

A GEOMETRIC CONDITION IMPLYING ENERGY EQUALITY FOR SOLUTIONS OF 3D NAVIER-STOKES EQUATION.

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ABSTRACT. We prove that every weak solution u to the 3D Navier-Stokes equation that belongs to the class $L^3L^{9/2}$ and ∇u belongs to $L^3L^{9/5}$ locally away from a $1/2$ -Hölder continuous curve in time satisfies the generalized energy equality. In particular every such solution is suitable.

1. INTRODUCTION

In this note we discuss the energy balance equality for weak solutions of the 3D Navier-Stokes equations. The system of the Navier-Stokes equations is given by

$$(1) \quad u_t + \operatorname{div}(u \otimes u) + \nabla p = \nu \Delta u,$$

$$(2) \quad \nabla \cdot u = 0,$$

where u is the velocity field, p the internal pressure. We focus primarily on the case of \mathbb{R}^3 . A weak solution to (1)–(2) is a pair of distributions $(u, p) \in \mathcal{D}'((0, T) \times \mathbb{R}^3)^4$ with $u \in L^2((0, T) \times \mathbb{R}^3)_{\text{loc}}$ such that

$$(3) \quad - \iint u \psi_t - \nu \iint u \Delta \psi = \iint (\operatorname{Tr}[(u \otimes u) \cdot \nabla \psi] + p \operatorname{div} \psi)$$

$$(4) \quad \iint u \cdot \nabla \phi = 0,$$

for all $(\psi, \phi) \in \mathcal{D}((0, T) \times \mathbb{R}^3)^4$, where \mathcal{D} stands for the space of C^∞ -smooth compactly supported functions. The classical existence theorem of Leray [9] states that given any divergence free initial condition $u_0 \in L^2(\mathbb{R}^3)$ one can find at least one weak solution (u, p) with

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$u \in L^\infty L^2 \cap L^2 H^1$ (here and throughout $L^r X = L^r([0, T]; X(\mathbb{R}^3))$), $u(t) \rightarrow u_0$ strongly in L^2 as $t \rightarrow 0$, the pressure is given by

$$(5) \quad p = \sum_{i,j=1}^3 R_i R_j (u_i u_j),$$

where R_i 's are the classical Riesz transforms, and the following energy inequality

$$(6) \quad \frac{1}{2} \int_{\mathbb{R}^3 \times \{t\}} |u|^2 + \nu \int_{t_0}^t \int_{\mathbb{R}^3} |\nabla u|^2 \leq \frac{1}{2} \int_{\mathbb{R}^3 \times \{t_0\}} |u|^2$$

holds for all $t \in (0, T]$ and a.e. $t_0 \leq t$ including $t_0 = 0$. Following Serrin [15] one can further tune u on a time set of measure zero to achieve weak continuity in L^2 . We denote $u \in C_w([0, T]; L^2(\mathbb{R}^3)) = C_w L^2$.

The lack of exact equality in (6) is a pitiful deficiency of Leray's solutions that to date remains unresolved. The main difficulty arises in the fact that the mollified velocity field u_δ may have a non-vanishing energy flux due to the nonlinear term, i.e. the equality

$$(7) \quad \lim_{\delta \rightarrow 0} \iint \text{Tr}[(u \otimes u)_\delta \cdot \nabla u_\delta] = 0$$

may fail. In [11] Lions showed that if $u \in L^4 L^4$ then the energy equality holds. Technique developed in [8] for general parabolic equations reproduces Lions' result as well. By interpolation with $L^\infty L^2$ one automatically obtains the range of conditions

$$(8) \quad u \in L^r L^s, \text{ for some } 2/r + 2/s = 1, s \geq 4.$$

These were shown to work in any spacial dimension by Shinbrot [16]. The above results are based on proving continuity of the trilinear form $u \cdot \nabla v \cdot w$ in $L^\infty L^2 \cap L^2 H^1 \cap L^r L^s$ making it possible to carry out the standard mollification argument in order to obtain (7). A different approximation procedure was proposed by Kukavica [7]. It allowed to essentially use cancellations present in the nonlinear term. The extra regularity requirement was thus relieved from the velocity field and put to the pressure giving the condition $p \in L^2((0, T) \times \mathbb{R}^3)_{\text{loc}}$. In view of (5) this is a weaker yet dimensionally the same version of Lions' condition.

