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Calculus for Middle School Teachers

Problems and Notes for MTHT 466

Bonnie Saunders **Fall 2010**

I. Infinity

Week 1. How big is Infinity?

Problem of the Week: The Chess Board Problem

There once was a humble servant who was also a chess master. He taught his king to play the game of chess. The king became fascinated by the game and offered the servant gold or jewels payment, but the servant replied that he only wanted rice: one grain for the first square of chess board, two on the second, four on the third, and so on with each square receiving twice as much as the previous square. The king quickly agreed. How much rice does the king owe the chess master? Suppose it was your job to pick up the rice. What might you use to collect it? A grocery sack, a wheelbarrow, or perhaps a Mac truck? Where might you store the rice?

1. What is the largest number your calculator will display?

2. What is the largest integer your calculator will display that has a 9 in the one's place? What happens if you add 1 to this number?

3. What is the largest number you can think of? Write it down, then write down a larger one.

4. (TEACHING PROJECT) Writing big numbers is made simpler by the use of exponents. Discuss definitions and basic rules for working with positive exponents.

5. The following are common errors made by middle school students when dealing with exponents. Identify the error. Why do you think the student made the error? Write a "teacher" explanation, using the basic rules of exponents, to help the students see the correct use of exponents.

   a) $9 \cdot 9 \cdot 9 \cdot 9 \cdot 9 = 99999$ b) $(5^2)^3 = 5^8$

   c) $3^2 \cdot 3^3 = 9^5$ d) $(2+3)^2 = 4+9$

Sets of numbers

$\mathbf{N} = \{1, 2, 3, 4, 5, ...\}$, Natural numbers.

$\mathbf{W} = \{0, 1, 2, 3, 4, 5, ...\}$, Whole numbers

$\mathbf{I} = \{... - 4, -3, -2, -1, 0, 1, 2, 3, 4, 5, ...\}$, Integers

$\mathbf{Q} = \{\frac{m}{n} : m \in \mathbf{I}, n \in \mathbf{I}, n \neq 0\}$, Rational numbers

$-$ Numbers that can be expressed as a ratio of two integers.

$\mathbf{R}$, Real numbers

6. Something to think about: Are there more @ signs or more $ signs? How do you tell without counting?

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@   @   @   @   @   @   @   @   @   @   @   @   @   @   @   @   @   @   @   @   @
$   $   $   $   $   $   $   $   $   $   $   $   $   $   $   $   $   $
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Definition of a one-to-one correspondence

A one-to-one correspondence between sets A and B is a pairing of each object in A with one and only one object in B.

Counting

Finite sets that can be put into a one-to-one correspondence necessarily have the same number of elements. A set that contains n objects can be put into a one-to-one correspondence with the set that contains the first n natural numbers. The process of making this particular one-to-one correspondence is called *counting*. When a child understands one to one correspondence, the child is ready to count.

7. Give examples of pairs of sets that can be put into one-to-one correspondence. Give examples of pairs of sets that can NOT be put into one-to-one correspondence.

8. Give an example of a pair of sets and two different ways that the two sets can be put into one-to-one correspondence.

Infinite sets

Sets with an infinite number of elements can be put into a one-to-one correspondence by using patterns or formulas.

9. Give an example of a pair of sets that have an infinite number of elements that can be put into one-to-one correspondence.

10. In what sense can we say that there are the same number of even whole numbers as there are whole numbers? In what sense can we say that there are twice as many whole numbers as even whole numbers?

11. In some sense, there is one more whole number than there are natural numbers. Explain the sense in which $\mathbf{W}$ and $\mathbf{N}$ contain the same number of numbers, that is show a one-to-one correspondence between these two sets.

Cardinality

If there is any way to put two sets into a one-to-one correspondence, we say that they have the same *cardinality*. We might say that the two sets have the same number of elements, but that is imprecise unless the set is finite.

12. Show a one-one-correspondence between $\mathbf{N}$ and each of the following sets. Show a pattern by making a table and use a formula.

a) the set of odd positive integers

b) the set of negative integers

c) the set of positive integers that are divisible by 3.

d) the set of powers of 10, that is $\{1, 10, 100, 1000, \cdots\}$

13. Show a one-one-correspondence between $\mathbf{W}$ and each of the sets in problem 12.

a) the set of odd positive integers

b) the set of negative integers

c) the set of positive integers that are divisible by 3.

d) the set of powers of 10, that is $\{1, 10, 100, 1000, \cdots\}$

14. CLASS DISCUSSION Are there more rational numbers than whole numbers? Discuss.

Handout

Dodge Ball

15. Write a short paragraph explaining the strategies for Player One and Player Two in the game of Dodge Ball.

16. CLASS DISCUSSION Are there more real numbers than rational numbers?

Sequences

A sequence is a list of numbers. They are often denoted with subscripts,

$$a_0, a_1, a_2, a_3, \cdots, a_n, \cdots$$

Sequences continue forever so that there is a one-to-one correspondence with W. The subscripts can also start with 1, instead of 0. (Or, in fact, any other integer, positive or negative) The subscript numbers are called *term numbers*. In elementary school, sequences are written down in tables. In middle school, children begin to learn how to write formulas for the numbers in the tables.

17. Joe has 107 baseball cards. Every week he buys 5 more cards. How many cards does he have after 3 weeks? How many cards does he have after one year? Make a table to help see the pattern. Write a formula to compute how many cards he has after any number of weeks. Explain your formula.

Arithmetic Sequences: constant consecutive difference

An *arithmetic sequence* is a sequence in which the difference between any two consecutive numbers in the sequence is constant. Arithmetic sequences are important in high school mathematics because formulas for arithmetic sequences correspond to equations for straight lines. The difference is that equations for straight lines use real numbers, not just integers, for the independent variable.

18. Write down three or four examples of arithmetic sequences. Write down three or four sequences that are not arithmetic sequences. Use tables or lists and formulas.

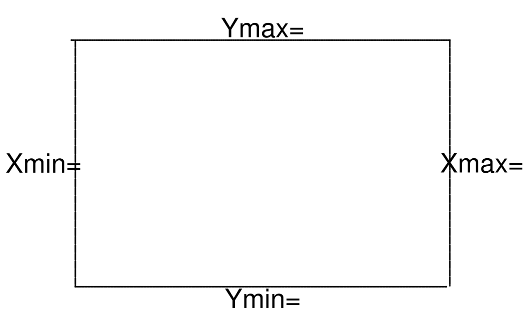
19. Remember an example of an arithmetic sequence from MATH 140 or MATH 141 that was constructed using pictures or objects. Make a table and write a formula for the sequence.

Using the calculator to make sequences

There are several ways to make sequences on a graphing caculator. One common way is to make a function on the $Y =$ screen and then to make a table.

20. Enter a function, Y_1, in your calculator and use it to make the table you created in problem 17.

21. Enter a function, Y_1, in your calculator and use it to make a table containing the arithmetic sequence $3, 10, 17, 24, 31, \ldots$. Graph the function with a reasonable window and record it below. Include the values for the window.



Y =

PRACTICE WITH LISTS

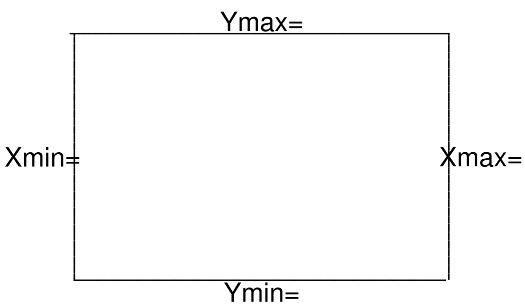
22. Make a list in your calculator containing the arithmetic sequence:

$$a_0 = 5, a_1 = 2, a_2 = -1, a_3 = -4, ...$$

What is the 1000^{th} term in this sequence? What is a_{1000}? Why are they not the same?

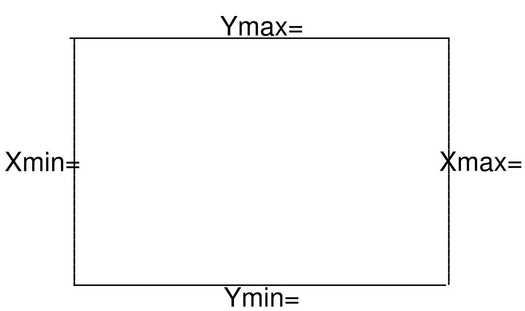
23. Make a list in your calculator containing each of these arithmetic sequences. Graph your results and answer the question.

a) $0.5, 0.75, 1.0, 1.25, ...$
If $0.5 = a_0$, what is the term number for 10.75?



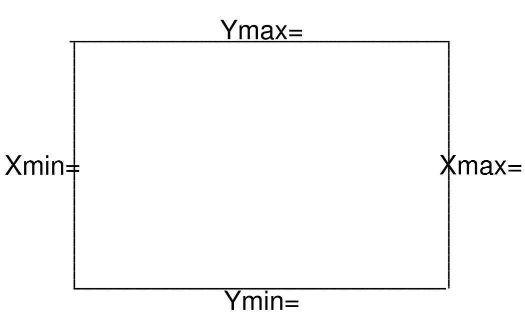
Y =

b) $0, 18, 36, 54, ...$
What number in this sequence is closest in value to 10000?



