The new tools are the *infinite asymptotic game* and the definition of *strategically Ramsey sets* in vector spaces over countable fields. Using these, one easily shows that under MA, $\mathbf{\Sigma}^1_2$ sets are strategically Ramsey, and a version of the basic determinacy result for infinite asymptotic games [10] connects the notions of weakly Ramsey and strategically Ramsey sets.

2. Notation

Let $\mathfrak{F}$ be a countable field and let E be a countable dimensional $\mathfrak{F}$-vector space with basis (e_n). We equip E with the discrete topology, whereby any subset is open, and equip its countable power E^∞ with the product topology. Since E is a

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The present paper is largely a reworking of the author's paper [11] with slicker proofs.

countable discrete set, E^∞ is a Polish space. Notice that a basis for the topology on E^∞ is given by sets of the form

$$N(x_0,\dots,x_k)=\{(y_n)\in E^\infty \bigm| y_0=x_0 \ \& \ \dots \ \& \ y_k=x_k\},$$

where $x_0,\dots,x_k\in E$. Let x,y,z,v be variables for *non-zero* elements of E. If $x=\sum a_ne_n\in E$, let supp $x=\{n \bigm| a_n\neq 0\}$ and set for $x,y\in E$,

$$x<y\Leftrightarrow \forall n\in \text{supp } x\ \forall m\in \text{supp } y\ \ n<m.$$

Similarly, if k is a natural number, we set

$$k<x\Leftrightarrow \forall n\in \text{supp } x\ \ k<n.$$

Analogous notation is used for finite subsets of $\mathbb{N}$. A finite or infinite sequence $(x_0,x_1,x_2,x_3,\dots)$ of vectors is said to be a block sequence if for all n, $x_n<x_{n+1}$.

Notice that, by elementary linear algebra, for all infinite dimensional subspaces $X\subseteq E$ there is a subspace $Y\subseteq X$ spanned by an infinite block sequence, called a *block subspace*. Henceforth, we use variables X,Y,Z,V,W to denote infinite dimensional block subspaces of E. Also, denote infinite block sequences by variables $\mathbf{x},\mathbf{y},\mathbf{z}$ and finite block sequences by variables $\vec{x},\vec{y},\vec{z}$.

If T is a set of finite block sequences, we let

$$[T]=\{\mathbf{x}\in E^\infty \bigm| \text{if } \vec{x} \text{ is a finite initial segment of } \mathbf{x}\text{, then } \vec{x}\in T\}.$$

3. Gowers' game and the infinite asymptotic game

Suppose $X\subseteq E$. We define *Gowers' game* G_X played below X between two players I and II as follows: I and II alternate (with I beginning) in choosing respectively infinite dimensional subspaces $Y_0,Y_1,Y_2,\ldots\subseteq X$ and vectors $x_0<x_1<x_2<\dots$ according to the constraint $x_i\in Y_i$:

I	Y_0		Y_1		Y_2		Y_3		$\dots$
II		x_0		x_1		x_2		x_3	$\dots$

Also, the *infinite asymptotic game* F_X played below X is defined as follows: I and II alternate (with I beginning) in choosing respectively natural numbers $n_0<n_1<n_2<\dots$ and vectors $x_0<x_1<x_2<\ldots\in X$ according to the constraint $n_i<x_i$:

I	n_0		n_1		n_2		n_3		$\dots$
II		x_0		x_1		x_2		x_3	$\dots$

In both games we say that the sequence $(x_n)_{n\in\mathbb{N}}$ is the *outcome* of the game. Moreover, if $\vec{x}$ is a finite block sequence, we define Gowers' game $G_X(\vec{x})$ and the infinite asymptotic game $F_X(\vec{x})$ as above except that the outcome is now $\vec{x}^\frown(x_0,x_1,x_2,\dots)$.

If X and Y are subspaces, where Y is spanned by an infinite block sequence $\mathbf{y}=(y_0,y_1,y_2,\dots)$, we write $Y\subseteq^* X$ if there is n such that $y_m\in X$ for all $m\geq n$. A simple diagonalisation argument shows that if $X_0\supseteq X_1\supseteq X_2\supseteq\dots$ is a decreasing sequence of block subspaces, then there is some $Y\subseteq X_0$ such that $Y\subseteq^* X_n$ for all n.

The aim of the games above is for each of the players to ensure that the outcome $\mathbf{x}$ lies in some predetermined set depending on the player. By the asymptotic nature of the game, it is easily seen that if $\mathbb{A}\subseteq E^\infty$ and $Y\subseteq^* X$, then if II has a strategy in G_X to play in $\mathbb{A}$, i.e., to ensure that the outcome is in $\mathbb{A}$, then II will have a strategy in G_Y to play in $\mathbb{A}$ too. Similarly, if I has a strategy in F_X to play in $\mathbb{A}$, then I also has a strategy in F_Y to play in $\mathbb{A}$.

