The compactness theorem for first order logic

Recall that a *theory* T of a first order language L is just a set of L-sentences.

Thm An L-theory T has a model if and only if any finite subset $T_0 \subseteq T$ has a model.

Corollary Suppose a theory T has arbitrarily large finite models. Then T has an infinite model.

Pf Let $\{c_n\}_{n \in \mathbb{N}}$ be distinct new constant symbols not in L. Let

$$S = T \cup \{ c_n \neq c_m \mid n, m \in \mathbb{N}, n \neq m \}.$$

Then, by the compactness theorem, S has a model.

$\square$

Non-standard models of arithmetic.

Let $L = \{0, 1, +, \cdot, <\}$, where $0, 1$ are constant symbols, $+, \cdot$ binary fct symbols, and $<$ a binary relation symbol.

Let also $\mathcal{N} = \langle N, 0, 1, +, \cdot, < \rangle$ be the standard L-structure and let T be the theory of $\mathcal{N}$, i.e.,

$$T = \{ F \mid F \text{ is an } L\text{-sentence } \& \ \mathcal{N} \models F \}.$$

So T consists of all L-sentences that are true of the natural numbers.

For simplicity, let $\underline{n}$ denote the term

$$\underbrace{((1+1)+1) \dots +1}_{n \text{ times}}.$$

Let also c be a new constant symbol.

Consider the theories

$$S_1 = T \cup \{ \underline{n} < c \mid n \in \mathbb{N} \}$$

$$S_2 = T \cup \{ \exists y \; \underline{n} \cdot y = c \mid n \in \mathbb{N} \}.$$

Thus, by the compactness theorem, both S_1, S_2 have a model.

Exercise

Show that if $\mathcal{M} \models S_2$, then also $\mathcal{M} \models S_1$.