Y =

c) $159, 148, 137, 126, ...$
What is the first negative number in this sequence?



Y =

d) Make up an arithmetic sequence such that the 95th term is 0. Show the graph.

Ymax=

Xmin= Xmax=

Ymin=

Y =

e) Make up an arithmetic sequence such that the first term is 23 and the 15th term is 51. Do you think that everyone will make up the same sequence?

Ymax=

Xmin= Xmax=

Ymin=

Y =

f) Make up an arithmetic sequence such that the fifth term is 40 and the 9th term is 12. Do you think that everyone will make up the same sequence?

Ymax=

Xmin= Xmax=

Ymin=

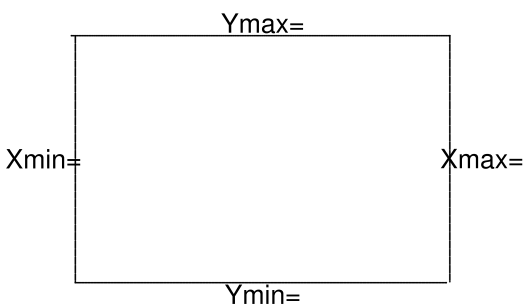
Y =

24. Consider the Problem of the Week and make a table to see the pattern. Write a formula to compute how many grains of sand there are on the nth square of the checkerboard. Explain your formula.

Geometric Sequences: constant consecutive ratios

A geometric sequence is a sequence in which the ratio between any two consecutive numbers is always the same. The sequence you made in problem 24 is a geometric sequence. The constant ratio is 2.

25. Enter a function, Y_1, in your calculator and use it to make a table containing the geometric sequence $1, 2, 4, 8, 16, \ldots$? Graph the function with a reasonable window and record it below. Include the values for the window.

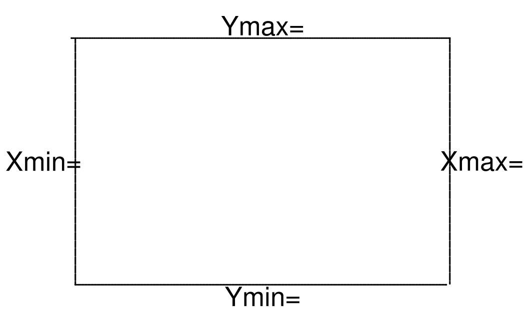


Y =

26. Consider the sequence that is the reciprocals of the numbers in the sequence in problem 25:

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$$

Graph it.



Y =

What is the smallest number in this sequence that your calculator can display exactly? What is the exact decimal expansion for the next number in the sequence?

MORE PRACTICE WITH LISTS

27. Make a list in your calculator containing each of these geometric sequences. Make a function and a table for each one. Plot and graph your results and answer the question.

a) $0.3, 0.09, 0.027, .0081, ...$
What number in this sequence is closest in value to 10^{-10}?

Ymax=

Xmin= Xmax=

Ymin=

Y =

b) $6, 18, 54, 162, ...$
What number in this sequence is closest in value to 10000?

Ymax=

Xmin= Xmax=

Ymin=

Y =

c) $15000, 3750, 937.5, ...$
What is the first number in this sequence that is less than 1?

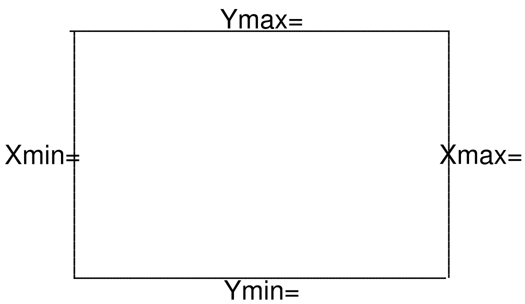
Ymax=

Xmin= Xmax=

Ymin=

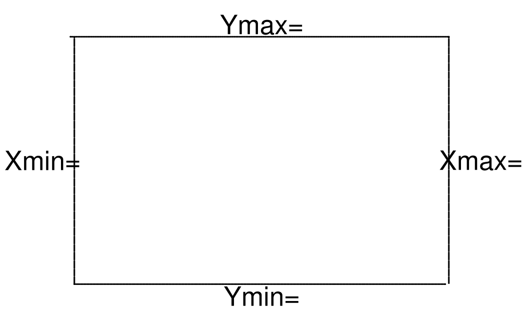
Y =

d) Make up a geometric sequence such that the 15th term is 1.
Do you think that everyone will make up the same sequence?



Y =

e) Make up an geometric sequence such that the third term is 3 and the 7th term is 48.
Do you think that everyone will make up the same sequence?

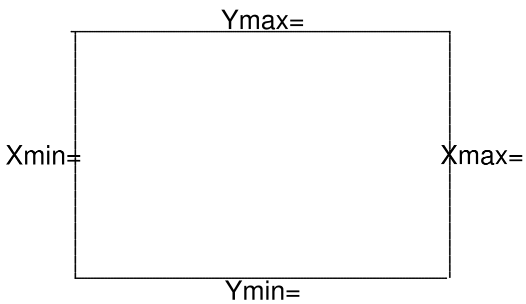


Y =

Other Kind of Sequences

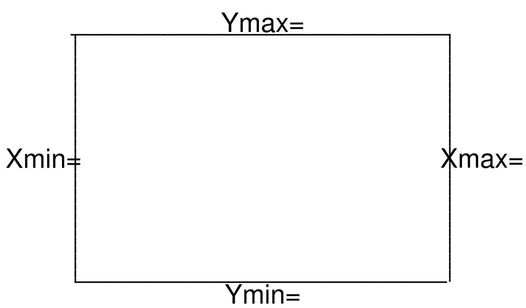
28. Decide on a likely pattern that continues each of these sequences. Make a table on your calculator for each of these sequences. Display the 500th term of each on your calculator and determine whether that value is exact or approximate (or too big or too small for your calculator to determine)?

a) $1, 4, 9, 16, \ldots$



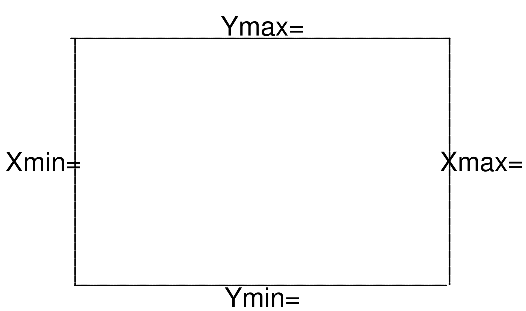
Y =

b) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$



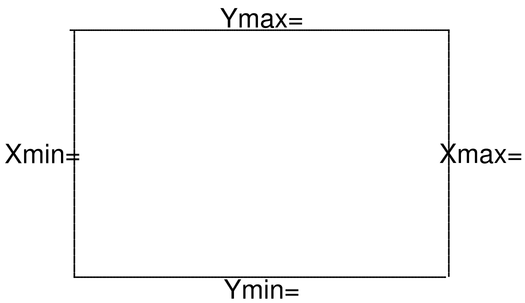
Y =

c) $0, 2, 6, 12, 20, \ldots$



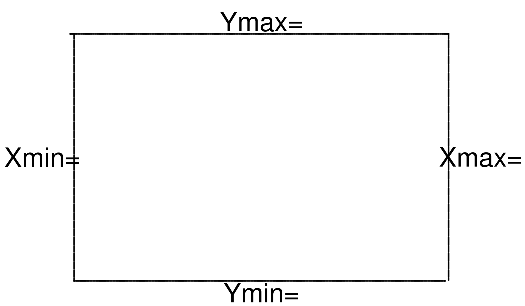
Y =

d) $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$



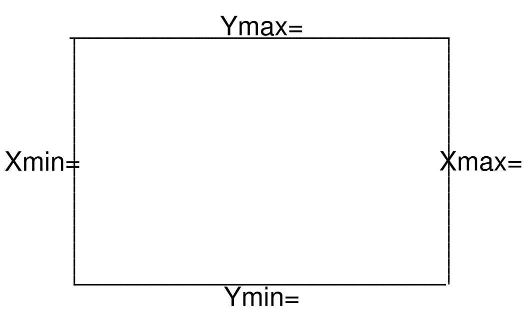
Y =

e) $1.0, 0.1, 0.01, 0.001, 0.0001, \ldots$



Y =

f) $1.0, 0.89, 0.78, 0.67, \ldots$



Y =

29. Conjecture and prove a formula for the nth term of an arithmetic sequence. Carefully describe all the variables in your formula.

30. Conjecture and prove a formula for the nth term of a geometric sequence. Carefully describe all the variables in your formula.

Week 2. How small is infinity? – Limits

Problem of the Week: A probability investigation

Suppose we have a box containing 5 white balls, 3 black balls, and 2 red balls. What is the probability that a ball, drawn at random, will be black? One at a time we add 3 balls – one of each color – to the box and recompute the probability of drawing a black ball. Is it possible that the probability will ever be $\frac{1}{3}$? What is the largest possible probability that can be realized?

