

COCYCLES AND MAÑE SEQUENCES WITH AN APPLICATION TO IDEAL FLUIDS

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ABSTRACT. Exponential dichotomy of a strongly continuous cocycle Φ is proved to be equivalent to existence of a Mañe sequence either for Φ or for its adjoint. As a consequence we extend some of the classical results to general Banach bundles. The dynamical spectrum of a product of two cocycles, one of which is scalar, is investigated and applied to describe the essential spectrum of the Euler equation in an arbitrary spacial dimension.

1. INTRODUCTION

Appearance of continuous cocycles on attractors of dissipative PDE, in particular the Navier-Stokes equation, has spurred development of the infinite-dimensional analogues of the classical results of Mather [17], Oseledets [18], Sacker and Sell [20], and others (see the historical account in [4]). New implementations of the theory in ideal fluid dynamics raised new questions that, as far as we know, have not been explicitly answered before. One of them includes precise formulation of the relationship between exponential dichotomy and existence of Mañe sequences or points. The purpose of this present note is partly to clarify known results in this direction by proving them in the most general settings, and partly to justify the cocycle related results claimed in [22]. The results have a direct physical interpretation in terms of shortwave instabilities of an ideal fluid and apply to describe the essential spectrum of non-dissipative advective equations. One particular example we will consider in the next section is the Euler equation on the torus. More examples and further discussion can be found in these recent papers [9, 10, 13, 22].

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2. STATEMENTS OF THE RESULTS

Let Θ be a locally compact Hausdorff space countable at infinity, and let X be a Banach space. Suppose $\varphi = \{\varphi_t\}_{t \in \mathbb{R}}$ is a continuous flow on Θ . A strongly continuous exponentially bounded cocycle Φ over the flow φ acting on the trivial bundle $\Theta \times X$ is a family of bounded linear operators $\{\Phi_t(\theta)\}_{t \geq 0, \theta \in \Theta} \subset \mathcal{L}(X)$ strongly continuous in t, θ , satisfying

$$\Phi_0(\theta) = \mathbf{I}, \quad \Phi_t(\varphi_s(\theta))\Phi_s(\theta) = \Phi_{t+s}(\theta),$$

for all $\theta \in \Theta$, $t, s \geq 0$, and such that $\sup_{0 \leq t \leq 1, \theta \in \Theta} \|\Phi_t(\theta)\| < \infty$. Generic cocycles appear as fundamental matrix solutions of systems of linear ODEs with variable coefficients. So, the solution $f(t)$ of the Cauchy problem

$$\begin{aligned} (1) \quad & f_t = \mathbf{a}(\varphi_t(\theta))f, \\ (2) \quad & f(0) = f_0 \end{aligned}$$

is given by $f(t) = \Phi_t(\theta)f_0$, where Φ is a cocycle over φ .

Following Chow and Leiva [6] we say that Φ has *exponential dichotomy* if there exists a continuous projector-valued mapping $\mathbf{P}(\theta) : X \rightarrow X$ such that for some $\varepsilon > 0$ and $M > 0$ one has

- (1) $\Phi_t(\theta)\mathbf{P}(\theta) = \mathbf{P}(\varphi_t(\theta))\Phi_t(\theta)$;
- (2) $\sup_{\theta \in \Theta} \|\Phi_t(\theta)|_{\text{Im } \mathbf{P}(\theta)}\| \leq M e^{-\varepsilon t}$;
- (3) the restriction $\Phi_t(\theta)|_{\text{Ker } \mathbf{P}(\theta)} : \text{Ker } \mathbf{P}(\theta) \rightarrow \text{Ker } \mathbf{P}(\varphi_t(\theta))$ is invertible, and

$$\|\Phi_t(\theta)x\| \geq M^{-1}e^{\varepsilon t}\|x\|,$$

holds for all $x \in \text{Ker } \mathbf{P}(\theta)$, $t \geq 0$, and $\theta \in \Theta$.

To every cocycle Φ we associate an evolution semigroup \mathbf{E} on the space of X -valued continuous functions vanishing at infinity, $C_0(\Theta; X)$, acting by the rule

$$(3) \quad \mathbf{E}_t f(\theta) = \Phi_t(\varphi_{-t}(\theta))f(\varphi_{-t}(\theta)).$$

The following Dichotomy Theorem of Mather [17], proved in the general settings by Rau [19], and Latushkin and Schnaubelt [15], relates exponential dichotomy to the semigroup \mathbf{E} .

Theorem 2.1. *The cocycle Φ has exponential dichotomy if and only if the semigroup \mathbf{E} is hyperbolic on $C_0(\Theta; X)$, i.e. $\sigma(\mathbf{E}_t) \cap \mathbb{T} = \emptyset$, $t > 0$.*

Restatement of exponential dichotomy in terms of local growth characteristics of the cocycle is our goal in this section. The well-known lemma due to Mañé says that in the case when Θ is compact, $\dim X < \infty$ and Φ is invertible, there exists a point $\theta_0 \in \Theta$ and vector $x_0 \in X$

such that $\sup_{t \in \mathbb{R}} \|\Phi_t(\theta_0)x_0\| < \infty$, provided 1 belongs to the approximate point spectrum of \mathbf{E}_1 . Thus, by virtue of Theorem 2.1, if Φ has exponential dichotomy, then no such (Mañe) point and vector exist.