Definition 1. *We say that a set $\mathbb{A} \subseteq E^\infty$ is* strategically Ramsey *if for all $V \subseteq E$ and all $\vec{z}$, there is $W \subseteq V$ such that either*

(a) *II has a strategy in $G_W(\vec{z})$ to play in $\mathbb{A}$, or*
(b) *I has a strategy in $F_W(\vec{z})$ to play in $\sim \mathbb{A}$.*

4. Analytic sets are strategically Ramsey

Lemma 2. *Open sets $\mathbb{U} \subseteq E^\infty$ are strategically Ramsey.*

Proof. Let $V \subseteq E$ and $\vec{z}$ be given. By a simple diagonalisation over all finite block sequences $\vec{x}$, we can find some $X \subseteq V$ such that for all $Y \subseteq X$ and $\vec{x}$,

II has a strategy in $G_Y(\vec{x})$ to play in $\mathbb{U}$ if and only if
II has a strategy in $G_X(\vec{x})$ to play in $\mathbb{U}$.

By a further diagonalisation over all finite block sequences $\vec{x}$, we can find some $Y \subseteq X$ such that for all $\vec{x}$,

if there is some $Z \subseteq Y$ such that for all $y \in Z$, II has no strategy in $G_X(\vec{x}\,\hat{}\,y)$ to play in $\mathbb{U}$,
then there is some n such that for all $y \in Y$, if $n < y$, then II has no strategy in $G_X(\vec{x}\,\hat{}\,y)$ to play in $\mathbb{U}$.

So, finally, combining these two properties, we have for all $\vec{x}$

if there is some $Z \subseteq Y$ such that for all $y \in Z$, II has no strategy in $G_Y(\vec{x}\,\hat{}\,y)$ to play in $\mathbb{U}$,
then there is some n such that for all $y \in Y$, if $n < y$, then II has no strategy in $G_Y(\vec{x}\,\hat{}\,y)$ to play in $\mathbb{U}$.

Now, let T be the set of all $\vec{x}$ such that II has no strategy in $G_Y(\vec{x})$ to play in $\mathbb{U}$. Since $\mathbb{U}$ is open, we have $[T] \cap \mathbb{U} = \emptyset$. Also, suppose that $\vec{x} \in T$. Then, as II has no strategy in $G_Y(\vec{x})$ to play in $\mathbb{U}$, there is some $Z \subseteq Y$ such that for all $y \in Z$, II has no strategy in $G_Y(\vec{x}\,\hat{}\,y)$ to play in $\mathbb{U}$, and so for some n and all $y \in Y$, if $n < y$, then II has no strategy in $G_Y(\vec{x}\,\hat{}\,y)$ to play in $\mathbb{U}$, i.e., $\vec{x}\,\hat{}\,y \in T$. Thus, if $\vec{z} \in T$, then T provides a quasi-strategy for I in $F_Y(\vec{z})$ to play in $[T] \subseteq \sim \mathbb{U}$. And if $\vec{z} \notin T$, then II has a strategy in $G_Y(\vec{z})$ to play in $\mathbb{U}$. Since V and $\vec{z}$ are arbitrary, this shows that $\mathbb{U}$ is strategically Ramsey. □

Lemma 3. *Suppose $\mathbb{A}_n \subseteq E^\infty$ and $\mathbb{B} = \bigcup_{n\in\mathbb{N}} \mathbb{A}_n$. Let $\vec{x}$ and $X \subseteq E$ be given. Then there is $Z \subseteq X$ such that either*

(a) *II has a strategy in G_Z to play (z_i) such that*

$$\exists n\, \forall V \subseteq Z \text{ I has no strategy in } F_V(\vec{x}\,\hat{}\,(z_0, \ldots, z_n)) \text{ to play in } \sim \mathbb{A}_n,$$

or

(b) *I has a strategy in $F_Z(\vec{x})$ to play in $\sim \mathbb{B}$.*

Proof. We say that a pair $(\vec{y}, n)$ *accepts* a block subspace Y if I has a strategy in $F_Y(\vec{y})$ to play in $\sim \mathbb{A}_n$. Also, $(\vec{y}, n)$ *rejects* Y if $\forall Z \subseteq Y$, $(\vec{y}, n)$ does not accept Z. Notice that acceptance and rejection are $\subseteq^*$-hereditary, so there is $Y \subseteq X$ such that for all $\vec{y}$ and n, either $(\vec{y}, n)$ accepts or rejects Y. Set

$$\mathbb{D} = \{(z_i) \mid \exists n\, (\vec{x}\,\hat{}\,(z_0, \ldots, z_n), n) \text{ rejects } Y\}$$

and notice that $\mathbb{D}$ is open. It follows, by Lemma 2, that there is $Z \subseteq Y$ such that either II has a strategy in G_Z to play in $\mathbb{D}$ or I has a strategy in F_Z to play in $\sim \mathbb{D}$.

