ON NEARLY EQUIVALENT FORMS FOR GSp(4)

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To the memory of Ilya Piatetski-Shapiro

Abstract. This is a note about the failure of multiplicity one.

The purpose of this note is to show the following:

Theorem 1. *Let F be a totally real number field, and n an even natural number. Then for every set of places S of cardinality n, there is a pair of automorphic cuspidal representations $\pi_1 = \otimes'_v \pi_{1,v}$ and $\pi_2 = \otimes'_v \pi_{2,v}$ such that*

- *for all $v \notin S$, $\pi_{1,v} \simeq \pi_{2,v}$;*
- *for all $v \in S$, $\pi_{1,v} \not\simeq \pi_{2,v}$.*

The idea is to combine the methods of the two papers [H-PS] and [P-TB]. Namely, we will use the following diagram

$$\begin{tikzcd}
\mathrm{GO}(2,2) \arrow[dr, "\theta"] & \\
& \mathrm{GSp}(4) \\
\mathrm{GO}(4) \arrow[uu, dotted, "JL"] \arrow[ur, "\theta"] &
\end{tikzcd}$$

The vertical arrow was constructed in §7 of [H-PS]. We will in fact use this diagram for representations of GSO instead of GO where representations of GSO are trivially extended to GO. A representation, local or global, of GSO(4) then corresponds to a pair (π_1, π_2) of representations of the multiplicative group of a non-split quaternion algebra D with the same central characters. Similarly a representation of GSO(2, 2) corresponds to a pair (ρ_1, ρ_2) of representations of GL(2) with the same central character. In each case we denote the representation of the

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group GSO by the corresponding pair of representations. Then the vertical arrow is shown to be

$$(\pi_1, \pi_2) \mapsto (\pi_1^{JL}, \pi_2^{JL})$$

locally or globally. One can then consider in the local or global situation the theta lift of the representations (π_1, π_2) and (π_1^{JL}, π_2^{JL}) to GSp(4).

Lemma 1. *In the local situation, the representations $\theta(\pi_1, \pi_2)$ and $\theta(\pi_1^{JL}, \pi_2^{JL})$ are non-zero and are inequivalent.*

Proof. In the archimedean situation this lemma is well-known. The non-vanishing of the local theta lifts follows from Remark 6.8 of [P-TB]. In order to prove that the two representations are inequivalent we will use Bessel models. Namely we will show the existence of a Bessel model for the representation $\theta(\pi_1^{JL}, \pi_2^{JL})$ that other representations cannot have. By Corollary 7.1 of [P-TB] the representation $\theta(\pi_1^{JL}, \pi_2^{JL})$ will have a (T, χ) Bessel model if and only if the two representations π_1^{JL}, π_2^{JL} have (T, χ)-Waldspurger models. It follows from Tunnell's Dichotomy Theorem [S, T] that if T is split, π_1^{JL} and π_2^{JL} will have (T, χ)-Waldspurger models. On the other hand, by the same corollary since π_1, π_2 do not have (T, χ)-Waldspurger models, $\theta(\pi_1, \pi_2)$ cannot have (T, χ)-Bessel models. □

Lemma 2. *Let D be a non-split quaternion algebra over a totally real number field F which is ramified at places in a set S. Let (π_1, π_2) be a pair of non-isomorphic automorphic representations of $D^\times(\mathbb{A}_F)$, and suppose that $\theta(\pi_1, \pi_2)$ is non-zero. Then*

- *for $v \in S$, $\theta(\pi_1, \pi_2)_v \not\simeq \theta(\pi_1^{JL}, \pi_2^{JL})_v$;*
- *for $v \notin S$, $\theta(\pi_1, \pi_2)_v \simeq \theta(\pi_1^{JL}, \pi_2^{JL})_v$.*

Proof. This is a direct consequence of the previous lemma provided that we know that $\theta(\pi_1^{JL}, \pi_2^{JL})$ is non-zero. This is well-known (see for example [Ro]). □

We can now present the proof of the main theorem.

Proof of Theorem 1. Given a set S as in the statement of the theorem, there is a quaternion algebra D ramified precisely at the places in S. By Lemma 2 it suffices to find a pair of non-equivalent automorphic representations (π_1, π_2) of $D^\times(\mathbb{A}_F)$ such that $\theta(\pi_1, \pi_2)$ is non-zero. In order to do this we will use Theorem 4.1 of [P-S] to find a pair (π_1, π_2) of non-equivalent representations such that both representations have a global (T, χ)-Waldspurger model for some torus T. By §13 of [P-TB] the representations $\theta(\pi_1, \pi_2)$ will have a (T, χ)-Bessel model and hence cannot be zero. We will take T to be any torus embedded in $D^\times$, and

$\chi = \otimes_v \chi_v$ any automorphic character of T. Let $v \notin S$ be a place where T_v is split. Let $\pi_{1,v}, \pi_{2,v}$ be two non-equivalent supercuspidal representations of $D_v^\times \simeq \mathrm{GL}_2(F_v)$ with the same central character as $\chi_v\big|_{Z_v}$. By Tunnell's Dichotomy Theorem [S, T] $\pi_{1,v}, \pi_{2,v}$ both contain χ_v when restricted to T_v. Theorem 4.1 of [P-S] then says that there are automorphic representations π_1, π_2 with (T, χ)-Waldspurger models and such that their respective local components at v are $\pi_{1,v}, \pi_{2,v}$. Since $\pi_{1,v}, \pi_{2,v}$ are non-equivalent, we have $\pi_1 \not\simeq \pi_2$, and we are done. □

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