After a recent progress on Onsager's conjecture for the Euler equations (see [3, 5, 6]) dimensionally sharper conditions were found in [3, 4]. Namely, the energy equality holds if $u \in L^3 B_{3,p}^{1/3}$ in the case of \mathbb{R}^3 or periodic domain, or if $u \in L^3 D(A^{5/12})$ in the case of a bounded domain. Here $B_{3,p}^{1/3}$ is the Besov space with smoothness $1/3$ integrability 3 and summability $1 \leq p < \infty$, and A denotes the Stokes operator. The

dimensional L^p -analogue of these spaces is $L^3L^{9/2}$, which lies outside of (8). In fact the cube of the dimension of $L^3L^{9/2}$ is the same as the dimension of the energy flux in (7), suggesting that this space might be the optimal one for any argument based on direct control of the flux. We thus conjecture that every weak solution to (1)–(2) in the class $u \in C_wL^2 \cap L^2H^1 \cap L^3L^{9/2}$ verifies the energy equality. In this paper we prove the following result in this direction.

Theorem 1.1. *Let $s \in C^{1/2}([0, T]; \mathbb{R}^3)$ and (u, p) be a weak solution to the NSE satisfying the following conditions*

- (i) $u \in C_wL^2 \cap L^2H^1 \cap L^3L^{9/2}$;
- (ii) $\nabla u \in L^3L^{9/5}((0, T) \times \mathbb{R}^3 \setminus \text{Graph}(s))_{\text{loc}}$,

and p is given by (5). Then (u, p) satisfies the generalized energy equality:

$$(9) \quad \int_{\mathbb{R}^3 \times \{t\}} |u|^2 \phi + 2\nu \int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 \phi = \int_{\mathbb{R}^3 \times \{0\}} |u|^2 \phi + \\ + \int_0^t \int_{\mathbb{R}^3} [|u|^2 (\phi_t + \nu \Delta \phi) + (|u|^2 + 2p) u \cdot \nabla \phi],$$

for all $\phi \in \mathcal{D}([0, T] \times \mathbb{R}^3)$, and $t \in [0, T]$.

A few remarks are in order. First, we note that for any weak solution (u, p) with $u \in L^\infty L^2 \cap L^2 H^1$, the pressure is restored via (5) up to an x -independent distribution. Thus, any condition $p \in L^r L^s$ would already imply (5), and hence the inclusions

$$(10) \quad p \in L^r L^s, \text{ for } 2/r + 3/s = 3, \quad 1 < s \leq 3.$$

This justifies the integrals in (9). Furthermore, considering the sequence $\phi(x/R)$ with $\phi = 1$ near 0 and $R \rightarrow 0$ we recover the global energy equality.

Following [2, 10] weak solutions with the natural bounds on u and p satisfying the generalized energy inequality in (9) for all non-negative ϕ are called suitable. The end result of the partial regularity theory developed in [2, 12, 13, 14] showed that the set of singular points of every such solution has zero one-dimensional parabolic Hausdorff measure. From this point of view the Hölder condition on s seems rather generous as the graph of a $C^{1/2}$ -function may have parabolic dimension as large as 2. Yet it is essential for the argument that the curve s is extended in time. For instance, assuming that s is a smooth curve on a slice $\mathbb{R}^3 \times \{t_0\}$ our argument necessitates the stronger condition $u \in L^3 L^6$, which already falls into the range of (8).

Finally let us note that condition (ii) is dimensionally the same as the $L^3L^{9/2}$ condition on u . As such it lies out of reach of the classical Prodi-Serrin condition [15] or the condition proposed in [1]. So, theoretically by requiring (ii) we do not exclude the possibility of having singularities away from the graph of s . In Section 3 we will continue our discussion of Theorem 1.1.