In the first case, II has a strategy in G_Z to play (z_i) such that

$$\exists n\, \forall V \subseteq Y \text{ I has no strategy in } F_V(\vec{x}\,\hat{}\,(z_0, \ldots, z_n)) \text{ to play in } \sim \mathbb{A}_n,$$

which immediately implies (a). So suppose instead that I has a strategy in F_Z to play in $\sim \mathbb{D}$, i.e., that I has a strategy in F_Z to play (z_i) such that

$$\forall n\, (\vec{x}\,\hat{}\,(z_0, \ldots, z_n), n) \text{ accepts } Z.$$

Thus, I has a strategy σ in F_Z to play (z_i) such that for all n, I has a strategy $\sigma_{(z_0,\ldots,z_n)}$ in $F_Z(\vec{x}\,\hat{}\,(z_0, \ldots, z_n))$ to play in $\sim \mathbb{A}_n$. By successively putting more and more strategies into play, I thus has a strategy in $F_Z(\vec{x})$ to play in $\bigcap_n \sim \mathbb{A}_n = \sim \mathbb{B}$, which gives us (b). Concretely, if at step $n+1$, $(z_0, \ldots, z_n)$ has been played, then I will respond with

$$\max\{\sigma(z_0, \ldots, z_n), \sigma_{(z_0)}(z_1, z_2, \ldots, z_n), \ldots, \sigma_{(z_0,\ldots,z_n)}(\emptyset)\}.$$

It follows that if (z_i) is the outcome of the game, then for all n, as II has responded to a stronger strategy than $\sigma_{(z_0,\ldots,z_n)}$ when playing $(z_{n+1}, z_{n+2}, \ldots)$, we see that $\vec{x}\,\hat{}\,(z_0, \ldots, z_n)\hat{}\,(z_{n+1}, z_{n+2}, \ldots) \in \sim \mathbb{A}_n$. Therefore, $\vec{x}\,\hat{}\,(z_i) \in \bigcap_n \sim \mathbb{A}_n$. □

Notice that both conclusions (a) and (b) in Lemma 3 are $\subseteq^*$-hereditary in Z.

Theorem 4. *Analytic sets are strategically Ramsey.*

Proof. Suppose $\mathbb{A} \subseteq E^\infty$ is analytic. Noting that for all $V \subseteq E$ and $\vec{z}$, $\mathbb{A}^V_{\vec{z}} = \{(x_i) \in V^\infty \mid \vec{z}\,\hat{}\,(x_i) \in \mathbb{A}\}$ is also an analytic subset of V^∞, we can by relativising to V^∞, suppose that $V = E$ and $\vec{z} = \emptyset$. Also, without loss of generality, $\mathbb{A} \neq \emptyset$.

As $\mathbb{A}$ is analytic, we can find a continuous surjection $F\colon \mathbb{N}^\mathbb{N} \to \mathbb{A}$ and set for every $s \in \mathbb{N}^{<\mathbb{N}}$, $\mathbb{A}_s = F[N_s]$, where $N_s = \{\alpha \in \mathbb{N}^\mathbb{N} \mid s \subseteq \alpha\}$. We note that $\mathbb{A}_s = \bigcup_{n\in\mathbb{N}} \mathbb{A}_{s\hat{}n}$. Let also $\mathbb{D}(s, \vec{x}, X)$ be the set

$$\{(z_i) \mid \exists n\, \forall W \subseteq X \text{ I has no strategy in } F_W(\vec{x}\,\hat{}\,(z_0, \ldots, z_n)) \text{ to play in } \sim \mathbb{A}_{s\hat{}n}\}.$$

Using Lemma 3 on each pair $(\vec{x}, s)$, we find $X \subseteq E$ such that for all $\vec{x}$ and all $s \in \mathbb{N}^{<\mathbb{N}}$ either

(a) II has a strategy in G_X to play in $\mathbb{D}(s, \vec{x}, X)$, or
(b) I has a strategy in $F_X(\vec{x})$ to play in $\sim \mathbb{A}_s$.

Suppose that I has no strategy in F_X to play in $\sim \mathbb{A} = \sim \mathbb{A}_\emptyset$. We describe a strategy for II in G_X to play in $\mathbb{A}$.

First, as II has a strategy in G_X to play in $\mathbb{D}(\emptyset, \emptyset, X)$, he follows this strategy until $(z_0, \ldots, z_{n_0})$ has been played such that I does not have a strategy in

$$F_X(z_0, \ldots, z_{n_0})$$

to play in $\sim \mathbb{A}_{n_0}$.

Thus, by the assumption on X, II must have a strategy in G_X to play in $\mathbb{D}((n_0), (z_0, \ldots, z_{n_0}), X)$. II now switches to follow this strategy until some further $(z_{n_0+1}, \ldots, z_{n_0+n_1+1})$ has been played such that I does not have a strategy in

$$F_X(z_0, \ldots, z_{n_0}, z_{n_0+1}, \ldots, z_{n_0+n_1+1})$$

to play in $\sim \mathbb{A}_{(n_0,n_1)}$.

So again, by the assumption on X, II must have a strategy in G_X to play in the set $\mathbb{D}((n_0, n_1), (z_0, \ldots, z_{n_0+n_1+1}), X)$. He again switches to follow this strategy until yet another $(z_{n_0+n_1+2},\ldots,z_{n_0+n_1+n_2+2})$ has been played such that I does not have a strategy in

$$F_X(z_0,\ldots,z_{n_0},z_{n_0+1},\ldots,z_{n_0+n_1},z_{n_0+n_1+1},\ldots,z_{n_0+n_1+n_2+2})$$

to play in $\sim\mathbb{A}_{(n_0,n_1,n_2)}$.