31. What is the smallest positive number your calculator will display?

32. What is the number closest to $\frac{1}{3}$ that your calculator will display? Is this number – the one displayed – greater than or less than $\frac{1}{3}$? What is the difference between the two numbers? Write down a number that is closer to $\frac{1}{3}$ than the one your calculator displays.

33. Repeat all parts of problem 32 for $\frac{2}{3}$.

34. What is the number closest to $\sqrt{2}$ that your calculator will display? Can you tell if this number – the one displayed – is greater than or less than $\sqrt{2}$?

35. Give an several examples of what is suggested by these rather imprecise statements:

$$\frac{1}{BIG} = LITTLE$$

$$\frac{1}{LITTLE} = BIG$$

$$\frac{1}{CLOSETO1} = CLOSETO1$$

36. (TEACHING PROJECT) Writing small numbers is made simpler by the use of negative exponents. Discuss definitions and basic rules for working with non-positive exponents. Research: Find a reference for these definitions and rules from a book or on the internet.

37. The following are common errors made by middle school students when dealing with negative exponents. Identify the error. Why do you think the student made the error? Write a "teacher" explanation, in terms of the definition of exponents, to help the students see the correct use of exponents.

a) $\frac{1}{10^{-3}} < 1$

b) $10^{\frac{1}{4}} = \frac{1}{10^4}$

c) $0.0002356 \cdot 10^{20} < 1$

d) $7.459817 \cdot 10^{10}$ is not an integer

e) $28911 \cdot 10^{-100} > 1$

Definition: $\lim_{n\to\infty} a_n = 0$

No matter how small a positive number you can think of, there is a value of n large enough so that $\frac{1}{n}$ is smaller than that number for every single value of n that is greater than N.

38. Explain: For any integers, n and N, $n \geq N$ if and only if $\dfrac{1}{n} \leq \dfrac{1}{N}$

39. Consider: $\lim_{n\to\infty} \dfrac{1}{n} = 0$. Think of a very smaller number and call it e. Find an integer N such that $\dfrac{1}{N} < e$? Think of an even smaller value for e and find N such that $\dfrac{1}{N} < e$. Identify a procedure for finding such an N given any e.

Rules for working with limits

The sub-sequence rule for working with limits

If $\lim_{n\to\infty} a_n = 0$, then the limit of any subsequence is also 0.

The squeeze rule for working with limits

If $0 \le b_n \le a_n$ for all values of n and if $\lim_{n\to\infty} a_n = 0$, then $\lim_{n\to\infty} b_n = 0$

The multiplication rule for working with limits

If $\lim_{n\to\infty} a_n = 0$ and c is some number, then $\lim_{n\to\infty} c \cdot a_n = 0$.

40. Which rule tells you that the limit is equal to 0?

a) $\lim_{n\to\infty} 10^{-n}$

b) The sequence $0.3, 0.03, 0.003, 0.0003, 0.00003, \ldots$.

c) $\lim_{n\to\infty} \frac{1}{\sqrt{n^3}}$

41. Find three more examples of sequences that converge to 0.

Zooming in

Beckmann activities: 2D, 2H, 2J

42. Are there any real numbers contained in all of the intervals, $[-\frac{1}{n}, \frac{1}{n}]$, for all values of n?

43. Draw "zoom-in" picture for each of these sequences of nested interval problems. In each case the nested intervals contain exactly one number. What is that number? Give the value as a fraction of whole numbers, not a decimal.

    a) $[.3, .4], [.33, .34], [.333, .334], \ldots$

    b) The middle third of the middle third of the middle third ... of $[0, 1]$

    c) The first tenth of the last tenth of $[0, 1]$,
    then the last tenth of that interval,
    then the first tenth of that interval,
    then the last tenth of that interval, ...

Sequences and limits

Definition: limits that are not zero

If $\lim_{n\to\infty} a_n - a = 0$, then we say that $\lim_{n\to\infty} a_n = a$ for any positive integer m.

44. Find an example for each of the following rules.

Rules for working with limits

If $\lim_{n\to\infty} a_{n+m} = a$ for some positive integer m, then $\lim_{n\to\infty} a_n = a$

For any real number b, if $\lim_{n\to\infty} a_n = a$, then $\lim_{n\to\infty} a_n + b = a + b$

For any real number c, if $\lim_{n\to\infty} a_n = a$, then $\lim_{n\to\infty} ca_n = ca$

If $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$, then $\lim_{n\to\infty} a_n + b_n = a + b$

If $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$, then $\lim_{n\to\infty} a_n b_n = ab$

If $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$ *and* if $b \neq 0$, then $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$

If $a_n < b_n$ for all values of n, then $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$

The Advanced Squeeze Rule for working with limits

If $a_n \le b_n \le c_n$ for all values of n and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = d$, then $\lim_{n\to\infty} b_n = d$

A sequence may no have a limit

There are two ways by which a sequence may not have a limit.

1. The terms in the sequence may increase (or decrease) beyond all bounds. In this case, we would write $\lim_{n\to\infty} a_n = \infty$ (or $\lim_{n\to\infty} a_n = -\infty$). Examples: $1, 2, 3, 4, \ldots$ (or $-2, -4, -8, -16, \ldots$)
2. Two different subsequences may have different limits. Example: $1.9, 4.99, 1.999, 4.9999, 1.99999, 4.999999 \ldots$

45. Give examples of sequences that do not have limits.

46. Identify a pattern for each of the following sequences. Identify the limit exactly and explain how you know it is the limit. If there is no limit, explain why it doesn't exist.

a) $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

b) $1.0, 0.1, 0.01, 0.001, \ldots$

c) $6.1, 6.01, 6.001, 6.0001, 6.00001, \ldots$

d) $5.9, 5.99, 5.999, 5.9999, \ldots$

e) $0.3, 0.33, 0.333, 0.3333, 0.33333, \ldots$

f) $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \ldots$

g) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots$

h) $2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots$

i) $1.1, 1.9, 1.01, 1.99, 1.001, 1.999, \ldots$

j) $2\frac{1}{2}, 3\frac{1}{3}, 4\frac{1}{4}, 5\frac{1}{5}, \ldots$

k) $1.8, 1.88, 1.888, 1.8888, \ldots$

47. USING A CALCULATOR TO GUESS A LIMIT. Find a pattern and use your calculator to guess whether or not there is a limit for each of these sequences and what the limit may be. You should find more values of the sequence than those that appear on this page.

a) $\frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{8}{13}, \ldots$

b) $\sqrt{\frac{1}{2}}, \sqrt{\sqrt{\frac{1}{2}}}, \sqrt{\sqrt{\sqrt{\frac{1}{2}}}}, \ldots$

c) $\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots$

d) $a_n =$ the fractional part of $(.375n)$

e) $4\sqrt{2-\sqrt{2}}, 8\sqrt{2-\sqrt{2+\sqrt{2}}}, 16\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}, \ldots$

f) $2^1, (\frac{3}{2})^2, (\frac{4}{3})^3, (\frac{5}{4})^4, \ldots$

g) $2\sin(90°), 3\sin(60°), 4\sin(45°), 5\sin(36°), \ldots$

h) $4\sin(90°), 8\sin(45°), 16\sin(22.5°), \ldots$

i) $4\tan(90°), 8\tan(45°), 16\tan(22.5°), \ldots$

Week 3. Adding an infinity of numbers

Problem of the Week: A nickel vs million

It's your first day on a new job and you have your choice of two ways to be paid:
METHOD #1: You get one nickel the first day. From then on everyday you get one and one half times what you got the day before.
METHOD #2: You get a million dollars every day.
Which pay schedule would you choose and why?

First some formulas for adding finite sums

Handout

Picture proofs

48. Confirm each of these formulas for $n = 1$ to $n = 20$ by making a table of values:

a) $1 + 2 + 3 + 4 + 5 + \ldots + n = \frac{1}{2}n(n+1)$ or $\displaystyle\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$

b) $1 + 2^2 + 3^2 + 4^2 + 5^2 + \ldots + n^2 = \frac{1}{6}n(n+1)(2n+1)$ or $\displaystyle\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n-1)$

c) $1 + 3 + 5 + \ldots + 2n - 1 = n^2$ or $\displaystyle\sum_{k=1}^{n} (2k-1) = n^2$

49. Confirm each of this statements, guess a general formula, write it with summation notation. Can you think of a picture proof for your formula?

$$1 + \frac{1}{2} = 2 - \frac{1}{2}$$

$$1 + \frac{1}{2} + \frac{1}{4} = 2 - \frac{1}{4}$$

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 2 - \frac{1}{8}$$

50. Confirm each of this statements, guess a general formula, write it with summation notation. Can you think of a picture proof for your formula?

$$1 + 2 = 2^2 - 1$$

$$1 + 2 + 4 = 2^3 - 1$$

$$1 + 2 + 4 + 8 = 2^3 - 1$$

Geometric Series

Handout

Hughes-Hallett: Section: 9.1: #15, #16, #17, #18

51. Draw a picture to show how to add an infinite number of numbers.

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots + \frac{1}{2^n} + \ldots = \sum_{n=1}^{\infty} \frac{1}{2^n}$$

52. Use base ten blocks to show that $0.\overline{3} = \frac{1}{3}$. How is this infinite series involved in your argument?