2. PROOF OF THEOREM 1.1

The proof is based on an approximation procedure. So, let us fix a mollifier $\beta \geq 0$, $\beta \in C_0^\infty(B_1)$ with $\int \beta = 1$, where $B_\rho = \{|x| \leq \rho\}$. For a distribution $u \in \mathcal{D}'(\mathbb{R}^3)^n$ we denote

$$u_\delta(x) = \delta^{-3} \int_{\mathbb{R}^3} \beta(y\delta^{-1}) u(x-y) dy.$$

If u is weak solution to the NSE and $u \in C_w([0, T]; L^2)$, then we have

$$(11) \quad \int_{\mathbb{R}^3 \times \{t\}} u\psi - \iint u\psi_t - \nu \iint u\Delta\psi \\ = \int_{\mathbb{R}^3 \times \{0\}} u\psi + \iint (\text{Tr}[(u \otimes u) \cdot \nabla\psi] + p \text{div} \psi)$$

holds for all $t \in [0, T]$ and $\psi \in \mathcal{D}([0, T] \times \mathbb{R}^3)^3$. Substituting

$$\psi = \delta^{-3} \beta((x - \cdot)\delta^{-1}) e_k,$$

where e_k are the vectors of the standard unit basis, we immediately obtain

$$u_\delta(t) = u_\delta(0) + \left\{ \int_0^t [u_k(s) * \beta' + (u_k(s)u_j(s)) * \beta_j'' + p(s) * \beta_k'''] ds \right\}_{k=1}^3,$$

for some $\beta', \beta_j'', \beta_j''' \in \mathcal{D}(\mathbb{R}^3)$. In view of $u \in L^\infty L^2 \cap L^2 L^6$ and (10), the function under the integral belongs to $L^r X$, where X is any Sobolev space $W^{\gamma,p}$, $s \geq 0$, $p \geq 2$, and $r < \infty$. This implies that u_δ is absolutely continuous in X with Freschet derivative $\partial_t u_\delta \in L^r X$. By the standard approximation argument, functions with such smoothness are allowed in (11). We therefore can substitute a test-function of the form $\psi = (u_\delta \phi)_\delta$, where $\phi \in \mathcal{D}([0, T] \times \mathbb{R}^3)$.

We now proceed with the construction of the appropriate test-function. In order to cut off the graph of s we first extend s beyond $[0, T]$ by defining

$$(12) \quad s^{\text{ext}}(t) = \begin{cases} s(0), & t < 0; \\ s(t), & 0 \leq t < T; \\ s(T), & t \geq T. \end{cases}$$

Clearly, $s^{\text{ext}} \in C^{1/2}(\mathbb{R}; \mathbb{R}^3)$. Second, we define

$$(13) \quad s_\varepsilon(t) = \int_{\mathbb{R}} \varepsilon^{-2} \alpha(\tau \varepsilon^{-2}) s^{\text{ext}}(t - \tau) d\tau,$$

for some mollifier α and $\varepsilon > 0$. The following approximation inequalities easily follow:

$$(14) \quad \sup_{0 \leq t \leq T} |s_\varepsilon(t) - s(t)| < \varepsilon;$$

$$(15) \quad \sup_{0 \leq t \leq T} |s'_\varepsilon(t)| \leq 1/\varepsilon.$$

Next we introduce a cut-off function $\chi \in C^\infty(\mathbb{R}^3)$ with $\chi \geq 0$, $\chi \equiv 0$ in B_2 and $\chi \equiv 1$ in $\mathbb{R}^3 \setminus B_3$. Denote

$$\chi_\varepsilon(x, t) = \chi\left(\frac{x - s_\varepsilon(t)}{\varepsilon}\right).$$

Notice that in view of (14),

$$(16) \quad \text{supp } \chi_\varepsilon \subset \{(x, t) : |x - s(t)| > \varepsilon\},$$

and χ_ε is infinitely smooth in time-space. Let us note the following inequality

$$(17) \quad \sup_{t \in [0, T]} \|D_x^\gamma \chi_\varepsilon\|_p \sim \varepsilon^{\frac{3}{p} - |\gamma|},$$

for any $1 \leq p \leq \infty$ and multiindex γ .