Continuing in this way and letting $m_k=(\sum_{j\leq k}n_j)+k$, the outcome of the game will be a sequence

$$\mathbf{z}=(z_0,z_1,z_2,\ldots,z_{m_0},\ldots,z_{m_1},\ldots,z_{m_2},\ldots)$$

such that for the sequence $\alpha=(n_0,n_1,n_2,\ldots)$ and all k, I does not have a strategy in $F_X(z_0,\ldots,z_{m_k})$ to play in $\sim\mathbb{A}_{(n_0,n_1,\ldots,n_k)}$. It follows that for all k, there must be an infinite block sequence $\mathbf{z}_k$ end-extending $(z_0,\ldots,z_{m_k})$ such that $\mathbf{z}_k\in\mathbb{A}_{(n_0,n_1,\ldots,n_k)}$. So for some $\beta_k\in N_{(n_0,n_1,\ldots,n_k)}$, we have $F(\beta_k)=\mathbf{z}_k$. But, by continuity of F, we have $F(\beta_k)\underset{k\to\infty}{\longrightarrow}F(\alpha)$, while $\mathbf{z}_k\underset{k\to\infty}{\longrightarrow}\mathbf{z}$, so $F(\alpha)=\mathbf{z}$ and hence $\mathbf{z}\in\mathbb{A}$. Therefore, this describes a strategy for II in G_X to play in $\mathbb{A}$. □

As is clear from the proof of Theorem 4, in the case when II has a strategy in G_X to play $\mathbf{z}\in\mathbb{A}$, he is at the same time able to continuously produce a witness $\alpha\in\mathbb{N}^{\mathbb{N}}$ such that $F(\alpha)=\mathbf{z}$. As it happens, this turns out to be useful in applications, so let us give a precise formulation of this fact.

Suppose $X\subseteq E$. We define the *unfolded Gowers game* H_X played below X between two players I and II as follows: I and II alternate (with I beginning) in choosing infinite dimensional subspaces $Y_0,Y_1,Y_2,\ldots\subseteq X$, respectively vectors $x_0<x_1<x_2<\ldots$ and $\epsilon_i\in\mathbb{N}\cup\{\#\}$, according to the constraint $x_i\in Y_i$ and with the demand that $\epsilon_i\in\mathbb{N}$ for infinitely many i.

I	Y_0		Y_1		Y_2		Y_3		$\ldots$
II		x_0,ϵ_0		x_1,ϵ_1		x_2,ϵ_2		x_3,ϵ_3	$\ldots$

We say that the pair of sequences $((x_i)_{i\in\mathbb{N}},(\epsilon_i)_{\epsilon_i\in\mathbb{N}})\in X^\infty\times\mathbb{N}^{\mathbb{N}}$ is the *outcome* of the game. (So essentially, when II plays $\epsilon_i=\#$, we can think of this as him delaying the decision on the next coordinate of the sequence $(\epsilon_n)_{\epsilon_n\in\mathbb{N}}$.)

Theorem 5. *Suppose $\mathbb{B}\subseteq E^\infty\times\mathbb{N}^{\mathbb{N}}$ is analytic and $\mathbb{A}=\mathrm{proj}_{E^\infty}(\mathbb{B})$. Then there is an $X\subseteq E$ such that either*

(a) *II has a strategy in H_X to play in $\mathbb{B}$, or*
(b) *I has a strategy in F_X to play in $\sim\mathbb{A}$.*

Proof. Without loss of generality, $\mathbb{B}\neq\emptyset$. Since $\mathbb{B}$ is analytic, there is a continuous surjection $f\colon\mathbb{N}^{\mathbb{N}}\to\mathbb{B}$. Let also $\pi\colon E^\infty\times\mathbb{N}^{\mathbb{N}}\to E^\infty$ denote the first coordinate projection, so that $F=\pi\circ f\colon\mathbb{N}^{\mathbb{N}}\to\mathbb{A}$ is a continuous surjection.

Now, by inspection of the proof of Theorem 4, we see that if there is no $X\subseteq E$ such that I has a strategy in F_X to play in $\sim\mathbb{A}$, then there is some $X\subseteq E$ such that II has a strategy in H_X to produce $\big((z_i)_{i\in\mathbb{N}},(\epsilon_i)_{i\in\mathbb{N}}\big)\in X^\infty\times(\mathbb{N}\cup\{\#\})^{\mathbb{N}}$ such that

$$F\big((\epsilon_i)_{\epsilon_i\in\mathbb{N}}\big)=\big((z_i)_{i\in\mathbb{N}}\big).$$

(Simply note that any coordinate of α is produced after finitely many steps of the game G_X, so it suffices to let $(\epsilon_i)_{\epsilon_i\in\mathbb{N}}=\alpha$.)

So for II to play in $\mathbb{B}$ in the game H_X, he plays an auxiliary game of H_X to find the $\big((z_i)_{i\in\mathbb{N}},(\epsilon_i)_{i\in\mathbb{N}}\big)$ such that $F\big((\epsilon_i)_{\epsilon_i\in\mathbb{N}}\big)=\big((z_i)_{i\in\mathbb{N}}\big)$ as above. Letting $\beta\in\mathbb{N}^{\mathbb{N}}$

be such that

$$f\big((\epsilon_i)_{\epsilon_i\in\mathbb{N}}\big)=\big((z_i)_{i\in\mathbb{N}},\beta\big),$$

for II to play in $\mathbb{B}$, it suffices to play the secondary sequence $(\delta_i)_{i\in\mathbb{N}}$ such that $(\delta_i)_{\delta_i\in\mathbb{N}}=\beta$. □