$$\frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \ldots + \frac{3}{10^n} + \ldots = \sum_{n=1}^{\infty} \frac{3}{10^n}$$

53. Give a convincing argument that $0.\overline{9} = 1$. How is this infinite series involved in your argument?

$$\frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} + \frac{9}{10^4} + \ldots + \frac{9}{10^n} + \ldots = \sum_{n=1}^{\infty} \frac{9}{10^n}$$

54. Show that for any real number, $x \neq 1$,

$$\sum_{k=0}^{N} x^k = 1 + x + x^2 + x^3 + \ldots + x^N = \frac{x^{N+1} - 1}{x - 1}$$

Summing an infinite number of numbers

We can compute the sum an infinite number of terms by taking limits, for example we say that

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \ldots + x^N + \ldots = \lim_{N\to\infty} \sum_{k=0}^{N} x^k$$

Looking at the result from 54 we can see that

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \ldots + x^N + \ldots = \lim_{N\to\infty} \frac{x^{N+1} - 1}{x - 1}$$

Now we just take the limit as $N \to \infty$. If $0 \leq x < 1$, then $\lim\limits_{N\to\infty} x^N = 0$, so, in the end, the limit on the right is $\dfrac{1}{1-x}$ and we can have

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \ldots + x^n + \ldots = \frac{1}{1-x}$$

You will see another formula that has another variable. This formula comes about by multiplying everything by a constant, r.

$$\sum_{n=0}^{\infty} rx^n = r + rx + rx^2 + rx^3 + rx^4 + \ldots + rx^n + \ldots = \frac{r}{1-x}$$

You can use these two formulas, or modify them slightly to do the next problem.

55. Write each series using summation notation and find the limit. In each case the left hand problem gives you a formula to use for the right hand problem.

a) $1 + x + x^2 + x^3 + \ldots + x^n + \ldots =$

$1 + .1 + .01 + .001 + .0001 + \ldots + .1^n + \ldots =$

b) $x^3 + x^4 + \ldots + x^n + \ldots =$

$\frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots =$

c) $1 + x^2 + x^4 + \ldots + x^{2n} + \ldots =$

$1 + \frac{1}{9} + \frac{1}{81} + \frac{1}{729} + \ldots =$

d) $x^3 + x^6 + x^9 + x^{12} + \ldots =$

$0.001 + 0.000001 + 0.000000001 + \ldots =$

56. Find the limit for
$$\frac{21}{100}+\frac{21}{10000}+\frac{21}{1000000}+\cdots+\frac{21}{10^{2n}}+\cdots$$
Explain why your answer is the fraction form for the repeating decimal $0.\overline{21}$.

57. Find the fraction form for each of these repeating decimals. Write the number as a fraction of whole numbers. Write it as a mixed fraction.

a) $0.\overline{4}$

b) $0.\overline{36}$

c) $3.\overline{21}$

d) $5.1\overline{21}$

e) $3.4\overline{21}$

f) $51.3\overline{14}$

Using the calculator to compute other infinite sums

58. Other infinite sums may or may not have limits like the geometric series does. Use your calculator (or other means) to conjecture whether or not any of these infinite sums are finite.

a) $1+2+3+4+5+\ldots+n+\ldots=$

b) $1+1+1+1+1+\ldots+1+\ldots=$

c) $\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\ldots+\frac{1}{n(n+1)}\ldots=$

d) $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{n}+\ldots=$

e) $1+\frac{2}{2}+\frac{3}{4}+\frac{4}{8}+\ldots+\frac{n}{2^n}+\ldots=$

f) $\frac{1}{3}+\frac{1}{8}+\frac{1}{15}+\frac{1}{24}+\ldots+\frac{1}{n^2-1}+\ldots=$

II. Functions

Week 4. Functions: graphs, tables and formulas

Problem of the Week: The Farmer's Fence

A field bounded on one side by a river is to be fenced on three sides so as to form a rectangular enclosure. If 197 feet of fencing is to be used, what dimensions will yield an enclosure of the largest possible area?

Functions with graphs that are lines

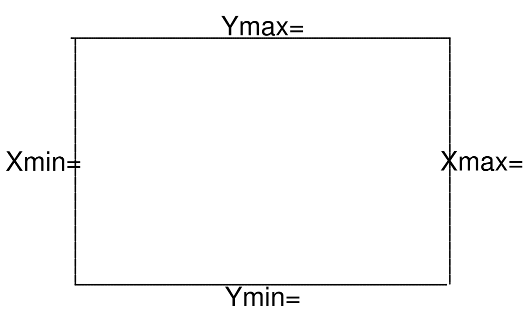
In Algebra I, students learn about equations for straight lines. But lines are geometric shapes and it is also important to understand the geometry of lines. Do problem 59 and problem 62 geometrically. That is, do not use an equation for the line until after you are asked to do so in the problems.

59. On a piece of graph paper with carefully labeled coordinate axes, sketch the line that goes through the two points $(1, 2)$ and $(3, 5)$. Give the exact coordinates for the following points on your graph. Show the points on your graph.

a) Three points on the line: one between the two given points and one on each side of the given points.
b) The point exactly half way between the two points.
c) The two points that divide the interval into three equal segments.
d) A point whose x–coordinate is negative and another point whose y–coordinate is negative.
e) A point whose x–coordinate is 0 and a point whose y–coordinate is 0.

60. Continuing with the line from problem 59, make a table of values, and describe the pattern. Write an equation that describes the line.

61. Find a way to graph the line from the problem 59 on your calculator. Sketch the calculator graph here, including the window values:



Y =

Use your calculator to generate a table of values for the line.

62. On a piece of graph paper with carefully labeled coordinate axes, sketch the line that goes through the the point $(2,1)$ and has slope equal to $\frac{1}{3}$. Give the exact coordinates for the following points on your graph. Show the points on your graph.

a) A point on the line on each side of the point $(2,1)$.

b) A point on the line where both the x–coordinate and y–coordinate are negative.

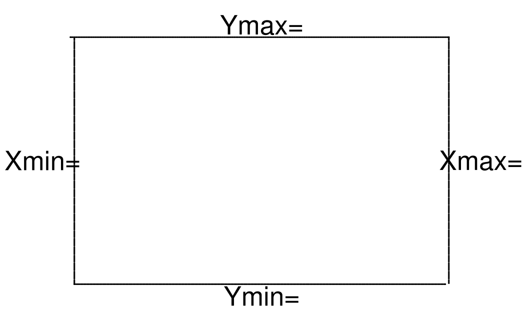
c) The point on the line whose x–coordinate is 0.

d) The point on the line whose y–coordinate is 0.

e) Two points on the line that are exactly $\sqrt{10}$ units from the point $(2,1)$. What is special about $\sqrt{10}$ for this problem?

63. Continuing with the line from problem 62, make a table of values, and describe the pattern. Write an equation that describes the line.

64. Find a way to graph the line from problem 62 on your calculator. Sketch the calculator graph here, including the window values:

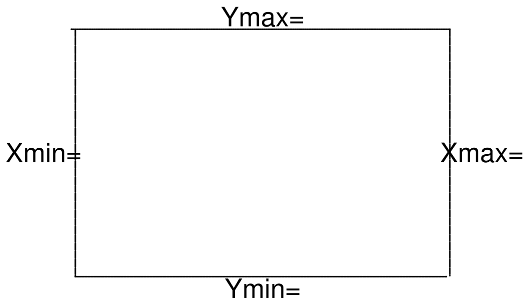


Y =

Use your calculator to generate a table of values for the line.

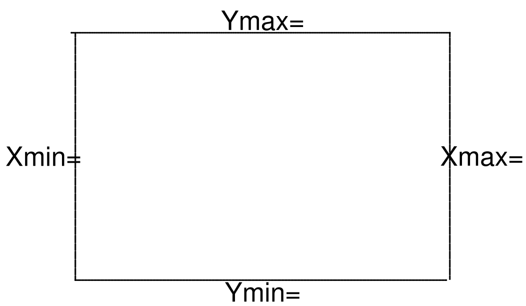
65. Sketch each line, then figure out how to draw each of these lines on your calculator:

a) A line that goes through the points $(-1, 1)$ and has slope, $\frac{2}{5}$.



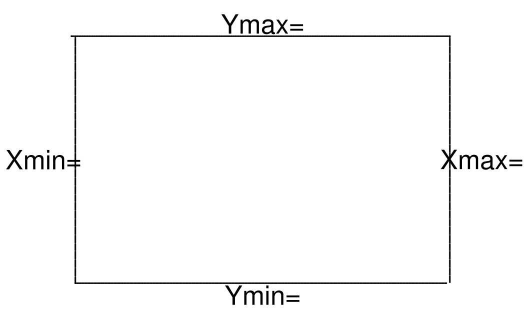
Y =

b) A line that goes through the point $(2, -3)$ and has slope 0.