Let us fix an arbitrary $\phi \in \mathcal{D}([0, T] \times \mathbb{R}^3)$ and define the following test-function

$$(18) \quad \psi = (u_\delta \phi \chi_\varepsilon)_\delta.$$

As we substitute this function into (11) we will adhere to the same order of limits as $\varepsilon, \delta \rightarrow 0$ in all our subsequent computations. Namely, first $\delta \rightarrow 0$ and then $\varepsilon \rightarrow 0$. Let us assign letters to the terms of equation (11) by writing it as

$$(19) \quad A - B - C = D + E.$$

We now examine each term separately.

First, let us notice that integration by parts carried out in B results in appearance of two terms that cancel out with A and B plus the following

$$\frac{1}{2} \int_{\mathbb{R}^3 \times \{t\}} |u_\delta|^2 \phi \chi_\varepsilon - \frac{1}{2} \int_{\mathbb{R}^3 \times \{0\}} |u_\delta|^2 \phi \chi_\varepsilon - \frac{1}{2} \int_0^t \int_{\mathbb{R}^3} |u_\delta|^2 (\phi \chi_\varepsilon)_t.$$

The first two integrals converge to the corresponding terms in (9), while the third integral is given by

$$\frac{1}{2} \iint |u_\delta|^2 (\phi \chi_\varepsilon)_t = \frac{1}{2} \iint |u_\delta|^2 \phi_t \chi_\varepsilon + \frac{1}{2} \iint |u_\delta|^2 \phi s'(t) \cdot \nabla \chi_\varepsilon.$$

Clearly, the first integral on the right hand side converges to its natural limit

$$\frac{1}{2} \iint |u|^2 \phi_t$$

producing the corresponding term in (9). The second integral converges to zero. Indeed, by (17) and Hölder, one obtains

$$\begin{aligned} \left| \iint |u_\delta|^2 \phi s'(t) \cdot \nabla \chi_\varepsilon \right| &\leq \int_0^t \left(\int_{|x-s_\varepsilon(t)| \leq 3\varepsilon} |u_\delta|^6 |\phi|^3 dx \right)^{1/3} |s'_\varepsilon(t)| \varepsilon dt \\ &\leq \int_0^t \left(\int_{|x-s_\varepsilon(t)| \leq 3\varepsilon} |u_\delta|^6 dx \right)^{1/3} dt \leq \int_0^t \left(\int_{|x-s_\varepsilon(t)| \leq 3\varepsilon + \delta} |u|^6 dx \right)^{1/3} dt, \end{aligned}$$

and the latter converges to zero as $\delta, \varepsilon \rightarrow 0$ since $u \in L^2 L^6$.

Let us now examine term C . We have

$$\begin{aligned} -C &= \nu \iint \nabla u_\delta \cdot \nabla (u_\delta \phi \chi_\varepsilon) = \nu \iint |\nabla u_\delta|^2 \phi \chi_\varepsilon - \frac{\nu}{2} \iint |u_\delta|^2 \Delta (\phi \chi_\varepsilon) \\ &= \nu \iint |\nabla u_\delta|^2 \phi \chi_\varepsilon - \frac{\nu}{2} \iint |u_\delta|^2 \Delta \phi \chi_\varepsilon \\ &\quad - \nu \iint |u_\delta|^2 \nabla \phi \cdot \nabla \chi_\varepsilon - \frac{\nu}{2} \iint |u_\delta|^2 \phi \Delta \chi_\varepsilon \\ &= C_1 - C_2 - C_3 - C_4. \end{aligned}$$

By the standard convergence theorems we see that C_1 and C_2 converge to the corresponding terms in (9), while in view of (17)

$$\begin{aligned} |C_3| &\leq \varepsilon \int_0^t \|u_\delta \nabla \phi\|_\delta^2 dt \rightarrow 0, \\ |C_4| &\leq \int_0^t \left(\int_{|x-s_\varepsilon(t)| \leq 3\varepsilon} |u_\delta|^6 |\phi|^3 dx \right)^{1/3} dt \rightarrow 0. \end{aligned}$$

Let us examine term E . We have

$$E = \iint \text{Tr}[(u \otimes u) \cdot \nabla \psi] + \iint p \text{div} \psi = F + G.$$