5. Infinite asymptotic games in normed vector spaces

Suppose now that $\mathfrak{F}$ is a countable *subfield* of $\mathbb{R}$ or $\mathbb{C}$ and $\|\cdot\|\colon E\to\mathbb{R}_{\geqslant 0}$ is a norm on E. For $X\subseteq E$, denote by $\mathcal{B}_X$ the unit ball of X and by $\mathfrak{B}(X)$ the set of block sequences (x_i) of X with $\|x_i\|\leq 1$. A set $\mathbb{A}\subseteq E^\infty$ is said to be *large* if for all $X\subseteq E$, $\mathbb{A}\cap\mathfrak{B}(X)\neq\emptyset$. Also, if $\Delta=(\delta_i)$ is a sequence of strictly positive real numbers, denoted by $\Delta>0$, and $\mathbb{A}\subseteq E^\infty$, we define

$$\mathbb{A}_\Delta=\{(z_i)\in E^\infty \bigm| \exists(x_i)\in\mathbb{A}\;\forall i\;\|x_i-z_i\|<\delta_i\},$$

$$\mathrm{Int}_\Delta(\mathbb{A})={\sim}({\sim}\,\mathbb{A})_\Delta=\{(x_i)\bigm|\forall(z_i)\;(\forall i\;\|x_i-z_i\|<\delta_i\to(z_i)\in\mathbb{A})\}.$$

To get a stronger statement in (b) of the definition of strategically Ramsey sets, we need to allow approximations. For this, we use a variant of a result from [10], though the proof given here is in the same spirit as that presented in [6].

Theorem 6. *Suppose there is a strategy σ for I in F_X to play in the set $\mathbb{B}\subseteq E^\infty$. Then for any sequence $\Delta>0$ there are intervals $I_0<I_1<I_2<\ldots$ of $\mathbb{N}$ such that for any block sequence $(x_i)\in\mathfrak{B}(X)$, if*

$$\forall n\;\exists m\;I_0<x_n<I_m<x_{n+1},$$

then $(x_i)\in\mathbb{B}_\Delta$.

Proof. Choose sets $\mathbb{D}_n\subseteq\mathcal{B}_X$ such that for each finite $d\subseteq\mathbb{N}$, the number of $x\in\mathbb{D}_n$ such that supp $x=d$ is finite, and for every $x\in\mathcal{B}_X$ there is some $y\in\mathbb{D}_n$ with supp $x=$ supp y and $\|x-y\|<\delta_n$. This is possible since the unit ball in $[e_i]_{i\in d}$ is totally bounded for all finite $d\subseteq\mathbb{N}$.

For each position $p=(n_0,y_0,\ldots,n_i,y_i)$ in F_X played according to σ in which $y_j\in\mathbb{D}_j$ for all j, we write $p<k$ if $n_j,y_j<k$ for all j. Notice that for all k there are only finitely many such p with $p<k$, so we can define

$$\alpha(k)=\max(k,\max\{\sigma(p)\bigm| p<k\})$$

and set $I_k=[k,\alpha(k)]$. The I_k are not necessarily successive, but their minimal elements tend to ∞. So, modulo passing to a subsequence, it is enough to show that if $(x_i)\in\mathfrak{B}(X)$ and

$$\forall n\;\exists m\;I_0<x_n<I_m<x_{n+1},$$

then $(x_i)\in\mathbb{B}_\Delta$.

Suppose such (x_i) is given. Find $y_i\in\mathbb{D}_i$ such that $\|x_i-y_i\|<\delta_i$ and supp $x_i=$ supp y_i for all i and let $0=b_0<b_1<b_2<\ldots$ be integers such that

$$I_{b_0}<y_0<I_{b_1}<y_1<I_{b_2}<y_2<\ldots.$$

We claim that there are natural numbers $n_i\leq\max I_{b_i}$ such that each

$$p_i=(n_0,y_0,\ldots,n_i,y_i)$$

is a position in F_X in which I has played according to σ. To see this, notice first that $n_0=\alpha(\emptyset)\in I_{b_0}$, so $p_0=(n_0,y_0)$ is played according to σ. Now, for the induction step, suppose that p_i is played according to σ, and notice that $p_i<\min I_{b_{i+1}}=b_{i+1}$.

We set $n_{i+1} = \sigma(p_i) \leq \alpha(b_{i+1}) = \max I_{b+1}$, whereby p_{i+1} is played according to σ. This finishes the induction and proves the claim.

Thus, $(n_0, y_0, n_1, y_1, \ldots)$ is a run of the game in which I has followed the strategy σ and so $(y_i) \in \mathbb{B}$, whereby $(x_i) \in \mathbb{B}_\Delta$. □

Theorem 7. *Suppose $\mathbb{A} \subseteq E^\infty$ is strategically Ramsey and for some $\Delta > 0$, $\mathrm{Int}_\Delta(\mathbb{A})$ is large. Then there is $X \subseteq E$ such that II has a strategy in G_X to play in $\mathbb{A}$.*

