

ON THE MOTION OF NON-DISSIPATIVE INHOMOGENEOUS FLUID-LIKE BODIES

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ABSTRACT. We study a system of equation governing evolution of inhomogeneous fluid-like bodies (granular fluid) disregarding effects of viscosity. A local well-posedness theory is developed on a bounded smooth domain with no-slip boundary condition on velocity and vanishing gradient of density. The cases of open space and periodic box are also considered, where the local existence and uniqueness of solutions is shown in Sobolev spaces up to the critical smoothness $\frac{n}{2} + 1$.

1. PROBLEM FORMULATION

We consider a system of partial differential equations:

$$(1) \quad \operatorname{div}_x \mathbf{v} = 0,$$

$$(2) \quad \partial_t \varrho + \mathbf{v} \cdot \nabla_x \varrho = 0,$$

$$(3) \quad \varrho \left(\partial_t \mathbf{v} + \operatorname{div}_x (\mathbf{v} \otimes \mathbf{v}) \right) + \beta \operatorname{div}_x (\nabla_x \varrho \otimes \nabla_x \varrho) = -\nabla_x p + \frac{\beta}{3} \nabla_x |\nabla_x \varrho|^2,$$

where $\beta > 0$ is a given constant, the unknowns $\varrho = \varrho(t, x)$, $\mathbf{v} = \mathbf{v}(t, x)$, (and $p = p(t, x)$) are functions of the time $t \in (0, T)$ and the spatial coordinate $x \in \Omega$ - a bounded regular domain in the Euclidean space \mathbb{R}^3 .

Problem (1) - (3) is supplemented with the boundary conditions

$$(4) \quad \mathbf{v} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

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and

$$(5) \quad \nabla_x \varrho|_{\partial\Omega} = \mathbf{0}.$$

The initial state of the system is determined by the initial conditions

$$(6) \quad \varrho(0, \cdot) = \varrho_0, \quad \mathbf{v}(0, \cdot) = \mathbf{v}_0,$$

where ϱ_0 and \mathbf{v}_0 satisfy the boundary conditions (4)–(5).

Applying the gradient to (2) we obtain the evolution equation for co-vector $\nabla_x \varrho$:

$$(7) \quad \partial_t \nabla_x \varrho + [\nabla_x \nabla_x \varrho] \mathbf{v} + (\nabla_x \mathbf{v})^\top \nabla_x \varrho = 0.$$

From (7) we obtain the balance relation

$$(8) \quad \partial_t |\nabla_x \varrho|^2 + \mathbf{v} \cdot \nabla_x |\nabla_x \varrho|^2 + 2 \nabla_x \mathbf{v} : (\nabla_x \varrho \otimes \nabla_x \varrho) = 0,$$

where $A : B$ denotes $\text{Tr}[A \cdot B^\top]$. It follows immediately from (8) that property (5) is transported by the flow provided all quantities are smooth. Thus the boundary condition (5) remains valid for any smooth solution of system (1) - (4) provided that it is imposed on the initial datum ϱ_0 . In particular, the density ϱ remains constant on each connected component of $\partial\Omega$. Saying differently, when dealing with smooth solution (i.e., those that have the velocity field at least in $L^1(0, T; W^{1,\infty}(\Omega))$) we could replace (5) by the requirement that $\nabla_x \varrho_0 = \mathbf{0}$ on $\partial\Omega$.

The system of governing equations (1)–(3) describes the motion of inhomogeneous incompressible fluids in which there are no entropy producing mechanisms. The model can be viewed as a generalization of the incompressible Euler system, considering $\varrho_0 \equiv \text{const} > 0$ in (6), and the incompressible inhomogeneous Euler system, obtained formally from (1)–(3) by setting $\beta = 0$.

In order to make the relation between (1)–(3) and the inhomogeneous Euler system more precise, let us recall that isothermal processes in fluid-like-bodies are characterized by the balance of mass and balance of linear momentum

$$(9) \quad \dot{\varrho} + \varrho \text{div}_x \mathbf{v} = 0 \quad \text{and} \quad \varrho \dot{\mathbf{v}} = \text{div}_x \mathbf{T}$$

completed by the reduced thermodynamical identity

$$(10) \quad \mathbf{T} : \mathbf{D}(\mathbf{v}) - \varrho \dot{\psi} = \zeta.$$

See for example Málek and Rajagopal [12] for details and as a general reference for the derivation of the model that we briefly summarize in the following lines.