We can write

$$(20) \quad F = \iint \operatorname{Tr}[(u \otimes u)_\delta \cdot \nabla(u_\delta \phi_{\chi_\varepsilon})]$$

$$(21) \quad = \iint \operatorname{Tr}[r_\delta(u, u) \cdot \nabla(u_\delta \phi_{\chi_\varepsilon})]$$

$$(22) \quad + \iint \operatorname{Tr}[(u - u_\delta) \otimes (u - u_\delta) \cdot \nabla(u_\delta \phi_{\chi_\varepsilon})]$$

$$(23) \quad + \iint \operatorname{Tr}[u_\delta \otimes u_\delta \cdot \nabla(u_\delta \phi_{\chi_\varepsilon})]$$

$$(24) \quad = F_1 + F_2 + F_3,$$

where

$$r_\delta(u, u)(x) = \delta^{-3} \int_{\mathbb{R}^3} \beta(y\delta^{-1})(u(x-y) - u(x)) \otimes (u(x-y) - u(x)) dy.$$

We will show that F_1 and F_2 already vanish in the limit of $\delta \rightarrow 0$ for a fixed $\varepsilon > 0$. Let us observe the following estimate

$$(25) \quad \|r_\delta(u, u)\|_{9/4} \leq \delta^{-3} \int_{\mathbb{R}^3} \beta(y\delta^{-1}) \|u(\cdot - y) - u(\cdot)\|_{9/2}^2 dy \stackrel{\text{def}}{=} R(t, \delta).$$

For F_1 we obtain

$$\begin{aligned} |F_1| &\leq \int_0^t R(t, \delta) \|\nabla(u_\delta \phi_{\chi_\varepsilon})\|_{9/5} dt \\ &\leq \left(\int_0^t R^{3/2}(t, \delta) dt \right)^{2/3} \left(\int_0^t \|\nabla(u_\delta \phi_{\chi_\varepsilon})\|_{9/5}^3 dt \right)^{1/3}. \end{aligned}$$

Observe that

$$R^{3/2}(t, \delta) \leq \delta^{-3} \int_{\mathbb{R}^3} \beta(y\delta^{-1}) \|u(\cdot - y) - u(\cdot)\|_{9/2}^3 dy,$$

and hence

$$\int_0^t R^{3/2}(t, \delta) dt \rightarrow 0,$$

as $\delta \rightarrow 0$, while in view of condition (ii) and (16), $\nabla(u_\delta \phi_{\chi_\varepsilon}) \in L^3 L^{9/5}$ uniformly as $\delta \rightarrow 0$ for any fixed $\varepsilon > 0$. Thus, $F_1 \rightarrow 0$. Similarly,

$$|F_2| \leq \int_0^t \|u - u_\delta\|_{9/2}^2 \|\nabla(u_\delta \phi_{\chi_\varepsilon})\|_{9/5} dt \rightarrow 0.$$

As to F_3 we have

$$F_3 = \frac{1}{2} \iint |u_\delta|^2 u_\delta \cdot \nabla(\phi_{\chi_\varepsilon}) = \frac{1}{2} \iint |u_\delta|^2 \chi_\varepsilon u_\delta \cdot \nabla \phi + \frac{1}{2} \iint |u_\delta|^2 \phi u_\delta \cdot \nabla \chi_\varepsilon.$$

Clearly, the first integral on the right hand side converges to

$$\frac{1}{2} \iint |u|^2 u \cdot \nabla \phi$$

giving us the corresponding term in (9). As to the second integral we estimate using the Hölder inequality and (17)

$$\begin{aligned} \left| \iint |u_\delta|^2 \phi u_\delta \cdot \nabla \chi_\varepsilon \right| &\leq \int_0^t \left(\int_{|x-s_\varepsilon(t)| \leq 3\varepsilon} |u_\delta|^{9/2} dx \right)^{2/3} dt \\ &\leq \int_0^t \left(\int_{|x-s_\varepsilon(t)| \leq 3\varepsilon + \delta} |u|^{9/2} dx \right)^{2/3} dt \rightarrow 0, \end{aligned}$$

as $\delta, \varepsilon \rightarrow 0$.