Y =

c) A line that has slope -0.6 and goes through the point $(1, 1)$



Y =

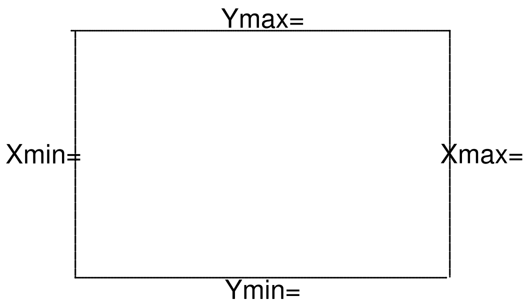
66. Explain why the equation

$$y = m(x - c) + d$$

describes the line through the point (c, d) that has slope m. Type this function into your "Y" page, using the letters m, c and d as well as x and use it to graph several different lines by changing the value of those three variables.

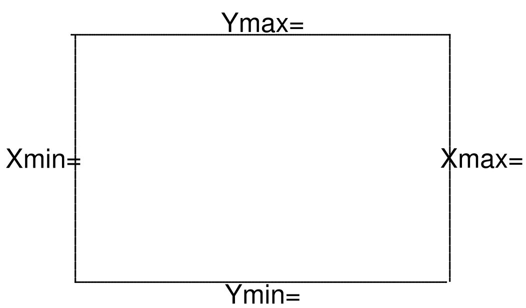
67. Figure out how to draw each of these lines on your calculator, then sketch a graph with a reasonable window.

a) A line that has slope 0.03 and goes through the point $(0.1, 1.01)$



Y

b) The line that goes through the two points $(1.5, 7.8)$ and $(-3.1, 10.9)$



Y

68. Sketch the line given by the equation, $y = 3x + 1$. Sketch the line that is perpendicular to this one and intersects it at the point $(0, 1)$. (Wait a minute! Is the point $(0, 1)$ really on the line?) Explain how to use the grid of the graph paper to be very accurate. Find an equation for this perpendicular line.

69. Suppose you are driving a constant speed from Chicago to Detroit, about 275 miles away. Sketch a graph of your distance from Chicago as a function of time. Why is this graph a straight line? What is the slope of the line? What are the units of the slope?

70. A car rental company offers cars at $\$40$ a day and 15 cents a mile. Its competitor's cars are $\$50$ a day and 10 cents a mile. Graph the cost of renting a car as a function of miles driven for each company. For each company, write a formula giving the cost of renting a car for a day as a function of the distance traveled. Explain why 15 cents a mile and 10 cents a mile each represent a slope. On your calculator, graph both functions on one screen. How should you decide which company is cheaper?

71. Consider a graph of Fahrenheit temperature, °F, against Celsius temperature, °C, and *assume that the graph is a straight line*. You know that $212°$F and $100°$C both represent the temperature at which water boils. Similarly, $32°$F and $0°$C represent water's freezing point. What is the slope of the graph? What are the units of the slope? What is the equation of the line? Use the equation to find what Fahrenheit temperature corresponds to $20°$C. Use the graph to find the same thing. What temperature is the same number of degrees in both Celsius and Fahrenheit? How can you see this on the graph?

Functions with graphs that are parabolas.

Definition: Parabola, quadratic

A *parabola* is the graph of a function which has the form

$$y = ax^2 + bx + c$$

Such functions are called *quadratic* functions. The right hand expression is called a *quadratic polynomial*.

72. Using the above definition, explain why the graph of $y = 3(x-1)(x-2)$ is a parabola. Where does this parabola intersect the x-axis? Graph it to check your answer.

Ymax=

Xmin= Xmax=

Ymin=

Y =

73. Explain why the graph of $y = 3(x-1)^2 + 2$ is a parabola. Explain how to tell that this parabola contains the point (1,2). What do you notice about the point $(1,2)$? Graph it to check.

Ymax=

Xmin= Xmax=

Ymin=

Y =

74. On your calculator, graph several different parabolas that cross the x-axis at $(1,0)$ and $(3,0)$. What do you notice about the vertex of all these parabolas? Is there a pattern? Look at more examples, changing the points where the parabola crosses the x-axis.

Ymax=

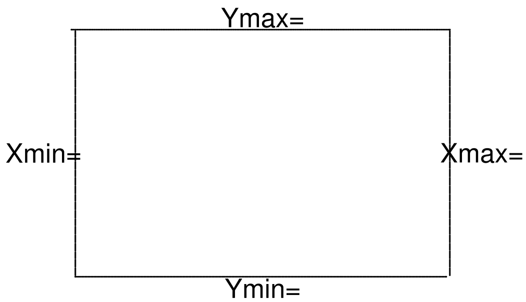
Xmin= Xmax=

Ymin=

Y =

75. Write a formula for the coordinates of the vertex of the parabola given generally by $y = a(x - z_1)(x - z_2)$.

76. On your calculator, graph a parabola that goes through the points $(-1, 0)$, $(0, 3)$ and $(1, 0)$.



Y =

77. Is it possible for a parabola to intersect the x-axis in only one point? If so, describe what would happen geometrically and find the equation for such a parabola.

78. Carefully explain FOIL. That is, explain why

$$(a + b)(c + d) = ac + ad + bc + bd.$$

What is the difference between explaining how to use FOIL and explaining why it works?

79. Where does the parabola $y = x^2 + 4x + 3$ cross the x-axis?
Where does the parabola $y = x^2 - 6x + 8$ cross the x-axis?
Where does the parabola $y = x^2 + x - 2$ cross the x-axis?
Where does the parabola $y = 3x^2 - 2x - 1$ cross the x-axis?

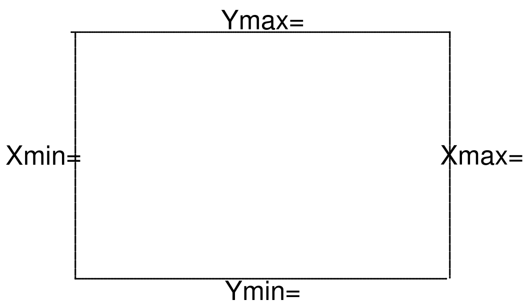
80. Find the points where the parabola $y = x^2 + 2x - 3$ crosses the x-axis by completing the square.
Find the points where the parabola $y = x^2 + 4x + 3$ crosses the x-axis.
Find the points where the parabola $y = x^2 + 4x - 3$ crosses the x-axis.
Find the points where the parabola $y = x^2 - x - 1$ crosses the x-axis.
Find the points where the parabola $y = 3x^2 + 4x - 6$ crosses the x-axis.

81. By completing the square, explain the quadratic formula. That is show why the two solutions to the equation $ax^2 + bx + c = 0$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

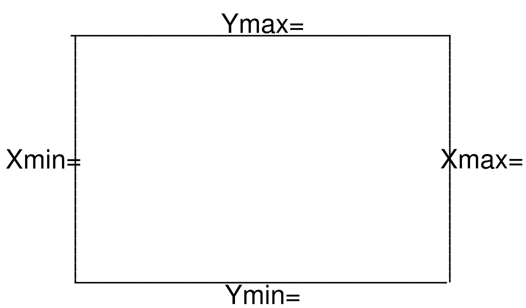
82. At what points does the parabola, $y = x^2 + x + 1$, cross the x-axis? Explain.

83. On your calculator, graph a parabola that has a vertex at the point $(-1, 3)$ and does not cross the x-axis.



Y =

84. Use your calculator to find an equation for the parabola that goes through the points$(-1, 2), (1, 4)$ and $(0, 1)$. Graph this parabola.



Y =

85. Explain the difference between solving the equation $ax^2 + bx + c = 0$ and graphing the equation $y = ax^2 + bx + c$. How does one help with doing the other?

86. Explain what factoring has to do with solving the equation $ax^2 + bx + c = 0$.

87. *The Algebra Nightmare* The two solutions to the equation $x^2 - 4x + 3 = 0$ are $x = 3$ and $x = 1$. And we know that $(x - 3)(x - 1) = x^2 - 4x + 3$. Something similar is true for all quadratic equations: We know that there are at most two solutions to a quadratic equation, $ax^2 + bx + c = 0$.
The two solutions are

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

and

$$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Show that

$$a(x - \frac{-b + \sqrt{b^2 - 4ac}}{2a})(x - \frac{-b - \sqrt{b^2 - 4ac}}{2a}) = ax^2 + bx + c$$

by multiplying out the left hand side and simplifying to get the right hand side.

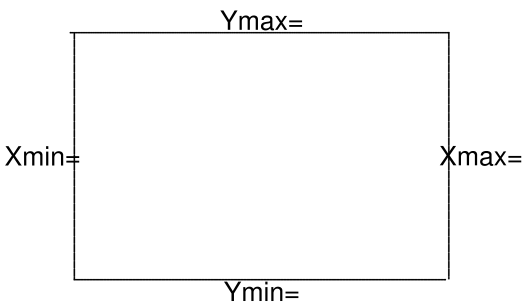
A famous quadratic function

If an object is thrown upward, dropped, or thrown downward and travels in a vertical line subject only to gravity (with wind resistance ignored), then the height s(t) of the object above the ground (in feet) after t seconds is given by

$$s(t) = -16t^2 + Vt + H$$

Where H is the initial height of the object at starting time ($t = 0$) and V is the initial velocity (speed in feet per second) of the object at starting time ($t = 0$).

88. Graph the function for the case when a rock is dropped from a height of 200 feet. Use the calculator to get a graph, then sketch it here. Include the units with your window values.