Proof. Suppose for a contradiction that for some $X \subseteq E$, I has a strategy in F_X to play in $\sim \mathbb{A} = E^\infty \setminus \mathbb{A}$. Then, using Theorem 6, we can find some $Y \subseteq X$ such that $\mathfrak{B}(Y) \subseteq (\sim \mathbb{A})_\Delta$, contradicting that $\mathrm{Int}_\Delta(\mathbb{A})$ is large. So since $\mathbb{A}$ is strategically Ramsey there is instead $X \subseteq E$ such that II has a strategy in G_X to play in $\mathbb{A}$. □

6. Strategically Ramsey sets under set theoretical hypotheses

Theorem 8. *The class of strategically Ramsey sets is closed under countable unions.*

Proof. Let $\mathbb{A}_n$ be strategically Ramsey for every n and set $\mathbb{B} = \bigcup_n \mathbb{A}_n$. Let $\vec{x}$ and $X \subseteq E$ be given. Since each $\mathbb{A}_n$ is strategically Ramsey, by diagonalising, there is some $Y \subseteq X$ such that for all $\vec{y}$ and n, either II has a strategy in $F_Y(\vec{y})$ to play in $\mathbb{A}_n$ or I has a strategy in $G_Y(\vec{y})$ to play in $\sim \mathbb{A}_n$. Also, by Lemma 3 there is $Z \subseteq Y$ such that either

(a) II has a strategy in G_Z to play (z_i) such that
$$\exists n\, \forall V \subseteq Z \text{ I has no strategy in } F_V(\vec{x}\,\hat{}\,(z_0, \ldots, z_n)) \text{ to play in } \sim \mathbb{A}_n,$$
or

(b) I has a strategy in $F_Z(\vec{x})$ to play in $\sim \mathbb{B}$.

Note that (a) implies that II has a strategy in G_Z to play (z_i) such that
$$\exists n \text{ II has a strategy in } G_Z(\vec{x}\,\hat{}\,(z_0, \ldots, z_n)) \text{ to play in } \mathbb{A}_n.$$

And, in this case, II first follows the strategy to play some $(z_0, \ldots, z_n)$ such that II has a strategy in $G_Z(\vec{x}\,\hat{}\,(z_0, \ldots, z_n))$ to play in $\mathbb{A}_n$ and thereafter continues with this other strategy. This, combined, is a strategy for II in $G_Z(\vec{x})$ to play in $\mathbb{B} = \bigcup_m \mathbb{A}_m$. □

Theorem 9 (MA_{ω_1})**.** *A union of $\aleph_1$ many strategically Ramsey sets is again strategically Ramsey.*

Proof. By Theorem 8, it is enough to consider well-ordered increasing unions of length ω_1. So suppose $\mathbb{A}_\xi \subseteq \mathbb{A}_\zeta \subseteq E^\infty$ are strategically Ramsey for all $\xi < \zeta < \omega_1$ and $\mathbb{B} = \bigcup_{\zeta<\omega_1} \mathbb{A}_\zeta$. Fix $\vec{x}$ and $X \subseteq E$. Since every $\mathbb{A}_\xi$ is strategically Ramsey, we can define a decreasing sequence $\ldots \subseteq^* X_\xi \subseteq^* \ldots \subseteq^* X_2 \subseteq^* X_1 \subseteq^* X_0 \subseteq X$ of length ω_1 such that for all $\xi < \omega_1$ either

(a) II has a strategy in $G_{X_\xi}(\vec{x})$ to play in $\mathbb{A}_\xi$, or
(b) I has a strategy in $F_{X_\xi}(\vec{x})$ to play in $\sim \mathbb{A}_\xi$.

If for some ξ, II has a strategy in $G_{X_\xi}(\vec{x})$ to play in $\mathbb{A}_\xi$, then II also has a strategy in $G_{X_\xi}(\vec{x})$ to play in $\mathbb{B} = \bigcup_{\zeta<\omega_1} \mathbb{A}_\zeta$ and we are done. So suppose instead that for every ξ, I has a strategy in $F_{X_\xi}(\vec{x})$ to play in $\sim \mathbb{A}_\xi$. By Lemma 5 in [5], under MA_{ω_1} there is a $Y \subseteq X$ such that $Y \subseteq^* X_\xi$ for all ξ. Thus, for every ξ, I has a strategy σ_ξ in $F_Y(\vec{x})$ to play in $\sim \mathbb{A}_\xi$.

Notice that σ_ξ is formally a function from the countable set D of finite block sequences $\vec{y}$ of Y to the set of natural numbers and hence a member of $\mathbb{N}^D$. By MA_{ω_1}, the family $\{\sigma\}_{\xi<\omega_1}$ cannot be $\leq^*$ unbounded in $\mathbb{N}^D$ and hence for some $\sigma \in \mathbb{N}^D$ we have $\sigma_\xi \leq^* \sigma$ for all ξ, i.e., for all ξ there is a finite set $p_\xi \subseteq D$ such that

$$\forall \vec{y} \in D \setminus p_\xi \quad \sigma_\xi(\vec{y}) \leq \sigma(\vec{y}).$$