In the above equations, ϱ stands for the density, \mathbf{v} is the velocity, $\mathbf{D}(\mathbf{v})$ the symmetric part of the velocity gradient, \mathbf{T} is the Cauchy stress, ψ is the

Helmholtz potential, ζ is the rate of entropy production and $\dot{z} = \partial_t z + \nabla_x z \cdot \mathbf{v}$ for any scalar quantity z .

The second law of thermodynamics is fulfilled if $\zeta \geq 0$. In what follows we set

$$(11) \quad \zeta = 0 \quad \text{and} \quad \psi = \tilde{\psi}(\varrho, \nabla_x \varrho).$$

Inserting (11) into (10), and using (9)₁ for incompressible fluids ($\text{div}_x \mathbf{v} = 0$) and (7), we obtain

$$(12) \quad \left(\mathbf{S} - \varrho \left(\frac{\partial \psi}{\partial \nabla_x \varrho} \otimes \nabla_x \varrho - \frac{1}{3} \left(\frac{\partial \psi}{\partial \nabla_x \varrho} \cdot \nabla_x \varrho \right) \mathbf{I} \right) \right) : \mathbf{D}(\mathbf{v}) = 0,$$

where $\mathbf{S} = \mathbf{T} - \left(\frac{1}{3} \text{Tr} \mathbf{T}\right) \mathbf{I} =: \mathbf{T} + p\mathbf{I}$ is the deviatoric part of the Cauchy stress and p is the mean normal stress (the pressure). It follows from (12) that

$$(13) \quad \mathbf{S} = \varrho \left(\frac{\partial \psi}{\partial \nabla_x \varrho} \otimes \nabla_x \varrho - \frac{1}{3} \left(\frac{\partial \psi}{\partial \nabla_x \varrho} \cdot \nabla_x \varrho \right) \mathbf{I} \right),$$

which implies that

$$(14) \quad \mathbf{T} = -p\mathbf{I} + \varrho \left(\frac{\partial \psi}{\partial \nabla_x \varrho} \otimes \nabla_x \varrho - \frac{1}{3} \left(\frac{\partial \psi}{\partial \nabla_x \varrho} \cdot \nabla_x \varrho \right) \mathbf{I} \right).$$

Choosing $\tilde{\psi}$ of the form

$$\tilde{\psi}(\varrho, \nabla_x \varrho) = \psi_1(\varrho) + \frac{\beta}{2\varrho} |\nabla_x \varrho|^2$$

and inserting it in (14), and then the result into (9), we finally obtain (1)–(3). Obviously, if we assume that $\tilde{\psi}$ in (11) depends on ϱ but is independent of $\nabla_x \varrho$ then we end-up with $\mathbf{T} = -p\mathbf{I}$. Inserting the last equation into (9) we obtain the inhomogeneous Euler equation.

From the mathematical viewpoint, system (1) - (3) is a rather non-standard hyperbolic system of first order equations. Our goal is to show local-in-time existence of strong solutions in the framework of Sobolev spaces for any sufficiently regular initial data ϱ_0, \mathbf{v}_0 satisfying the boundary conditions (4), (5). Problem (1) - (6) is closely related to the inhomogeneous Euler system studied by Beirão da Veiga [2], Beirão da Veiga and Valli [3], [4]. Besides, there is a vast amount of literature devoted to the classical incompressible Euler system, see Bourguignon and Brezis [5], Ebin and Marsden [7], Kato and Lai [9], Temam [14], among others.