It remains to examine the pressure term G . We have

$$G = \iint p_\delta \chi_\varepsilon u_\delta \cdot \nabla \phi + \iint p_\delta \phi u_\delta \cdot \nabla \chi_\varepsilon = G_1 + G_2.$$

Since $p \in L^{3/2} L^{9/4}$ and $u \in L^3 L^{9/2}$ the local L^2 -pairing between u and p is continuous. So,

$$G_1 \xrightarrow{\delta \rightarrow 0} \iint p \chi_\varepsilon u \cdot \nabla \phi \xrightarrow{\varepsilon \rightarrow 0} \iint p u \cdot \nabla \phi.$$

As for G_2 we apply the following estimate

$$\begin{aligned} |G_2| &\leq \|p_\delta\|_{L^{3/2} L^{9/4}} \|\nabla \chi_\varepsilon\|_{L^\infty L^3} \left[\int_0^t \left(\int_{|x-s_\varepsilon(t)| \leq 3\varepsilon} |u_\delta|^{9/2} dx \right)^{2/3} dt \right]^{1/3} \\ &\leq C \left[\int_0^t \left(\int_{|x-s_\varepsilon(t)| \leq 3\varepsilon + \delta} |u|^{9/2} dx \right)^{2/3} dt \right]^{1/3} \rightarrow 0. \end{aligned}$$

This finishes the proof.

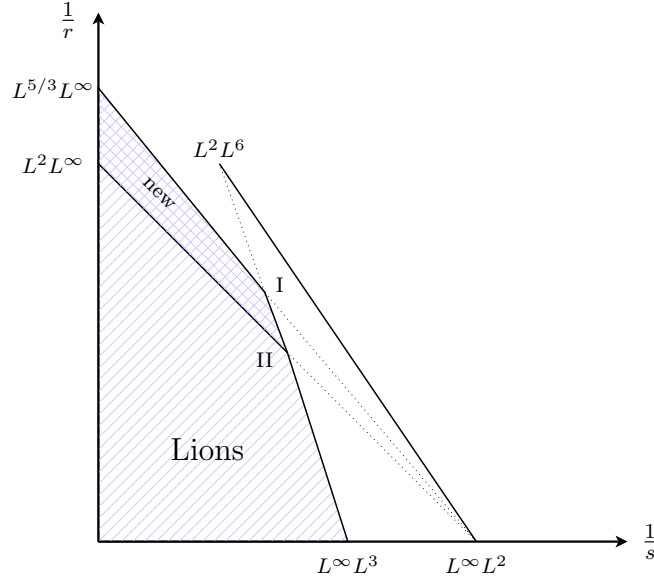
3. EXTENTIONS

First, we note that one can incorporate an external divergence-free force f as long as $\iint f \psi \rightarrow \iint f u$ in the limit as $\delta, \varepsilon \rightarrow 0$. For this purpose $f \in H^{-1}$ appears to be sufficient.

Second, by extrapolation from $L^3 L^{9/2}$ along the line starting at $L^\infty L^2$ or $L^2 L^6$ or any other space in between we obtain the convex range of $L^r L^s$ -spaces determined by

$$(26) \quad 5/3r + 2/s \leq 1, \quad 1/r + 3/s \leq 1,$$

for $r \geq 5/3$ and $s \geq 3$, which can be used in (i) to substitute $L^3 L^{9/2}$. Figure 1 graphically demonstrates the region where this range is not

FIGURE 1. Here, $I = L^3 L^{9/2}$, $II = L^4 L^4$

covered by the previously known results. The complementary condition on ∇u is given by

$$(27) \quad \nabla u \in L^{\frac{r}{r-2}} L^{\frac{s}{s-2}},$$

for those $r, s > 2$ that are in the range (26). For some r and s , however, condition (27) already implies (ii) by interpolation with $\nabla u \in L^2 L^2$ or it may be strong enough to imply regularity via the Prodi-Serrin condition or [1]. We leave details for the reader.