The analogue of Mañe's lemma in the general settings was proved in [15], where points had to be replaced by so-called Mañe sequences.

Definition 2.2. A sequence of pairs $\{(\theta_n, x_n)\}_{n=1}^\infty$, where $\theta_n \in \Theta$ and $x_n \in X$, is called a *Mañe sequence* of the cocycle Φ if $\{x_n\}_{n=1}^\infty$ is bounded and there are constants $C > 0$ and $c > 0$ such that for all $n \in \mathbb{N}$

$$(4a) \quad \|\Phi_n(\theta_n)x_n\| > c,$$

$$(4b) \quad \|\Phi_k(\theta_n)x_n\| < C, \text{ for all } 0 \leq k \leq 2n.$$

In order to completely characterize the dichotomy in terms of Mañe sequences, one is lead to consider the adjoint operator \mathbf{E}_1^* defined on the space of regular X^* -valued measures of bounded variation, since if $\sigma_{ap}(\mathbf{E}_1) \cap \mathbb{T} = \emptyset$, then $\sigma_p(\mathbf{E}_1^*) \cap \mathbb{T} \neq \emptyset$.

Theorem 2.3. *The following conditions are equivalent:*

- (i) Φ is not exponentially dichotomic;
- (ii) There is a Mañe sequence either for the cocycle Φ or for its adjoint Ψ .

We recall that the adjoint cocycle $\Psi = \Phi^*$ is the cocycle over the inverse flow $\{\varphi_{-t}\}_{t \in \mathbb{R}}$ acting on $\Theta \times X^*$ by the rule

$$\Psi_t(\theta) = \Phi_t^*(\varphi_{-t}(\theta)).$$

It is the cocycle that generates the adjoint evolution semigroup \mathbf{E}_t^* , and it inherits the continuity and boundedness properties from the original cocycle.

We note that in case X is a Hilbert space, Theorem 2.3 can be deduced from the analogue of the Dichotomy Theorem 2.1 on the space $L^2(\Theta, \mathfrak{m}, X)$ over an appropriately chosen φ -invariant measure \mathfrak{m} (see [1, 5, 14]). In this case one takes advantage of the apparent reflexivity of the space and the C^* -algebra technique developed in [2]. As a corollary of Theorem 2.3 and its proof we will obtain the full analogue of the Dichotomy Theorem on $L^2(\Theta, \mathfrak{m}, X)$ for any, reflexive or not, Banach space X .

We now show an example of how Theorem 2.3 applies to spectral problems of fluid dynamics.

We consider the linearized Euler equation on the torus \mathbb{T}^n :

$$(5) \quad v_t = -(u_0 \cdot \nabla)v - (v \cdot \nabla)u_0 - \nabla p,$$

$$(6) \quad \nabla \cdot v = 0,$$

where $u_0 \in [C^\infty(\mathbb{T}^n)]^n$ is a given equilibrium solution to the nonlinear equation. It can be shown that (5)–(6) generates a C_0 -semigroup \mathbf{G}_t on the space L^2 of divergence-free fields, and in fact, on any energy Sobolev space H^m . In contrast to the point spectrum, the essential spectrum of \mathbf{G}_t is related to so-called shortwave instabilities of the flow u_0 . Those are instabilities created by localized highly oscillating disturbances of the form

$$v_\delta(x) = b_0(x)e^{i\xi_0 \cdot x/\delta}, \quad \delta \ll 1.$$

Propagation of such disturbances along the corresponding streamline of the flow u_0 can be described by the WKB-type asymptotic formula

$$(7) \quad v(x, t) = b(x, t)e^{iS(x, t)/\delta} + O(\delta).$$

In this formula the amplitude b and frequency $\xi = \nabla S$ are governed by evolution laws which can be obtained by direct substitution of the ansatz (7) into the linearized Euler equation (5). In the Lagrangian coordinates associated with the flow u_0 , those laws become free of partial differentiation, which allows one to view them as a finite-dimensional dynamical system of the form (1). Specifically, in this case $\Theta = \mathbb{T}^n \times \mathbb{R}^n \setminus \{0\}$, φ_t is the flow on Θ generated by the bicharacteristic system of equations describing evolution of the material particle x and frequency ξ :

$$(8) \quad x_t = u_0(x),$$

$$(9) \quad \xi_t = -\partial u_0^\top \xi,$$

and the amplitude equation for $b(t)$ is given by

$$(10) \quad b_t = \partial u_0(x)b + \langle \partial u_0(x)b, \xi \rangle \xi |\xi|^{-2},$$

subject to incompressibility condition $b \perp \xi$ (see [8, 22] for details).