Y =

Use the graph to answer this question: How long does it take the rock to fall?

89. How high would the rock go if it were given an initial upward velocity of 20 feet per second?

90. How high will a tennis ball go if it is thrown from a height of 4 feet with an initial velocity of 80 feet per second? How long will it take to get there? Do you think you could throw it hard enough to get that initial velocity? What velocity is needed to throw the ball 10 feet in the air? (Surely that can be done.)

Week 5. More functions

Review:
Workshop Geometry: Unit 1, Lesson 1 -- the definition of a circle
Workshop Geometry: Unit 2, Lesson 4 -- the Pythagorean theorem

Problem of the Week: The closest point
Find the point on the line $y = 3x + 2$ that is closest to the point $(2, 3)$. (This does NOT mean the closest point that has integer coordinates -- you will need to consider other points.)

Distance functions and circles

Definition:
The expression $|x - y|$ (we say "the absolute value of $x - y$") means the distance, on a number line, between the two numbers x and y.

91. Explain why the expression $|x|$ is equal to x if x is positive; is equal to $-x$ if x is negative; and is equal to 0 if x is 0.

92. Graph the function $y = |x|$ on your calculator. Explain the results.

Ymax=

Xmin= Xmax=

Ymin=

Y =

93. Graph the function $y = |x - 2|$ on your calculator. Explain the results.

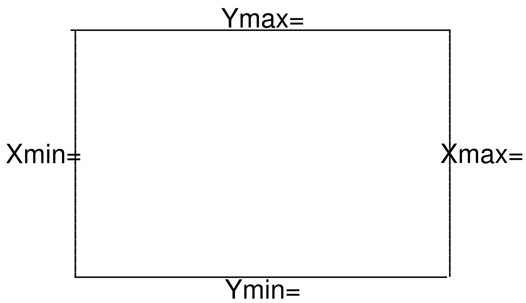
Ymax=

Xmin= Xmax=

Ymin=

Y =

94. Graph the function $y = |x + 7|$ on your calculator. Explain the results.



Y =

95. Suppose you are driving a constant speed, 60mph, from Chicago to Detroit, about 275 miles away. When you are 120 miles from Chicago you pass through Kalamazoo, Michigan. Sketch a graph of your distance from Kalamazoo as a function of time.

96. Use the Pythagorean Theorem to find the distance between the two points $(1, 3)$ and $(-1, 5)$.

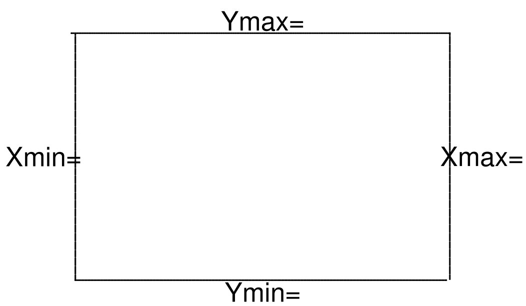
97. Use the Pythagorean Theorem to find a formula for the distance between the two points (c, d) and (a, b).

Definition:
A circle is the set of all points that are equidistance from a fixed point. That fixed point is called the center of the circle. The common distance of a point on the circle from the center is called the radius of the circle.

98. Use the Pythagorean Theorem to find an equation for the circle that has a center at $(2, 3)$ and a radius of 2.

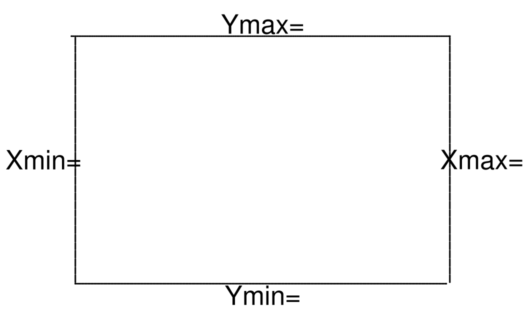
You can draw a circle on your graphing calculator in two ways. Using functions, you must enter two functions in order to get both the top and the bottom halves of the circle.

99. On your calculator: draw the circle centered at $(4, 5)$ with a radius 5.



Y =

100. On your calculator: draw the graph $(x-2)^2 + (y+3)^2 = 4$. What is the center of this circle? What is the radius of the circle?

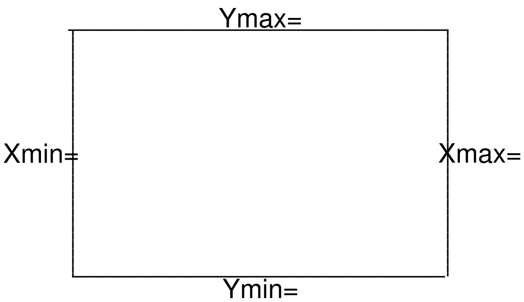


Y =

101. On a piece of graph paper, sketch the graph of these circles. Use a compass to make a good circle. Label the $x-$ and $y-$ intercepts. Graph the equation on the calculator to check your sketch.
a) $(x+6)^2 + y^2 = 4$ b) $(x-5)^2 + (y+2)^2 = 5$

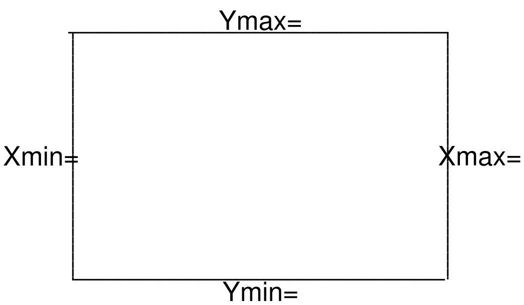
102. Do these problems on a sheet of graph paper. First sketch the circle, then find an equation for the circle. Check your answer by graphing the circle on your calculator.

a) Center $(2, 2)$; passes through the origin.



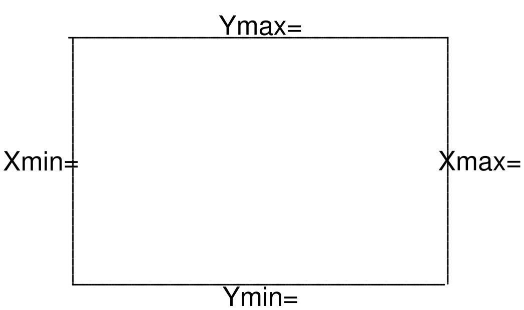
Y =

b) Center $(-1, -3)$; passes through $(-4, -2)$.



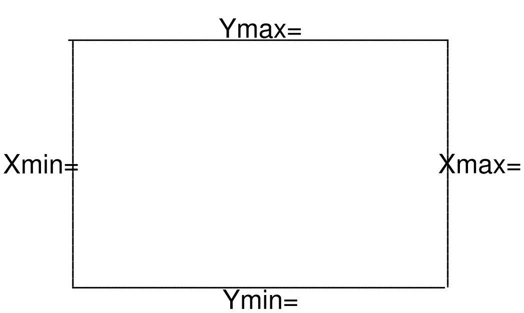
Y =

c) Center $(1, 2)$; intersects $x-$axis at -1 and 3.



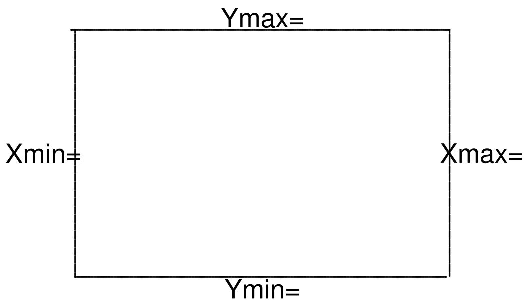
Y =

d) Center $(3, 1)$; diameter 2



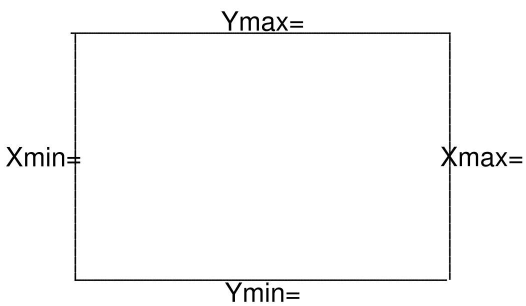
Y =

e) Center $(-5,4)$; intersects the $x-$axis in exactly one point.



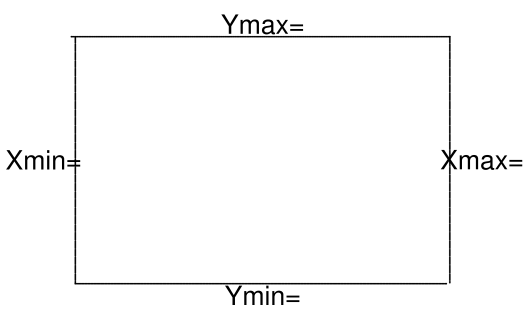
Y =

f) Center $(2,-6)$; intersects the $y-$axis in exactly one point.