In light of these results, the initial-boundary value problem (1) - (6) seems to be well-understood. However, there are additional difficulties related to the presence of the quadratic term $\text{div}_x(\nabla_x \varrho \otimes \nabla_x \varrho)$ that make the proof

rather delicate. At the first glance, we face a problem with “loss of derivatives” as $\operatorname{div}_x(\nabla_x \varrho \otimes \nabla_x \varrho)$ is apparently not controlled by the standard energy norm. On the other hand, it is not difficult to check (see Section 2 below) that difficult term disappears, at least at the level of *a priori* estimates, because of cancelation of certain terms in the energy balance. This fact was also observed by Lin et al. [10], where the authors establish local existence of regular solutions for the Cauchy problem for a system closely related to (1) - (3). However, the technique used by Lin et al. [10] does not apply directly to our case because the system of equations considered in [10] does not contain the density in the material derivative in (3). Accordingly, the resulting problem in [10] may be rewritten as a *symmetric hyperbolic* system in terms of two new variables - the velocity \mathbf{v} and the density gradient $\nabla_x \varrho$ - that can be solved by a standard approximation scheme. Such a reduction is impossible in the present setting due to the dependence on ϱ . As a matter of fact, introducing *three* phase variables \mathbf{v} , ϱ , $\nabla_x \varrho$ would lead to a system very similar to the inhomogeneous ideal magnetohydrodynamics equations studied by Secchi [13]. It is worth noting that the corresponding local existence result obtained by Secchi [13] requires the density gradient $\nabla_x \varrho$ to be small - a rather restrictive hypothesis in the context of (1) - (6).

Our approach is based on *a priori* estimates derived in Section 2. Solutions are then constructed by means of the Galerkin method proposed by Temam [14]. This technique put in the abstract framework was later developed by Kato and Lai [9]. In contrast with [9], however, our function spaces framework is based on spaces of solenoidal functions and a result of Ghidaglia [8] concerning regularity of solutions to certain elliptic problems posed in these spaces, see Section 3.

The existence theory established in Section 3 for the framework of Sobolev spaces of integer order (in smooth bounded domains) is extended in Section 4 to the framework of the fractional Sobolev spaces (embedded into $W^{1,\infty}(\Omega)$ for the velocity). We however restrict for simplicity to the spatially periodic or Cauchy problem here. Our procedure is based on approximation of initial data by functions in higher integer Sobolev spaces (then the theory established in Section 3 is applicable) and new uniform estimates in fractional Sobolev spaces. Using them we can easily take the limit and conclude the existence result.

2. A PRIORI ESTIMATES

A priori estimates are derived under the principal hypothesis that all quantities appearing in (1) - (6) are smooth enough. They are purely formal and serve only to identify the function spaces framework used in the

existence proof. Most of them depend effectively on the length of the time interval $(0, T)$ and/or on the size of the initial data.

For simplicity and without loss of generality, we set $\beta = 1$ in what follows.

2.1. Uniform unconditioned estimates. We assume that initially the density satisfies uniform estimates

$$(15) \quad 0 < m_0 \leq \varrho_0(x) \leq M_0 < \infty, \text{ for all } x \in \Omega.$$

As ϱ satisfies the transport equation (2) we immediately obtain that

$$(16) \quad 0 < m_0 \leq \varrho(t, x) \leq M_0 < \infty, \text{ for all } x \in \Omega, t > 0.$$

Multiplying Eq. (3) by \mathbf{v} and using (2) and (8) we deduce after integration over Ω the energy balance

$$(17) \quad \int_{\Omega} (\varrho |\mathbf{v}|^2 + |\nabla_x \varrho|^2) (t, x) \, dx = \int_{\Omega} (\varrho_0 |\mathbf{v}_0|^2 + |\nabla_x \varrho_0|^2) (x) \, dx,$$

for any $t > 0$. Unfortunately, estimates (16 - 17) seem to be the only *a priori* bounds that are independent of the length of the time interval. Note that (8) yields

$$(18) \quad |\nabla_x \varrho(t, x)|^2 \leq c \exp \left(\int_0^t \|\nabla_x \mathbf{v}\|_{L^\infty(\Omega; \mathbb{R}^3)} \, d\tau \right) \sup_{x \in \Omega} |\nabla_x \varrho_0(x)|^2$$

for any $t > 0, x \in \Omega,$

however, *uniform* bounds on the velocity gradient $\nabla_x \mathbf{v}$ are out of reach of our approach.

2.2. Pressure equation. A proper control of the pressure plays a crucial role in problems involving the incompressibility constraint (1). Unlike its thermodynamic counterpart that is an explicitly given function of the density, the pressure in “incompressible” problems is viewed as a kind of Lagrange multiplier maintaining (1) in force and implicitly related to the motion.