Let \mathbf{B} stand for the cocycle generated by the amplitude equation (10), and let χ_t denote the integral flow of u_0 , i.e. the solution of (8). It can be shown that in terms of \mathbf{B} and χ the asymptotic formula (7) takes the form

$$(11) \quad v(x, t) = \mathbf{G}_t v_\delta(x) = \mathbf{B}_t(\chi_{-t}(x), \xi_0)v_0(\chi_{-t}(x)) + O(\delta),$$

as $\delta \rightarrow 0$ (see [22, 23]). Thus, if the cocycle \mathbf{B} has growing solutions, then the semigroup \mathbf{G}_t and hence the flow u_0 is linearly unstable to shortwave perturbations.

Suppose now that \mathbf{B} is not dichotomic. Then in view of Theorem 2.3 either \mathbf{B} or \mathbf{B}^* has a Mañé sequence. Since \mathbf{G}_t corresponds to \mathbf{B}^* through a formula similar to (11), and $\sigma(\mathbf{G}_t) = \sigma(\mathbf{G}_t^*)$, we can assume for definiteness that \mathbf{B} has a Mañé sequence $\{(x_n, \xi_n), b_n\}$. We consider a vector field $b_n(x)$ localized near x_n and aligned with b_n up to a term

of order $O(\delta)$ so that $v_{\delta,n} = b_n(x)e^{i\xi_n \cdot x/\delta}$ is divergence-free. Choosing $\delta = \delta_n$ small enough we obtain

$$(12) \quad \mathbf{G}_k v_{\delta,n} = \mathbf{B}_k(\chi_{-k}(x), \xi_n) b_n(\chi_{-k}(x)) e^{i\xi_n \cdot \chi_{-k}(x)/\delta} + O(\delta),$$

for all $0 \leq k \leq 2n$. Thus, denoting $z_n = v_{\delta(n),n}$ we fulfill the sufficient condition for hyperbolicity stated in Lemma 3.1 below, which implies that $1 \in |\sigma(\mathbf{G}_t)|$. Given the fact that the constructed sequence z_n is weakly-null, we can even conclude that $1 \in |\sigma_{ess}(\mathbf{G}_t)|$, the essential spectrum in the Browder sense.

Definition 2.4. Let us recall that the *dynamical spectrum* of a cocycle Φ is the set of all points $\lambda \in \mathbb{R}$ such that $\{e^{-\lambda t} \Phi_t\}$ has no exponential dichotomy. We denote this set by Σ_Φ .

Generally, for a cocycle Φ with compact fiber-maps, its dynamical spectrum is the union of disjoint segments, which may tend to $-\infty$ or be infinite on the left (see [4, 16, 20]). Moreover, the number of segments is limited to the spacial dimension of X if the latter is finite.

After rescaling, Theorem 2.3 states that $\lambda \in \Sigma_\Phi$ if and only if either $e^{-\lambda t} \Phi_t$ or its adjoint has a Mañe sequence. Thus, going back to our example we obtain the following inclusion

$$(13) \quad \exp\{t\Sigma_{\mathbf{B}}\} \subset |\sigma_{ess}(\mathbf{G}_t)|,$$

while, on the other hand, as shown in [22, 23],

$$(14) \quad |\sigma_{ess}(\mathbf{G}_t)| \subset \exp\{t[\min \Sigma_{\mathbf{B}}, \max \Sigma_{\mathbf{B}}]\}.$$

In view of the above discussion the physical meaning of a Mañe sequence becomes more transparent in the context of fluid dynamics: it shows exactly what particle in what frequency has to be excited to destabilize the flow. The dynamical spectrum $\Sigma_{\mathbf{B}}$, in turn, provides the range of all possible rates at which the excitations grow exponentially.

On the Sobolev space H^m of divergence-free fields, the norm of $\mathbf{G}_k v_{\delta,n}$ behaves like $\|\partial \chi_k^{-\top}(x_n) \xi_n\|^m \|\mathbf{B}_k(x_n, \xi_n) b_n\|$, as $\delta \rightarrow 0$. So, in this case one is naturally lead to consider the augmented cocycle

$$\mathbf{B}\mathbf{X}_t^m(x, \xi) = \|\partial \chi_t^{-\top}(x) \xi\|^m \mathbf{B}_t(x, \xi).$$

Via a similar reasoning as above we can obtain the following inclusions

$$(15) \quad \exp\{t\Sigma_{\mathbf{B}\mathbf{X}^m}\} \subset |\sigma_{ess}(\mathbf{G}_t)| \subset \exp\{t[\min \Sigma_{\mathbf{B}\mathbf{X}^m}, \max \Sigma_{\mathbf{B}\mathbf{X}^m}]\}.$$

The influence of the scalar cocycle $\mathbf{X}_t^m = \|\partial \chi^{-\top}(x) \xi\|^m$ on the whole spectrum of $\mathbf{B}\mathbf{X}^m$ is growing with m provided \mathbf{X}^m itself has a non-trivial spectrum, or equivalently, u_0 has exponential stretching of trajectories. Since $\Sigma_{\mathbf{X}^m}$ is one connected segment expanding as $m \rightarrow \infty$, it

will fill all possible gaps in $\Sigma_{\mathbf{B}\mathbf{X}^m}$ for m large enough. Whenever this happens we obtain the identity

$$(16) \quad \exp\{t\Sigma_{\mathbf{B}\mathbf{X}^m}\} = |\sigma_{ess}(\mathbf{G}_t)|.$$