Treating the nonlinear terms F as in [3] one can lower the order of derivative in condition (ii) by cost of increasing the integrability exponent. At extreme one gets

$$(28) \quad u \in L^3 B_{3,p}^{1/3}([0, T] \times \mathbb{R}^3 \setminus \text{Graph}(s))_{\text{loc}},$$

where "loc" means that for any $\phi \in D([0, T] \times \mathbb{R}^3 \setminus \text{Graph}(s))$, $\phi u \in L^3 B_{3,p}^{1/3}$.

We also notice that the argument does not make use of the global estimates on u and p . Thus if we are to pursue only the local energy inequality (9) one can restate condition (i) in the local sense with an additional assumption $p \in L^{3/2} L_{\text{loc}}^{9/4}$. The letter does not seem to follow directly from the corresponding conditions on u without extra smoothness assumptions on the initial condition (see [17]).

Lastly, we note that Theorem 1.1 is valid on a smooth bounded domain as well with the same requirement $p \in L^{3/2} L^{9/4}$.

REFERENCES

- [1] H. Beirão da Veiga. A new regularity class for the Navier-Stokes equations in \mathbf{R}^n . *Chinese Ann. Math. Ser. B*, 16(4):407–412, 1995. A Chinese summary appears in *Chinese Ann. Math. Ser. A* **16** (1995), no. 6, 797.
- [2] L. Caffarelli, R. Kohn, and L. Nirenberg. Partial regularity of suitable weak solutions of the Navier-Stokes equations. *Comm. Pure Appl. Math.*, 35(6):771–831, 1982.
- [3] A. Cheskidov, P. Constantin, S. Friedlander, and R. Shvydkoy. Energy conservation and Onsager’s conjecture for the Euler equations. 2007. submitted.
- [4] A. Cheskidov, S. Friedlander, and R. Shvydkoy. On the energy equality for weak solutions of the 3D Navier-Stokes equations. 2007. submitted.
- [5] Peter Constantin, Weinan E, and Edriss S. Titi. Onsager’s conjecture on the energy conservation for solutions of Euler’s equation. *Comm. Math. Phys.*, 165(1):207–209, 1994.
- [6] Jean Duchon and Raoul Robert. Inertial energy dissipation for weak solutions of incompressible Euler and Navier-Stokes equations. *Nonlinearity*, 13(1):249–255, 2000.
- [7] Igor Kukavica. Role of the pressure for validity of the energy equality for solutions of the Navier-Stokes equation. *J. Dynam. Differential Equations*, 18(2):461–482, 2006.
- [8] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural’ceva. *Lineinye i kvazi-lineinye uravneniya parabolicheskogo tipa*. Izdat. “Nauka”, Moscow, 1968.
- [9] Jean Leray. Sur le mouvement d’un liquide visqueux emplissant l’espace. *Acta Math.*, 63(1):193–248, 1934.
- [10] Fanghua Lin. A new proof of the Caffarelli-Kohn-Nirenberg theorem. *Comm. Pure Appl. Math.*, 51(3):241–257, 1998.
- [11] J. L. Lions. Sur la régularité et l’unicité des solutions turbulentes des équations de Navier Stokes. *Rend. Sem. Mat. Univ. Padova*, 30:16–23, 1960.
- [12] Vladimir Scheffer. Partial regularity of solutions to the Navier-Stokes equations. *Pacific J. Math.*, 66(2):535–552, 1976.
- [13] Vladimir Scheffer. Hausdorff measure and the Navier-Stokes equations. *Comm. Math. Phys.*, 55(2):97–112, 1977.
- [14] Vladimir Scheffer. The Navier-Stokes equations on a bounded domain. *Comm. Math. Phys.*, 73(1):1–42, 1980.
- [15] James Serrin. The initial value problem for the Navier-Stokes equations. In *Nonlinear Problems (Proc. Sympos., Madison, Wis., pages 69–98*. Univ. of Wisconsin Press, Madison, Wis., 1963.
- [16] Marvin Shinbrot. The energy equation for the Navier-Stokes system. *SIAM J. Math. Anal.*, 5:948–954, 1974.
- [17] Hermann Sohr and Wolf von Wahl. On the regularity of the pressure of weak solutions of Navier-Stokes equations. *Arch. Math. (Basel)*, 46(5):428–439, 1986.

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