Dividing (3) by ϱ and applying div_x to both sides of the result we deduce

$$(19) \quad -\operatorname{div}_x \left(\frac{1}{\varrho} \nabla_x p \right) = \operatorname{div}_x \left(\frac{1}{\varrho} \operatorname{div}_x \left(\nabla_x \varrho \otimes \nabla_x \varrho - \frac{1}{3} |\nabla_x \varrho|^2 \mathbb{I} \right) \right) \\ + \operatorname{div}_x \operatorname{div}_x (\mathbf{v} \otimes \mathbf{v}).$$

Moreover, as \mathbf{v} satisfies the impermeability boundary condition (4), we have

$$\varrho \mathbf{n} \otimes \mathbf{v} : \nabla_x \mathbf{v} = \varrho \mathbf{v} \cdot \nabla_x (\mathbf{v} \cdot \mathbf{n}) - \varrho \mathbf{v} \otimes \mathbf{v} : \nabla_x \mathbf{n} = -\varrho \mathbf{v} \otimes \mathbf{v} : \nabla_x \mathbf{n}$$

on $\partial\Omega$. Therefore, after a simple manipulation, we conclude from (3) and (5) that

$$(20) \quad \nabla_x p \cdot \mathbf{n} = \varrho \nabla_x \mathbf{n} : [\mathbf{v} \otimes \mathbf{v}] \text{ on } \partial\Omega.$$

It appears that the highest derivative of density in (19) is of order 3, which gives a gain of one derivative of ϱ over p . This will not be sufficient to close estimates below. We will show however that the highest order derivative can be absorbed by the pressure giving rise to a modified pressure. More specifically, let us consider

$$(21) \quad \Pi = p + \frac{2}{3} |\nabla_x \varrho|^2.$$

Using Π , the momentum equation (3) takes the form

$$(22) \quad \begin{aligned} \partial_t \mathbf{v} + \operatorname{div}_x(\mathbf{v} \otimes \mathbf{v}) + \frac{1}{\varrho} \operatorname{div}_x(\nabla_x \varrho \otimes \nabla_x \varrho) &= \\ &= -\frac{1}{\varrho} \nabla_x \Pi + \frac{1}{\varrho} \nabla_x |\nabla_x \varrho|^2. \end{aligned}$$

Taking divergence we obtain

$$(23) \quad \begin{aligned} -\operatorname{div}_x \left(\frac{1}{\varrho} \nabla_x \Pi \right) &= \operatorname{div}_x \operatorname{div}_x(\mathbf{v} \otimes \mathbf{v}) \\ &+ \operatorname{div}_x \left(\frac{1}{\varrho} \operatorname{div}_x(\nabla_x \varrho \otimes \nabla_x \varrho) \right) \\ &- \operatorname{div}_x \left(\frac{1}{\varrho} \nabla_x |\nabla_x \varrho|^2 \right). \end{aligned}$$

The velocity tensor can be written as

$$\operatorname{div} \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) = \nabla_x \mathbf{v} : (\nabla_x \mathbf{v})^\top.$$

For the remaining part of the right-hand side of (23) we have

$$\begin{aligned} &\operatorname{div}_x \left(\frac{1}{\varrho} \operatorname{div}_x(\nabla_x \varrho \otimes \nabla_x \varrho) \right) - \operatorname{div}_x \left(\frac{1}{\varrho} \nabla_x |\nabla_x \varrho|^2 \right) \\ &= \operatorname{div} \left(\frac{1}{\varrho} \nabla_x \varrho \cdot \nabla_x \nabla_x \varrho \right) + \operatorname{div} \left(\frac{1}{\varrho} \nabla_x \varrho \Delta \varrho \right) - \operatorname{div} \left(\frac{1}{\varrho} \nabla_x |\nabla_x \varrho|^2 \right) \end{aligned}$$

in view of the identity $\nabla_x \varrho \cdot \nabla_x \nabla_x \varrho = \frac{1}{2} \nabla_x |\nabla_x \varrho|^2$,