According to Theorem 2.3, to every point of the set $\Sigma_{\mathbf{B}\mathbf{X}^m}$ there corresponds a Mañé sequence. The fact that this set gets larger with m and eventually becomes connected implies that there is an increasing number of Mañé sequence needed to serve points of $\Sigma_{\mathbf{B}\mathbf{X}^m}$. Physically, this means that in a finer norm, such as the norm of H^m , fluid has more spots sensitive to shortwave perturbations than it does in the basic energy norm. Although the above statements apply in any spacial dimension, in the more tractable case of $n = 2$ a much stronger result was obtained by Koch [12]. It shows that any non-isochronic stationary flow in a flat domain is nonlinearly instable in the Hölder classes $C^{1,\alpha}$.

Motivated by the example of the Euler equation, in Section 4 we pose the general question of how the spectrum of a cocycle Φ changes under multiplication by another scalar cocycle \mathbf{C} . We will show that it is contained in the arithmetic sum of Σ_Φ and $\Sigma_{\mathbf{C}}$, and we give a sufficient condition for $\Sigma_{\mathbf{C}\Phi}$ to be connected. This condition applied to the Euler equation will yield a lower bound on m for which (16) holds. This will completely justify the result claimed in [22].

Finally, we remark that all our arguments are local, and as such can be generalized to an arbitrary continuous Banach bundle.

3. CHARACTERIZATION OF EXPONENTIAL DICHOTOMY

In this section we present the proof of Theorem 2.3 and use it to show the analogue of the Dichotomy Theorem on L^p spaces.

The proof relies on the following lemma, which we state slightly more generally than it is needed at the moment. However, it will be used later to its full extent.

Lemma 3.1. *Let Z be a Banach space and $\mathbf{T} \in \mathcal{L}(Z)$. Suppose there is a bounded sequence of vectors $\{z_k\}_{k=1}^\infty$ and a subsequence of natural numbers $\{n_k\}_{k=1}^\infty \subset \mathbb{N}$ such that*

- (a) $\lim_{k \rightarrow \infty} n_k^{-1} \log \|\mathbf{T}^{n_k} z_k\| \geq \lambda_1$;
- (b) $\lim_{k \rightarrow \infty} n_k^{-1} \log \|\mathbf{T}^{2n_k} z_k\| \leq \lambda_1 + \lambda_2$.

Then the following statements are true

- (i) *If $\lambda_1 \leq \lambda_2$, then $[\lambda_1, \lambda_2] \cap \log |\sigma(\mathbf{T})| \neq \emptyset$;*
- (ii) *If $\lambda_2 \leq \lambda_1$, then $[\lambda_2, \lambda_1] \subset \log |\sigma(\mathbf{T})|$.*

Proof. To prove (i) let us assume, on the contrary, that $[\lambda_1, \lambda_2] \cap \log |\sigma(\mathbf{T})| = \emptyset$. Then there is $\varepsilon > 0$ such that

$$[\lambda_1 - \varepsilon, \lambda_2 + \varepsilon] \cap \log |\sigma(\mathbf{T})| = \emptyset.$$

Let Z_s and Z_u denote the spectral subspaces corresponding to the parts of the spectrum below $\lambda_1 - \varepsilon$ and above $\lambda_2 + \varepsilon$, respectively. For n large enough we have

$$(17) \quad \|\mathbf{T}^n|_{Z_s}\| < e^{n(\lambda_1 - \varepsilon)};$$

$$(18) \quad \|\mathbf{T}^n z\| \geq e^{n(\lambda_2 + \varepsilon)} \|z\|, \quad z \in Z_u.$$

Let $z_k = z_k^s + z_k^u$. Then, by (b) and (18),

$$\lambda_1 + \lambda_2 \geq \lim_{k \rightarrow \infty} n_k^{-1} \log \|\mathbf{T}^{2n_k} z_k^u\| \geq \lambda_2 + \varepsilon + \lim_{k \rightarrow \infty} n_k^{-1} \log \|\mathbf{T}^{n_k} z_k^u\|.$$

So,

$$\lim_{k \rightarrow \infty} n_k^{-1} \log \|\mathbf{T}^{n_k} z_k^u\| \leq \lambda_1 - \varepsilon.$$

In combination with (17) this gives

$$\lim_{k \rightarrow \infty} n_k^{-1} \log \|\mathbf{T}^{n_k} z_k\| \leq \lambda_1 - \varepsilon,$$

which contradicts condition (b).