Y =

g) Endpoints of diameter are $(3,3)$ and $(1,-1)$



Y =

103. Find the point on the circle $x^2 + y^2 = 25$ that is closest to the point $(1,3)$.

Exponential functions

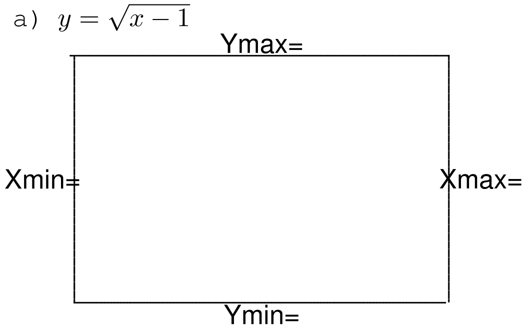
Handout

Hughes-Hallett: Section 1.2

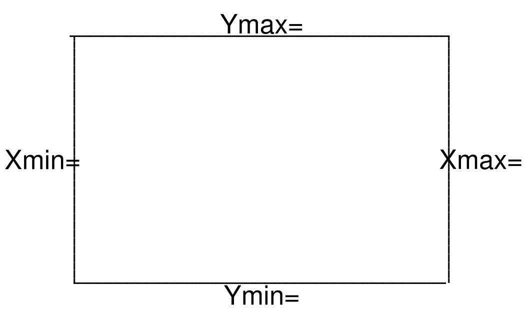
Other functions

104. For each of these functions, first make a table of values, then use your calculator to graph the function. From the graph check the (x, y) pairs in your table. Finally, for each function figure out for what values of x the expression is a real number.

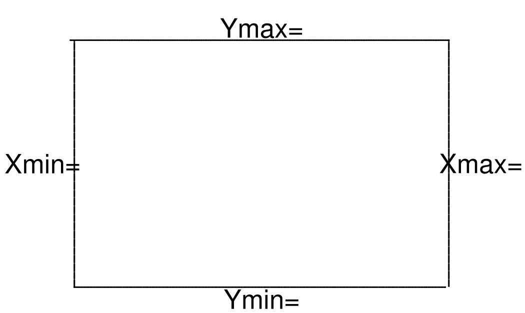
a) $y = \sqrt{x-1}$



b) $y = (x-1)^3$



c) $y = \sqrt{x^2 - 9}$



105. Functions may be created in many ways. With your calculator, you can explore many different functions just by typing a legitimate calculator expression using the x variable into the function page (Y). Try something interesting. View the graph with different windows and generate tables to review.

Week 6. Transformations in a coordinate system

Review:

Workshop Geometry: Unit 1, lesson 2: Pay particular attention to the parts about using grid paper. Come to class with plenty of graph paper

Problem of the Week: Architectural Arches

See handout, Discovery Project 4

Translations and three Reflections

Whenever possible, do these problems in three ways: making a table, using a graph and modifying a formula. Use your calculator liberally.

106. Start with the circle $\{(x,y) : x^2 + y^2 = 9\}$. Translate the circle so that the center is at $(3,4)$. Write an equation for the translated circle.

107. Start with the graph of the absolute value function, $\{(x,y) : y = |x|\}$. Translate the graph 4 units to the right. What is the formula for the translated function? Can you give a geometric interpretation of this new function in terms of distances? Make a table of values for both the original and the translated function.

108. Start with the graph, $\{(x,y) : y = |x-2|\}$. Translate the graph 4 units to the left. What is the formula for the translated function? Make a table of values for both the original and the translated function.

109. Start with the function, $\{(x,y) : y = |x|\}$. Translate the graph so that the vertex is at $(a,0)$. What is the equation of the new function? What is the difference in the graph if a is positive or negative?

110. Start with the function, $\{(x,y) : y = |x|\}$. Translate the graph so that the vertex is at $(0,b)$. What is the equation of the new function? What is the difference in the graph if b is positive or negative?

111. Start with the parabola, $\{(x,y) : y = x^2\}$. Translate the graph 3 units to the right. What is the formula for the translated function? Make a table of values for both. Explain how the table of values shows the translation.

112. Start with the parabola, $\{(x,y) : y = x^2\}$. Translate the graph 2 units to the left. What is the formula for the translated function? Make a table of values for both. Explain how the table of values shows the translation.

113. Start with the parabola, $\{(x,y) : y = x^2\}$. Translate the graph 3 units to down. What is the formula for the translated function? Make a table of values for both. Explain how the table of values shows the translation.

114. Start with the parabola, $\{(x,y) : y = x^2\}$. Translate the graph 3 units to up and 2 units to the left. What is the formula for the translated function? Make a table of values for both. Explain how the table of values shows the translation.

115. Start with the parabola, $\{(x,y) : y = x^2\}$. Translate the graph so the vertex of the parabola is now at $(2,-4)$. What is the formula for the translated function? Make a table of values for both. Explain how the table of values shows the translation.

116. Start with the parabola, $\{(x,y) : y = x^2\}$. Translate the graph so that the vertex is at (h,k). What is the equation of the new parabola?

117. Use problem 116 to find the equation of a parabola that has it's vertex at the point $(-1,10)$.

118. Find the number c such that the vertex of the parabola $y = x^2 + 8x + c$ lies on the x-axis.

119. Find a formula for a function whose graph is the reflection about the x-axis of $\{(x,y) : y = x^2\}$. Make a table of values for both. Explain how the table of values shows the reflection.

120. By reflecting and then translating the graph of $\{(x,y) : y = x^2\}$. Find the formula for a parabola that turns down and has a vertex at $(1,4)$.

121. In the previous problem, what happens if you first translate and then reflect. Do you get the same result?

122. Find a formula for a function whose graph is a parabola that turns up and has a vertex at the point (k,h).

123. Find a formula for a function whose graph is the reflection about the y-axis of $\{(x,y) : y = x^3\}$.

124. Find a formula for a function whose graph is the reflection about the y-axis of $\{(x,y) : y = \sqrt{x-1}\}$.

125. A function whose graph is symmetric about the y-axis is called an "even" function. Give two examples of even functions.

Stretching and Enlargements

126. The graph of a parabola is obtained from the graph of $y = x^2$ by vertically stretching away from the x-axis by a factor of 2. What is the equation for this parabola?

127. The graph of a parabola is obtained from the graph of $y = x^2$ by vertically shrinking towards the x-axis by a factor of 2. What is the equation for this parabola?

128. For parabolas, explain why stretching in the y-direction (away from the x-axis) looks like shrinking in the x-direction (towards the y-axis).

129. Start with a circle, $x^2+y^2 = 1$, and stretch it by a factor of 3 in the x-direction. What is the equation of the resulting ellipse?

130. Start with a circle, $x^2 + y^2 = 1$, and stretch it by a factor of 3 in the x-direction and by a factor of 2 in the y-direction. What is the equation of the resulting ellipse?

131. Start with $f(x) = x^2 + 2$, then write the rule of a function whose graph is the graph of f but shifted 5 units to the left and 4 units up.

132. Start with $f(x) = x^2 + 2$, then write the rule of a function whose graph is the graph of f but first shrunk by a factor of 2 towards the y-axis and then shifted 5 units to the left and 4 units up. do you get the same resulting function if you shrink after the translations?

Reflections through $y = x$

133. Start with the line $y = 2x$. Reflect the graph through the line $y = x$. Write a function to graph the reflected line. Repeat for the following lines item $y = 3x$ item $y = 2x + 3$. item $y = -x + 1$. item $y = 5x - 2$. Make a conjecture about the slope of the reflected line?

134. Start with the parabola $y = x^2$. Reflect the graph through the line $y = x$. Can you write a formula for the function this reflected graph?

135. Explain the graph of the function, $f(x) = \sqrt{x-1}$, in terms of a reflection and a translation.

136. Start with the parabola $y = x^2 + 5$. Reflect the graph through the line $y = x$. Can you write a function to graph this reflection? Can you write two functions that will graph this reflection?

137. Start with the parabola $y = -2x^2 + 1$. Reflect the graph through the line $y = x$. Can you write a function to graph this reflection? Can you write two functions that will graph this reflection?

138. Explain the graph of the function, $f(x) = 1 + \sqrt{x-3}$, in terms of translations and a reflection.

Week 7. Exam week

III. Area and Integrals

Week 8. Areas of regions bounded by curves

Review:
Workshop Geometry: page 203 #6, #7; page 243 #6.

Problem of the Week: Circle Problem
Workshop Geometry page 244, #11

Scaling area

The stretching area principle:
If a geometric figure is stretched or shrunk by a factor of r in one direction, then the area increases (decreases) by a factor of r.

The scaling area principle:
If an geometric figure is enlarged (or reduced) by a factor of r, then the area increases (decreases) by a factor of r^2.

Handout
Scaling Area and Streching Area worksheets

139. Use the stretching area principle to conclude the area of a rectangle, assuming you know the area of a square.

140. Explain why the scaling principle is true for a square.

141. An ellipse is given by an equation of the form: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. Use stretching arguments to find a formula for the area of an ellipse.

142. Use the scaling area principle to explain: The ratio of the areas of two circles is the ratio of the squares of their radii.

143. Using the scaling area principle rather than computing areas with formulas, do the review problems from Workshop Geometry

Finding areas with the calculator

Your graphing calculator enables you to find the area bounded by the graph of a function, the x-axis, and two vertical lines. Use this feature of your calculator to find the following areas.