$$\begin{aligned} &= \operatorname{div} \left(\frac{1}{\varrho} \nabla_x \varrho \Delta \varrho \right) - \frac{1}{2} \operatorname{div} \left(\frac{1}{\varrho} \nabla_x |\nabla_x \varrho|^2 \right) \\ &= -\frac{1}{\varrho^2} |\nabla_x \varrho|^2 \Delta \varrho + \frac{1}{\varrho} |\Delta \varrho|^2 + \frac{1}{\varrho} \nabla_x \varrho \cdot \nabla_x \Delta \varrho \\ &\quad - \frac{1}{2} \left(-\frac{1}{\varrho^2} \nabla_x \varrho \cdot \nabla_x |\nabla_x \varrho|^2 + \frac{1}{\varrho} \Delta |\nabla_x \varrho|^2 \right) \end{aligned}$$

in view of the identity $\Delta |\nabla_x \varrho|^2 = 2 \nabla_x \varrho \cdot \nabla_x \Delta \varrho + 2 |\nabla_x^{(2)} \varrho|^2$,

$$= \frac{|\Delta \varrho|^2 - |\nabla_x^{(2)} \varrho|^2}{\varrho} + \frac{1}{\varrho^2} \left(\frac{1}{2} \nabla_x \varrho \cdot \nabla_x |\nabla_x \varrho|^2 - |\nabla_x \varrho|^2 \Delta \varrho \right).$$

Here $\nabla_x^{(2)} \varrho$ denote the full second gradient $(\partial_{i,j} \varrho)_{i,j}$.

We may rewrite (23), (20) in the form of an inhomogeneous elliptic Neumann problem

$$(24) \quad -\operatorname{div}_x \left(\frac{1}{\varrho} \nabla_x \Pi \right) = \nabla_x \mathbf{v} : (\nabla_x \mathbf{v})^\top + \frac{|\Delta \varrho|^2 - |\nabla_x^{(2)} \varrho|^2}{\varrho} \\ + \frac{1}{\varrho^2} \left(\frac{1}{2} \nabla_x \varrho \cdot \nabla_x |\nabla_x \varrho|^2 - |\nabla_x \varrho|^2 \Delta \varrho \right)$$

$$(25) \quad \nabla_x \Pi \cdot \mathbf{n}|_{\partial\Omega} = \varrho \nabla_x \mathbf{n} : [\mathbf{v} \otimes \mathbf{v}]|_{\partial\Omega},$$

and we fix the pressure by prescribing the mean value as $\int_\Omega \Pi = 0$ for a.a. $t \in (0, T)$.

Note that, as a consequence of (5), the density ϱ is constant on the boundary with its value being determined by the initial data.

2.2.1. Elliptic estimates. Before we start, we introduce a family of Hilbert spaces $H^m(\Omega)$ of functions defined in Ω and having m generalized derivatives square integrable. The spaces H^m are endowed with the scalar product

$$\langle u, v \rangle_{H^m} = \int_\Omega \sum_{|\alpha| \leq m} \partial^\alpha u \partial^\alpha v \, dx.$$

The spaces of vector valued functions $H^m(\Omega; \mathbb{R}^3)$ and $H^m(\partial\Omega)$ are defined in a similar way.

In order to deduce suitable estimates for the pressure Π , we consider the elliptic problem

$$(26) \quad -\operatorname{div}_x \left(\frac{1}{\varrho} \nabla_x \Pi \right) = h \text{ in } \Omega, \quad \nabla_x \Pi \cdot \mathbf{n}|_{\partial\Omega} = g,$$

where h, g satisfy the appropriate compatibility conditions for (26) to admit a unique solution. The standard elliptic theory yields the *a priori bounds*

$$(27) \quad \|\nabla_x \Pi\|_{L^2(\Omega)} \leq c \left(\|h\|_{H^{-1}(\Omega)} + \|g\|_{H^{-1/2}(\partial\Omega)} \right).$$

It also implies the estimate

$$\|\nabla_x \Pi\|_{H^{m+1}(\Omega)} \leq c(\varrho) (\|h\|_{H^m(\Omega)} + \|g\|_{H^{m+1/2}(\partial\Omega)}),$$

for all $m \geq 0$ (see [11, Theorem 5.2]). In order to recover the dependence of c on ϱ we write

$$-\Delta \Pi = -\frac{\nabla_x \varrho}{\varrho} \nabla_x \Pi + \varrho h \text{ in } \Omega.$$