To prove (ii) let us assume that $\lambda_2 < \lambda_1$ and fix any $\lambda \in [\lambda_2, \lambda_1]$. Let us denote $\delta = \lambda_1 - \lambda \geq 0$. We consider a new bounded sequence

$$w_k = e^{-n_k \delta} z_k.$$

For this sequence the following conditions are verified:

$$\lim_{k \rightarrow \infty} n_k^{-1} \log \|\mathbf{T}^{n_k} w_k\| \geq \lambda,$$

$$\lim_{k \rightarrow \infty} n_k^{-1} \log \|\mathbf{T}^{2n_k} w_k\| \leq \lambda_2 + \lambda \leq 2\lambda.$$

Applying (i) with $\lambda_1 = \lambda_2 = \lambda$ we obtain $\lambda \in \log |\sigma(\mathbf{T})|$. \square

Proof of Theorem 2.3. Let us assume (ii). Suppose $\{(\theta_n, x_n)\}_{n=1}^\infty$ is a Mañe sequence for the cocycle Φ . For each n let us find an open neighborhood of θ_n , denoted U_n , such that for all $\theta \in U_n$,

$$(19a) \quad \|\Phi_n(\theta_n) - \Phi_n(\theta)\| < c/2,$$

$$(19b) \quad \|\Phi_{2n}(\theta)\| < 2C.$$

Let $\phi_n \in C_0(\Theta)$ be a scalar function of unit norm supported on U_n such that $\phi_n(\theta_n) = 1$. Then, by (19) and (4), we obtain

$$\|\mathbf{E}_n(\phi_n(\cdot)x_n)\| > c/2,$$

$$\|\mathbf{E}_{2n}(\phi_n(\cdot)x_n)\| < 2C.$$

So, Lemma 3.1 applies with $\lambda_1 = \lambda_2 = 0$ to show that \mathbf{E} is not hyperbolic, and hence, by virtue of Theorem 2.1, Φ is not exponentially dichotomic.

If there is a Mañe sequence for the adjoint cocycle Ψ , then by the previous argument applied to \mathbf{E}^* on $C_0(\Theta; X^*)$, we find that Ψ is not exponentially dichotomic. Hence, Φ is not dichotomic either, as seen directly from the definition.

To show the converse implication, let us assume (i). By Theorem 2.1 one has $1 \in |\sigma(\mathbf{E}_1)|$. There are two possibilities that follow from this – either there is an approximate eigenvalue or there is a point of the residual spectrum on the unit circle.

In the first case there is a normalized sequence of functions $f_n \in C_0(\Theta; X)$ such that

$$(20) \quad \|\mathbf{E}_k f_n - e^{i\alpha k} f_n\| \leq \frac{1}{2},$$

for some $\alpha \in \mathbb{R}$ and all $k = 1, \dots, 2n$. Let us choose points $\theta'_n \in \Theta$ so that $\|f_n(\theta'_n)\| = 1$. By (20), we have

$$\|\Phi_n(\varphi_{-n}(\theta'_n))f_n(\varphi_{-n}(\theta'_n))\| \geq 1/2,$$

and

$$\|\Phi_k(\cdot)f_n(\cdot)\| \leq 2, \quad 1 \leq k \leq 2n.$$

Choosing $\theta_n = \varphi_{-n}(\theta'_n)$ and $x_n = f_n(\varphi_{-n}(\theta'_n))$ we fulfill the conditions of Definition 2.2.

In the second case, let $e^{i\alpha}$ be a point of the residual spectrum of \mathbf{E}_1 . Hence, there exists $\nu \in \mathcal{M}(\Theta, X^*)$, with $\|\nu\| = 1$, a regular Borel X^* -valued measure of bounded variation, such that

$$(21) \quad \mathbf{E}_n^* \nu = e^{i\alpha n} \nu, \quad n \in \mathbb{N}.$$

Recall that the norm in $\mathcal{M}(\Theta; X^*)$ is given by the total variation

$$(22) \quad \|\nu\| = \sup \left\{ \sum_{i=1}^N \|\nu(A_i)\| : \bigcup_{i=1}^N A_i = \Theta, A_i \cap A_j = \emptyset \right\}.$$

We also consider the semivariation of a set $A \subset \Theta$ defined by

$$|\nu|(A) = \sup\{|x^{**}\nu|(A) : x^{**} \in X^{**}\},$$

and we recall the following inequality [7, p.4]:

$$(23) \quad |\nu|(A) \leq 4 \sup\{\|\nu(B)\| : B \subset A\}.$$

Going back to our proof, let us fix $n \in \mathbb{N}$. By the continuity of Ψ and φ , using the topological assumption on Θ , we can find a partitioning of Θ into Borel sets $\{A_j\}_{j \in J}$ such that for every $j \in J$,

$$(24) \quad \|\Psi_n(\varphi_{-n}(\theta')) - \Psi_n(\varphi_{-n}(\theta''))\| < c_0,$$

$$(25) \quad \|\Psi_{2n}(\varphi_{-n}(\theta')) - \Psi_{2n}(\varphi_{-n}(\theta''))\| < c_0,$$

holds for all $\theta', \theta'' \in A_j$, and where the constant $c_0 > 0$ is to be specified later.