By reason of cardinality, there is some $p \subseteq D$ such that for an unbounded set $S \subseteq \omega_1$ we have $p_\xi = p$ for all $\xi \in S$. Now let n_0 be large enough such that $n_0 \not< y_0$ for all $\vec{y} = (y_0, \ldots, y_m) \in p$. We modify σ so that $\sigma(\emptyset) = n_0$ and otherwise leave it unaltered. Then σ is a strategy for I in $F_Y(\vec{x})$ to play in $\sim \mathbb{B} = \bigcap_{\xi<\omega_1} \sim \mathbb{A}_\xi = \bigcap_{\xi \in S} \sim \mathbb{A}_\xi$. To see this, suppose that (z_i) is the outcome of a game in which I has followed σ. Then as $n_0 < z_0$, we must have $(z_0, \ldots, z_m) \notin p$ for all m, and hence for all $\xi \in S$ and m, $\sigma(z_0, \ldots, z_m) = \sigma_\xi(z_0, \ldots, z_m)$. If follows that for every $\xi \in S$, I has followed the strategy σ_ξ and hence $(z_i) \notin \mathbb{A}_\xi$. □

Since $\mathbf{\Sigma}^1_2$ sets are unions of $\aleph_1$ many Borel sets, we have the following strengthening of a result of Bagaria and López-Abad [3]. They essentially proved the conclusion of Theorem 7 for $\mathbf{\Sigma}^1_2$ sets, but only under a hypothesis relatively consistent with the existence of a large cardinal. On the other hand, our hypothesis, namely MA_{ω_1}, is equiconsistent with ZF, which permits the use of absoluteness arguments.

Corollary 10 (MA_{ω_1})**.** $\mathbf{\Sigma}^1_2$ *sets are strategically Ramsey.*

We do not know if the axiom of projective determinacy suffices to prove that all projective sets are strategically Ramsey, though we very much suspect so. Again, Bagaria and López-Abad [4] proved that under PD, projective sets are weakly Ramsey.

7. Adversarial games

In this section we consider adversarial versions of Gowers' game and the infinite asymptotic game in which both players contribute to the outcome. Unfortunately, we can in this case only prove the Ramsey principle for open and closed sets. Simpler adversarial games were first considered by A. M. Pelczar [9], where a specific instance of Theorem 11 below was used to prove that any space saturated with subsymmetric sequences must contain a minimal subspace. Related uses of Theorem 11 can be found in [6].

Suppose $X \subseteq E$. We define the game A_X played below X between two players I and II as follows: I and II alternate in choosing block subspaces $Z_0, Z_1, Z_2, \ldots \subseteq X$ and vectors $x_0 < x_1 < x_2 < \ldots \in X$, respectively integers $n_0 < n_1 < n_2 < \ldots$ and vectors $y_0 < y_1 < y_2 < \ldots \in X$ according to the constraints $n_i < x_i$ and $y_i \in Z_i$:

I		$n_0 < x_0, Z_0$		$n_1 < x_1, Z_1$		$n_2 < x_2, Z_2$	$\ldots$
II	n_0		$y_0 \in Z_0, n_1$		$y_1 \in Z_1, n_2$		$\ldots$

We say that the sequence $(x_0, y_0, x_1, y_1, \ldots)$ is the *outcome* of the game.

If $\vec{x}$ is a finite block sequence of *even* length, the game $A_X(\vec{x})$ is defined as above except that the outcome is now $\vec{x}^\frown(x_0, y_0, x_1, y_1, \ldots)$.

On the other hand, if $\vec{x}$ is a finite block sequence of *odd* length, $A_X(\vec{x})$ is defined in a similar way as before except that I begins

I	Z_0		$n_0 < x_0, Z_1$		$n_1 < x_1, Z_2$		$\dots$
II		$y_0 \in Z_0, n_0$		$y_1 \in Z_1, n_1$		$y_2 \in Z_2, n_2$	$\dots$

and the *outcome* is now $\vec{x}\,\hat{}\,(y_0, x_0, y_1, x_1, \dots)$ rather than $\vec{x}\,\hat{}\,(x_0, y_0, x_1, y_1, \dots)$.

We define the game B_X in a similar way to A_X except that we now have I playing integers and II playing block subspaces:

I		$x_0 \in Z_0, n_0$		$x_1 \in Z_1, n_1$		$x_2 \in Z_2, n_2$	$\dots$
II	Z_0		$n_0 < y_0, Z_1$		$n_1 < y_1, Z_2$		$\dots$

with $x_i \in Z_i \subseteq X$ and $n_i < y_i \in X$. Again, the *outcome* is $(x_0, y_0, x_1, y_1, \dots)$.

If $\vec{x}$ is a finite block sequence of *even* length, the game $B_X(\vec{x})$ is defined as above except that the outcome is now $\vec{x}\,\hat{}\,(x_0, y_0, x_1, y_1, \dots)$.

On the other hand, if $\vec{x}$ is a finite block sequence of *odd* length, $B_X(\vec{x})$ is defined by letting I begin

I	n_0		$x_0 \in Z_0, n_1$		$x_1 \in Z_1, n_2$		$\dots$
II		$n_0 < y_0, Z_0$		$n_1 < y_1, Z_1$		$n_2 < y_2, Z_2$	$\dots$

and the *outcome* is now $\vec{x}\,\hat{}\,(y_0, x_0, y_1, x_1, \dots)$.

Thus, in both games A_X and B_X, one should remember that I is the *first* to play a vector. And in A_X, I plays block subspaces and II plays tail subspaces, while in B_X, II takes the role of playing block subspaces and I plays tail subspaces.