Again, by [11, Theorem 5.2] one has for all $m \geq 0$,

$$(28) \quad \begin{aligned} \|\nabla_x \Pi\|_{H^{m+1}(\Omega)} &\lesssim \|\varrho h\|_{H^m(\Omega)} + \left\| \frac{\nabla_x \varrho}{\varrho} \cdot \nabla_x \Pi \right\|_{H^m(\Omega)} \\ &\quad + \|g\|_{H^{m+1/2}(\partial\Omega)}. \end{aligned}$$

Here and throughout we use the notation $A \lesssim B$ to denote $A \leq cB$, for some absolute $c > 0$. Moreover, $H^m(\Omega)$ is an algebra for $m \geq 2$, in which case we get

$$\left\| \frac{\nabla_x \varrho}{\varrho} \cdot \nabla_x \Pi \right\|_{H^m(\Omega)} \leq c \|\varrho\|_{H^{m+1}(\Omega)}^2 \|\nabla_x \Pi\|_{H^m(\Omega)},$$

where, by interpolation,

$$\|\nabla_x \Pi\|_{H^m(\Omega)} \leq \|\nabla_x \Pi\|_{H^{m+1}(\Omega)}^{1-\lambda} \|\nabla_x \Pi\|_{L^2(\Omega)}^\lambda, \quad \lambda = \frac{1}{(m+1)}.$$

Combining the previous estimates we may infer that

$$(29) \quad \begin{aligned} \|\nabla_x \Pi\|_{H^{m+1}(\Omega)} &\lesssim \|\varrho\|_{H^m(\Omega)} \|h\|_{H^m(\Omega)} + \|g\|_{H^{m+1/2}(\partial\Omega)} \\ &\quad + \|\varrho\|_{H^{m+1}(\Omega)}^{2/\lambda} (\|h\|_{H^{-1}(\Omega)} + \|g\|_{H^{-1/2}(\partial\Omega)}) \end{aligned}$$

for any $m \geq 2$.

2.2.2. Pressure estimates. Applying (29) to solutions of problem (24) - (25) we conclude that

$$(30) \quad \begin{aligned} \|\nabla_x \Pi\|_{H^{m+1}(\Omega)} &\lesssim \|\varrho\|_{H^m(\Omega)} \left(\|\mathbf{v}\|_{H^{m+1}(\Omega; \mathbb{R}^3)}^2 + \|\varrho\|_{H^{m+2}(\Omega)}^2 + \|\varrho\|_{H^{m+2}(\Omega)}^3 \right) \\ &\quad + \|\varrho\|_{H^{m+1}(\Omega)}^{2/\lambda} \left(\|\mathbf{v}\|_{H^{m+1}(\Omega; \mathbb{R}^3)}^2 + \|\varrho\|_{H^{m+2}(\Omega)}^2 + \|\varrho\|_{H^{m+2}(\Omega)}^3 \right) \\ &\quad + \|\mathbf{v}\|_{H^{m+1}(\Omega; \mathbb{R}^3)}^2, \end{aligned}$$

for any integer $m \geq 2$. We can see that the pressure Π enjoys the same regularity as ϱ and gains one spatial derivative with respect to \mathbf{v} .

2.3. Density estimates. Let us fix a multi-index α with $|\alpha| \leq m$, and a $j \in \{1, 2, 3\}$. Applying $\partial_x^\alpha \partial_j$ (here, we denote $\partial_j := \partial_{x_j}$ for simplicity) to the continuity equation (2) we obtain

$$(31) \quad \partial_t(\partial_x^\alpha \partial_j \varrho) + \mathbf{v} \cdot \nabla_x (\partial_x^\alpha \partial_j \varrho) + \partial_x^\alpha \partial_j \mathbf{v} \cdot \nabla_x \varrho = h_\alpha^1,$$

where

$$h_\alpha^1 = \mathbf{v} \cdot \nabla_x (\partial_x^\alpha \varrho) + \partial_x^\alpha \mathbf{v} \cdot \nabla_x \varrho - \partial_x^\alpha (\mathbf{v} \cdot \nabla_x \varrho).$$