By (23), for every $j \in J$, there is a set $B_j \subset A_j$ such that

$$(26) \quad 4\|\nu(B_j)\| > |\nu|(A_j).$$

Let us fix arbitrary tag points $\theta_j \in B_j$. According to (21) and (24) – (25), we have

$$(27) \quad \Psi_n(\varphi_{-n}(\theta_j)) \frac{\nu(\varphi_{-n}(B_j))}{|\nu|(A_j)} = \frac{\nu(B_j)}{|\nu|(A_j)} + v_n^j,$$

$$(28) \quad \Psi_{2n}(\varphi_{-n}(\theta_j)) \frac{\nu(\varphi_{-n}(B_j))}{|\nu|(A_j)} = \frac{\nu(\varphi_n(B_j))}{|\nu|(A_j)} + u_n^j,$$

where

$$(29) \quad \|v_n^j\|, \|u_n^j\| < c_0 \frac{\|\nu(\varphi_{-n}(B_j))\|}{|\nu|(A_j)}.$$

Let us denote $\eta = |\nu|(\Theta)$. We claim that there exists $j = j(n) \in J$ such that

$$(30) \quad \|\nu(\varphi_{-n}(B_{j(n)}))\| \leq \frac{4}{\eta} |\nu|(A_{j(n)}),$$

$$(31) \quad \|\nu(\varphi_n(B_{j(n)}))\| \leq \frac{4}{\eta} |\nu|(A_{j(n)}).$$

Indeed, suppose there is no such $j(n)$. Then for each $j \in J$ either (30) or (31) fails. So, by the subadditivity of semivariation, we obtain

$$\begin{aligned} \eta = |\nu|(\Theta) &\leq \sum_{j \in J} |\nu|(A_j) \leq \frac{\eta}{4} \sum_{j \in J} \|\nu(\varphi_{-n}(B_j))\| + \\ &\quad + \frac{\eta}{4} \sum_{j \in J} \|\nu(\varphi_n(B_j))\| \leq 2\frac{\eta}{4} \|\nu\| = \frac{\eta}{2}, \end{aligned}$$

a contradiction.

Let us put $\theta_n = \varphi_{-n}(\theta_{j(n)})$ and $x_n^* = \frac{\nu(\varphi_{-n}(B_{j(n)}))}{|\nu|(A_{j(n)})}$. In view of (30), $\{x_n^*\}$ is a bounded sequence. Also, by (30) and (29), we have

$$\|v_n^{j(n)}\|, \|u_n^{j(n)}\| < \frac{4c_0}{\eta}.$$

So, by (26), (27), (28), (31), and (23),

$$(32) \quad \|\Psi_n(\theta_n)x_n^*\| \geq \frac{1}{4} - \frac{4c_0}{\eta},$$

$$(33) \quad \|\Psi_{2n}(\theta_n)x_n^*\| \leq \frac{4}{\eta} + \frac{4c_0}{\eta}.$$

It suffices to take $c_0 = \eta/32$. □

In the compact case existence of a Mañe sequence is equivalent to existence of a Mañe point (see, for example, [4]). So, in this case Theorem 2.3 can be restated as follows.

Corollary 3.2. *Suppose $\dim X < \infty$ and Θ is compact. Then Φ is exponentially dichotomic if and only if either Φ or Ψ has a Mañe point.*

Another fact that follows directly from Lemma 3.1 is that any Lyapunov index of the cocycle Φ belongs to the dynamical spectrum Σ_Φ (see also Johnson, Palmer and Sell [11]).

Indeed, suppose

$$\lambda = \lim_{k \rightarrow \infty} n_k^{-1} \log \|\Phi_{n_k}(\theta)x\|,$$

for some $\theta \in \Theta$ and $x \in X$. Then by the same construction as in the proof of Theorem 2.3 we find functions f_k such that

$$\lambda = \lim_{k \rightarrow \infty} n_k^{-1} \log \|\mathbf{E}_{n_k} f_k\|,$$

$$2\lambda = \lim_{k \rightarrow \infty} n_k^{-1} \log \|\mathbf{E}_{2n_k} f_k\|.$$

Applying Lemma 3.1 with $\lambda_1 = \lambda_2 = \lambda$ we obtain $\lambda \in \log |\sigma(\mathbf{E}_t)|$.

As another consequence of Theorem 2.3 we prove the analogue of the Dichotomy Theorem 2.1 for L^p -spaces.

Let \mathbf{m} be a Borel φ -quasi-invariant measure on Θ . We define \mathbf{E} on $L^p(\Theta, \mathbf{m}, X)$, $1 \leq p < \infty$, by the rule

$$(34) \quad \mathbf{E}_t f(\theta) = \left(\frac{d(\mathbf{m} \circ \varphi_{-t})}{d\mathbf{m}} \right)^{1/p} \Phi_t(\varphi_{-t}(\theta)) f(\varphi_{-t}(\theta)),$$

where the expression under the root is the Radon-Nikodim derivative (we refer to [4] for a detailed discussion).