144. Use your calculator to compute the area of a rectangle with dimensions 3X7.

145. Use your calculator to find the area of a right triangle if the two legs have length 5cm and 3cm.

146. Use the absolute value function on your calculator to find the approximate area of an equilateral triangle that has sides length $5\sqrt{2}$ inches. Find the exact area.

147. Use the absolute value function on your calculator to find the approximate area of an equilateral triangle that has sides length 10 inches. Find the exact area.

148. Use your calculator to find the approximate area of a circle that has radius 3 centimeters.

149. Confirm the formula of the area of an ellipse by computing examples with your calculator.

Area of a circle: a limits problem

150. Find an approximate the area of a circle by covering the circles with units squares and counting the squares.

151. Write a program to approximate the area of a circle that has radius 1 without using the formula for the area of a circle or using the value of π.

152. How would you modify the program to approximate the area of a circle that has radius r?

153. Modify your program to approximate the area of the ellipse given by $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. Check your answer against the formula found in problem 141.

Week 9. Parabolas and Area

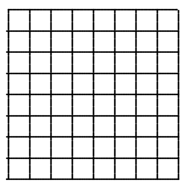
Review: From Week 1

Formula for the sum of squares. Show that

$$\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n-1)$$

Problem of the Week: How Many Squares?

There are 64 1x1 squares in this checkerboard. How many squares of all different sizes can you find in this picture?



The area bounded, in part, by a parabola.

154. Make a guess: what is the area bounded by $y = x^2$, the x-axis and the line $x = 1$.

155. Approximate the area bounded by $y = x^2$, the x-axis and the line $x = 1$ using four subdivisions of the interval $[0, 1]$ on the x-axis.

156. Approximate the area bounded by $y = x^2$, the x-axis and the line $x = 1$ using five subdivisions of the interval $[0, 1]$ on the x-axis.

157. Approximate the area bounded by $y = x^2$, the x-axis and the line $x = 1$ using six subdivisions of the interval $[0, 1]$ on the x-axis.

158. Find an expression for the area bounded by $y = x^2$, the x-axis and the line $x = 1$ using n subdivisions of the interval $[0, 1]$ on the x-axis.

159. Find the area bounded by $y = x^2$, the x-axis and the line $x = 1$. In other words, compute $\int_0^1 x^2 dx$.

Draw a picture and then find the area.

160. What is the area under $f(x) = x$ on the interval [0,1]? In other words, compute $\int_0^1 x dx$

161. What is the area under $f(x) = 1 - x^2$ on the interval [0,1]? In other words, compute $\int_0^1 (1 - x^2)dx$

162. What is the area under $f(x) = 1 - x^2$ on the interval [-1,1]? In other words, compute $\int_{-1}^1 (1 - x^2)dx$

163. What is the area under $f(x) = x^2 + 3$ on the interval [0,1]? In other words, compute $\int_0^1 (x^2 + 3)dx$

164. Do you think it is true that $\int_0^1 3x^2 dx = 3 \int_0^1 x^2 dx$? Explain your thinking

165. Compute $\int_1^3 4x + 1dx$

166. Compute $\int_{-1}^1 |x|dx$

167. Compute $\int_{-2}^0 (x^2 + 2x + 1)dx$

Changing scale in the x direction

168. What is the area under $f(x) = x^2$ on the interval [0,4]? In other words, compute $\int_0^4 x^2 dx$

169. Compute $\int_0^2 x^2 dx$

170. What is the area under $f(x) = x^2$ on the interval [0.5,1]? In other words, compute $\int_{0.5}^1 x^2 dx$

171. Find a formula for $\int_0^t x^2 dx$.

Rules for integrals

If $A \leq C \leq B$, then $\int_A^B f(x)dx = \int_A^C f(x)dx + \int_C^B f(x)dx$

$$\int_A^B af(x) + bg(x)dx = a\int_A^B f(x)dx + b\int_A^B g(x)dx$$

$$\int_{cA}^{cB} f(cx)dx = \frac{1}{c}\int_A^B f(x)dx$$

Use these rules to do the next few problems.

172. Evaluate $\int_0^1 (x^2 + x)dx$.

173. Find a formula for $\int_0^1 (ax^2 + bx + c)dx$.

174. Find a formula for $\int_A^B (x^2 + 3x + 5)dx$.

175. Using the formulas you developed, compute $\int_1^2 1 - x^2 dx$. Explain why the answer is negative.

176. Carefully graph the following functions shade the area(s) indicated by the integral and compute the integral:

a) $\int_{-1}^{0} (2x^2 + 3x + 2)dx$.

b) $\int_{0}^{4} (x^2 - 4x + 3)dx$.

c) $\int_{-1}^{1} (3x^2 - 2x + 1)dx$.

Integrals that are NOT areas

Hughes-Hallett: Section 5.1

Definition Integral

Hughes-Hallett: Section 5.2 – Computing using the definition

177. Write a program to approximate integrals on your calculator. Test the program on many integrals.

Week 10. Review and Midterm

IV. Derivatives

Week 11. Tangent lines

Problem of the Week: atop Willis Tower

How far can you see from the SkyDeck viewing area(1353 feet high) of the Sears Tower? If you were allowed to stand on the top of the Sears Tower (1454 feet), how much further could you see?

Review: From Week 4

178. Confirm that the equation of the line that goes through the point, (a, b) and has slope m can be written as $y = m(x-a)+b$.

179. Find conditions on the coefficients, a, b, and c, so that the equation, $ax^2+bx+c=0$, has exactly one solution.

Tangent line to circles

180. Find the best explanation for this fact: The tangent line to a circle is perpendicular to a radius of the circle.

For the following, sketch the circles and the tangent lines before computing. Afterwards, graph the function and the tangent line on the calculator to confirm your answers.

181. Find equations for the lines that are tangent to the circle, $x^2+y^2 = 1$ at the points $(0, 1)$, $(1, 0)$, $(-1, 0)$, and $(0, -1)$.

182. Find an equation for the line that is tangent to the circle, $x^2+y^2 = 25$ at the point $(3, 4)$.

183. Find an equation for the line that is tangent to the circle, $x^2 + y^2 = 4$ at the point $(\sqrt{2}, -\sqrt{2})$.

184. Find an equation for the line that is tangent to the circle, $(x+1)^2 + (y-1)^2 = 25$ at the point $(2, 5)$.

optional
Tangent lines from a point to a circle

185. When is a triangle inscribed in a circle a right triangle?

186. Find all the tangent lines to the circle, $x^2 + y^2 = 1$, that go through the point $(1, 1)$.

187. Find all the tangent lines to the circle, $x^2 + y^2 = 1$, that go through the point $(0, 4)$.

Tangent line to parabolas

For the following, use the fact that the tangent line intersects a parabola in only one point to get algebraic equations to solve.

188. Find the equation of the tangent line to the curve at the given point. Graph both the function and the tangent line to confirm your answer.

a) $y = x^2$ at $(1, 1)$.

b) $y = x^2$ at $(-2, 4)$.

c) $y = 3x^2 - 2x + 1$ at $(0, 1)$.

189. Find the equation of the tangent line to the curve at the given point. Graph both the function and the tangent line to confirm your answer. Use transformations to make it easy

a) $y = x^2 + 2x + 1$ at $(0, 1)$.

b) $y = 3x^2$ at $(1, 3)$.

c) $y = x^2 + 1$ at $(2, 5)$.

190. Find the equation of the tangent line to the curve at the given point. Graph both the function and the tangent line to confirm your answer. Use transformations to make it easy

a) $y = x^2 + 2x$ at $(0, 0)$.

b) $y = 3x^2 + 5$ at $(1, 8)$.

c) $y = \sqrt{x}$ at $(1, 1)$.

Derivative at a point.

Definition: the derivative at a point.
For a curve, $y = f(x)$, we name the slope of the tangent line at a point $(a, f(a))$, $f'(a)$, and call it the derivative of f at the point a.

191. Find a curve that does not have a tangent line at some point.

192. Show that $f'(a) = 2a$ if $f(x) = x^2$ for all values of a.

Week 12. Rate of change

Problem of the Week: Buildling an Odometer

Hungerford pg 204

Instantaneous rate of change

Hughes-Hallett Section 2.1

The Derivative Function

Hughes-Hallett Section 2.3

Practice with derivatives

Week 13. Applications

Problem of the Week: At the beach

A swimmer at the beach is 10 feet from shore when he spies the ice cream man on the shore 20 feet down the shore line (The ice cream man is right on the water's edge.) To minimize the time to get to the ice cream, should the swimmer swim a bee line to the ice cream man or should she first swim to shore and then run along the beach? Where exactly should she come ashore? Her speed on land is $8\frac{ft}{sec}$ but she swims half that speed.

Optimization Applications

Hungerford, Section 2.4

Several more problems

Week 14. Changing Areas

Problem of the Week: Filling a parabola

A parabola (in flatland) that is the shape of $y = x^2$ is being filled with water. What is the rate of change of the height of the water when the water is h units high?

Shapes and changing rates

Integrals changing rate

Week 15. The Fundmental Theorem of Calculus

The Fundamental Theorem of Calculus and Calculating integrals

Practice for Final

Practice for Final