Suppose $\mathbb{A} \subseteq E^\infty$, $Y \subseteq^* X$ and $\vec{x}$ are given. Then one easily sees that if II has a strategy in $A_X(\vec{x})$ to play in $\mathbb{A}$, then II also has a strategy in $A_Y(\vec{x})$ to play in $\mathbb{A}$. Similarly, if I has a strategy in $B_X(\vec{x})$ to play in $\mathbb{A}$, then I also has a strategy in $B_Y(\vec{x})$ to play in $\mathbb{A}$. Also, if II has a strategy in $A_X(\vec{x})$ to play in $\mathbb{A}$, then II also has a strategy in $B_X(\vec{x})$ to play in $\mathbb{A}$.

Theorem 11. *Suppose $\mathbb{A} \subseteq E^\infty$ is open or closed. Then there is $X \subseteq E$ such that either*

(1) *II has a strategy in A_X to play in $\mathbb{A}$, or*
(2) *I has a strategy in B_X to play in $\sim \mathbb{A}$.*

Proof. Suppose first that $\mathbb{A}$ is open. We say that

(a) $(\vec{x}, X)$ is *good* if II has a strategy in $A_X(\vec{x})$ to play in $\mathbb{A}$.
(b) $(\vec{x}, X)$ is *bad* if $\forall Y \subseteq X$, $(\vec{x}, Y)$ is not good.
(c) $(\vec{x}, X)$ is *worse* if it is bad and either
  (1) $|\vec{x}|$ is odd and $\exists n\, \forall y \in X\ (n < y \to (\vec{x}\,\hat{}\,y, X)$ is bad), or
  (2) $|\vec{x}|$ is even and $\forall Y \subseteq X\ \exists x \in Y\ (\vec{x}\,\hat{}\,x, X)$ is bad).

One checks as always that good, bad and worse are all $\subseteq^*$-hereditary.

Lemma 12. *If $(\vec{x}, X)$ is bad, then there is some $Z \subseteq X$ such that $(\vec{x}, Z)$ is worse.*

Proof. By diagonalisation, we can find some $Y \subseteq X$ such that for all $\vec{y}$, $(\vec{y}, Y)$ is either good or bad.

Assume first that $|\vec{x}|$ is even. Since $(\vec{x}, Y)$ is bad, we have $\forall V \subseteq X$ II has no strategy in $A_V(\vec{x})$ to play in $\mathbb{A}$. So $\forall V \subseteq X\ \exists x \in V$ such that II has no strategy in

$A_V(\vec{x}\hat{\ }x)$ to play in $\mathbb{A}$, and hence such that $(\vec{x}\hat{\ }x, V)$ is not good. Thus,

$$\forall V \subseteq X\ \exists x \in V\ (\vec{x}\hat{\ }x, Y) \text{ is bad},$$

and so already $(\vec{x}, Y)$ is worse.

Now suppose instead that $|\vec{x}|$ is odd and, towards a contradiction, that there is no $Z \subseteq Y$ such that $(\vec{x}, Z)$ is worse. Then, as $(\vec{x}, Y)$ is bad, $\forall Z \subseteq Y\ \exists y \in Z\ (\vec{x}\hat{\ }y, Z)$ is not bad and thus also $\forall Z \subseteq Y\ \exists y \in Z\ (\vec{x}\hat{\ }y, Y)$ is good. So

$$\forall Z \subseteq Y\ \exists y \in Z\ \text{ II has a strategy in } A_Y(\vec{x}\hat{\ }y) \text{ to play in } \mathbb{A},$$

and hence II also has a strategy in $A_Y(\vec{x})$ to play in $\mathbb{A}$, contradicting that $(\vec{x}, Y)$ is bad. □

Diagonalising, we now find $X \subseteq E$ such that for all $\vec{x}$, either $(\vec{x}, X)$ is good or worse. Assume that II has no strategy in A_X to play in $\mathbb{A}$, whereby $(\emptyset, X)$ is worse. Then, by unraveling the definition of worse and using that bad and worse coincide below X, one sees that I has a strategy in B_X to produce block sequences $(z_0, z_1, z_2, \ldots)$ so that for all m, $(z_0, z_1, \ldots, z_m, X)$ is worse. In particular, for no m does II have a strategy in $A_X(z_0, \ldots, z_m)$ to play in $\mathbb{A}$, and so, as $\mathbb{A}$ is open, we must have $(z_0, z_1, z_2, \ldots) \in \sim \mathbb{A}$. So I has a strategy in B_X to play in $\sim \mathbb{A}$, which finishes the proof for open sets.

Now if instead $\mathbb{A}$ is closed, set

$$\mathbb{B} = \{x\hat{\ }\mathbf{x} \mid x \in E \ \&\ \mathbf{x} \notin \mathbb{A}\} = E \times \sim \mathbb{A},$$

which is open. So find some $X \subseteq E$ such that either

(1) II has a strategy in A_X to play in $\mathbb{B}$, or
(2) I has a strategy in B_X to play in $\sim \mathbb{B}$.

Now if II has a strategy in A_X to play in $\mathbb{B}$, then I has a strategy in B_X to play in $\sim \mathbb{A}$. And if I has a strategy in B_X to play in $\sim \mathbb{B}$, then II has a strategy in A_X to play in $\mathbb{A}$, which is what needed proof. □

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