Theorem 3.3. *Let \mathbf{E} be defined by (34) on the space $L^p(\Theta, \mathbf{m}, X)$, with $1 \leq p < \infty$, where \mathbf{m} is a Borel φ -quasi-invariant measure such that $\mathbf{m}(U) > 0$ for every open set U . Then Φ has exponential dichotomy if and only if \mathbf{E} is hyperbolic.*

Proof. Suppose that Φ has exponential dichotomy, then the spaces

$$(35) \quad Z_s = \{f \in L^p(\Theta, \mathfrak{m}, X) : f(\theta) \in \text{Im } \mathbf{P}(\theta)\},$$

$$(36) \quad Z_u = \{f \in L^p(\Theta, \mathfrak{m}, X) : f(\theta) \in \text{Ker } \mathbf{P}(\theta)\}$$

define, respectively, exponentially stable and unstable subspaces for \mathbf{E} such that $L^p(\Theta, \mathfrak{m}, X) = Z_s \oplus Z_u$. Hence, \mathbf{E} is hyperbolic.

Suppose Φ has no exponential dichotomy. Let us assume that Φ has a Mañe sequence. Then the same construction as in the proof of Theorem 2.3, with localized scalar functions $\phi_n \in L^p(\Theta, \mathfrak{m})$, $\|\phi_n\|_p = 1$, shows that \mathbf{E} is not hyperbolic.

If the adjoint cocycle Ψ has a Mañe sequence, then we regard the corresponding functions $\varphi_n(\theta)x_n^*$ as elements of $L_{w^*}^q(\Theta, \mathfrak{m}, X^*)$, the space of weak*-measurable q -integrable functions with values in X^* . This space is the dual of $L^p(\Theta, \mathfrak{m}, X)$, provided $p^{-1} + q^{-1} = 1$ (see [3]).

From Lemma 3.1 we conclude that the operator \mathbf{E}_1^* is not hyperbolic over $L_{w^*}^q(\Theta, \mathfrak{m}, X^*)$. Hence, \mathbf{E}_1 is not hyperbolic over $L^p(\Theta, \mathfrak{m}, X)$. \square

4. SCALAR MULTIPLE OF A COCYCLE

Let Θ, φ, X , and Φ be as before, and let $\mathbf{C} = \{\mathbf{C}_t(\theta)\}_{t \geq 0, \theta \in \Theta}$ be a scalar cocycle over the same flow φ acting on $\Theta \times \mathbb{C}$. Then the product $\mathbf{C}\Phi = \{\mathbf{C}_t(\theta)\Phi_t(\theta)\}_{t \geq 0, \theta \in \Theta}$ defines another cocycle on $\Theta \times X$. An example of how products of this type arise in the equations of fluid dynamics was presented in Section 2.

Lemma 4.1. *One has the following inclusion*

$$(37) \quad \Sigma_{\mathbf{C}\Phi} \subset \Sigma_{\mathbf{C}} + \Sigma_{\Phi}.$$

Proof. Let $\rho \in \Sigma_{\mathbf{C}\Phi}$. Then by Theorem 2.3 there exists a Mañe sequence, say, for $e^{-\rho t}\mathbf{C}_t\Phi_t$ (the case of adjoint cocycle is treated similarly). Let $\{\theta_n, x_n\}_{n=1}^\infty$ be that sequence. Then we have

$$(38) \quad |\mathbf{C}_n(\theta_n)| \|\Phi_n(\theta_n)x_n\| > ce^{\rho t},$$

$$(39) \quad |\mathbf{C}_{2n}(\theta_n)| \|\Phi_{2n}(\theta_n)x_n\| < Ce^{2\rho t},$$

for all $n \in \mathbb{N}$. Let us extract a subsequence $\{n_k\}_{k=1}^\infty$ such that the limits

$$(40) \quad \lim_{k \rightarrow \infty} n_k^{-1} \log |\mathbf{C}_{n_k}(\theta_{n_k})| = \lambda_1,$$

$$(41) \quad \lim_{k \rightarrow \infty} n_k^{-1} \log |\mathbf{C}_{2n_k}(\theta_{n_k})| = \lambda_1 + \lambda_2,$$

$$(42) \quad \lim_{k \rightarrow \infty} n_k^{-1} \log \|\Phi_{n_k}(\theta_{n_k})x_{n_k}\| = \mu_1,$$

$$(43) \quad \lim_{k \rightarrow \infty} n_k^{-1} \log \|\Phi_{2n_k}(\theta_{n_k})x_{n_k}\| = \mu_1 + \mu_2,$$

exist. By (38) and (39), we have

$$(44) \quad \lambda_1 + \mu_1 \geq \rho,$$

$$(45) \quad \lambda_1 + \lambda_2 + \mu_1 + \mu_2 \leq 2\rho.$$

Let us consider two cases: $\lambda_1 \leq \lambda_2$ and $\lambda_1 > \lambda_2$. If $\lambda_1 \leq \lambda_2$, then by Lemma 3.1, there is $\lambda \in [\lambda_1, \lambda_2] \cap \Sigma_{\mathbf{C}}$. From (44) and (45), we have $\lambda_2 + \mu_2 \leq \rho$. So, $\mu_2 \leq \rho - \lambda_2 \leq \rho - \lambda_1 \leq \mu_1$. In this case Lemma 3.1 implies that $[\mu_2, \mu_1] \subset \Sigma_{\Phi}$. We choose $\mu = \rho - \lambda \in [\mu_2, \mu_1]$ to satisfy $\rho = \lambda + \mu$.

If $\lambda_2 < \lambda_1$, then $[\lambda_2, \lambda_1] \subset \Sigma_{\mathbf{C}}$. From the above we still have $\mu_2 \leq \rho - \lambda_2$ and $\mu_1 \geq \rho - \lambda_1$. If $\mu_1 \leq \mu_2$, then we find a point $\mu \in [\mu_1, \mu_2] \cap \Sigma_{\Phi}$, and choose $\lambda = \rho - \mu \in [\lambda_2, \lambda_1]$. If $\mu_1 > \mu_2$, then $[\mu_2, \mu_1] \subset \Sigma_{\Phi}$ and $[\mu_2, \mu_1] \cap [\rho - \lambda_1, \rho - \lambda_2] \neq \emptyset$. Choosing $\mu \in [\mu_2, \mu_1] \cap [\rho - \lambda_1, \rho - \lambda_2]$ we get $\lambda = \rho - \mu \in [\lambda_2, \lambda_1]$. This finishes the argument. \square

Now let us assume that both cocycles Φ and \mathbf{C} are invertible so that their spectra are bounded from above and below. We denote $\mu_{\max}^{\Phi} = \max \Sigma_{\Phi}$ and $\mu_{\min}^{\Phi} = \min \Sigma_{\Phi}$. Similar notation will be used for other cocycles.

Lemma 4.2. *Suppose $\rho \in [\mu_{\min}^{\mathbf{C}\Phi}, \mu_{\max}^{\mathbf{C}\Phi}] \setminus \Sigma_{\mathbf{C}\Phi}$. Then the following inequalities hold:*

$$(46) \quad \mu_{\max}^{\mathbf{C}} + \mu_{\min}^{\Phi} < \rho < \mu_{\min}^{\mathbf{C}} + \mu_{\max}^{\Phi}.$$

Proof. Let \mathbf{P} , $\varepsilon > 0$ and M be as in the definition of the dichotomy. Let us fix any $\lambda \in \Sigma_{\mathbf{C}}$. Then by Theorem 2.3 there exists a Mañé sequence $\{\theta_n\}$ for \mathbf{C} :

$$(47) \quad |\mathbf{C}_n(\theta_n)| > ce^{n\lambda},$$

$$(48) \quad |\mathbf{C}_{2n}(\theta_n)| < Ce^{2n\lambda}.$$

Given that $\text{Ker } \mathbf{P}(\theta) \neq \{0\}$ for every $\theta \in \Theta$, we can find a unit vector $x_n \in \text{Ker } \mathbf{P}(\theta_n)$ for every n . Then in view of (48) we have

$$Ce^{n(2\lambda + 2\mu_{\max}^{\Phi} + \varepsilon)} \geq |\mathbf{C}_{2n}(\theta_n)| \|\Phi_{2n}(\theta_n)x_n\| \geq M^{-1}e^{2n(\rho + \varepsilon)}.$$

Thus, $\rho \leq \lambda + \mu_{\max}^{\Phi} - \varepsilon$ for all $\lambda \in \Sigma_{\mathbf{C}}$. This proves the right side of (46). The left side is proved similarly using (47). \square

As an immediate consequence of Lemma 4.2 we obtain the following sufficient condition for $\Sigma_{\mathbf{C}\Phi}$ to be connected.

Theorem 4.3. *Suppose the cocycles \mathbf{C} and \mathbf{B} are invertible. The dynamical spectrum $\Sigma_{\mathbf{C}\Phi}$ has no gaps provided the diameter of $\Sigma_{\mathbf{C}}$ is greater than the diameter of Σ_{Φ} , i.e.*

$$(49) \quad \mu_{\max}^{\mathbf{C}} - \mu_{\min}^{\mathbf{C}} \geq \mu_{\max}^{\Phi} - \mu_{\min}^{\Phi}.$$

Going back to our example with the Euler equation, let us denote

$$\lambda_{\max} = \mu_{\max}^{\mathbf{X}}, \quad \lambda_{\min} = \mu_{\min}^{\mathbf{X}}.$$

Then $\Sigma_{\mathbf{X}^m} = m[\lambda_{\min}, \lambda_{\max}]$. Assume that $\lambda_{\max} > 0$, and hence by incompressibility, $\lambda_{\min} < 0$. In this case condition (49) turns into

$$(50) \quad |m| \geq \frac{\mu_{\max}^{\mathbf{B}} - \mu_{\min}^{\mathbf{B}}}{\lambda_{\max} - \lambda_{\min}}.$$

So, if $|m|$ is large enough, then we have identity (16) over the Sobolev space H^m . In fact, if the cocycle \mathbf{B} has trivial dynamical spectrum, such as in the case of a parallel shear flow u_0 or $n = 2$ in the vorticity formulation, then $\mu_{\max}^{\mathbf{B}} = \mu_{\min}^{\mathbf{B}}$, and (16) holds for any $m \neq 0$.

We refer to [21, 22] for more details on the description of the essential spectrum for the Euler and other similar equations.

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