Multiplying (31) by $\partial_x^\alpha \partial_j \varrho$ yields

$$\partial_t |\partial_x^\alpha \partial_j \varrho|^2 + \mathbf{v} \cdot \nabla_x |\partial_x^\alpha \partial_j \varrho|^2 + 2\partial_x^\alpha \partial_j \mathbf{v} \cdot \nabla_x \varrho \partial_x^\alpha \partial_j \varrho = 2\partial_x^\alpha \partial_j \varrho h_\alpha^1,$$

which in turn implies

$$(32) \quad \partial_t \left(\frac{1}{\varrho} |\partial_x^\alpha \partial_j \varrho|^2 \right) + \operatorname{div}_x \left(\frac{1}{\varrho} |\partial_x^\alpha \partial_j \varrho|^2 \mathbf{v} \right) = -\frac{2}{\varrho} \partial_x^\alpha \partial_j \mathbf{v} \cdot \nabla_x \varrho \partial_x^\alpha \partial_j \varrho + \frac{2}{\varrho} \partial_x^\alpha \varrho h_\alpha^1.$$

Notice that the highest derivative of \mathbf{v} contained in h_α^1 is of order m , while the highest derivative of ϱ is of order $m + 1$. So, we have the bound

$$(33) \quad \int_\Omega \left| \frac{2}{\varrho} \partial_x^\alpha \partial_j \varrho h_\alpha^1 \right| dx \leq c \|\varrho\|_{H^{m+1}(\Omega)}^2 \|\mathbf{v}\|_{H^m(\Omega; \mathbb{R}^3)}.$$

2.4. Velocity estimates. Similarly to the preceding part, we differentiate momentum equation (22) α -times to deduce

$$(34) \quad \partial_t(\partial_x^\alpha \mathbf{v}) + (\mathbf{v} \cdot \nabla_x)(\partial_x^\alpha \mathbf{v}) = -\partial_x^\alpha \left(\frac{1}{\varrho} \nabla_x \Pi \right)$$

$$(35) \quad + \partial_x^\alpha \left(\frac{1}{\varrho} \nabla_x |\nabla_x \varrho|^2 \right)$$

$$(36) \quad - \frac{1}{\varrho} \operatorname{div}_x (\nabla_x \varrho \otimes \partial_x^\alpha \nabla_x \varrho)$$

$$(37) \quad - \frac{1}{\varrho} \operatorname{div}_x (\partial_x^\alpha \nabla_x \varrho \otimes \nabla_x \varrho)$$

$$(38) \quad + \mathbf{h}_\alpha^2,$$

where \mathbf{h}_α^2 contains at most $m + 1$ derivatives of ϱ and m derivatives of \mathbf{v} . Notice that terms (35) and (37) add up to the following

$$\begin{aligned}
& \partial_x^\alpha \left(\frac{1}{\varrho} \nabla_x |\nabla_x \varrho|^2 \right) - \frac{1}{\varrho} \operatorname{div}_x (\partial_x^\alpha \nabla_x \varrho \otimes \nabla_x \varrho) \\
&= \frac{2}{\varrho} \nabla_x (\nabla_x \varrho \cdot \nabla_x \partial^\alpha \varrho) - \frac{1}{\varrho} (\nabla_x \varrho \cdot \nabla_x \nabla_x \partial^\alpha \varrho) + \mathbf{h}_\alpha^3 \\
&= \frac{2}{\varrho} \nabla_x (\nabla_x \varrho \cdot \nabla_x \partial^\alpha \varrho) - \frac{1}{\varrho} \nabla_x (\nabla_x \varrho \cdot \nabla_x \partial^\alpha \varrho) + \mathbf{h}_\alpha^4 \\
&= \frac{1}{\varrho} \nabla_x (\nabla_x \varrho \cdot \nabla_x \partial^\alpha \varrho) + \mathbf{h}_\alpha^4.
\end{aligned}$$

We thus can rewrite (34) through (38) as follows

$$\begin{aligned}
(39) \quad \partial_t (\partial_x^\alpha \mathbf{v}) + (\mathbf{v} \cdot \nabla_x) (\partial_x^\alpha \mathbf{v}) &= -\partial_x^\alpha \left(\frac{1}{\varrho} \nabla_x \Pi \right) \\
&+ \frac{1}{\varrho} \nabla_x (\nabla_x \varrho \cdot \nabla_x \partial^\alpha \varrho) \\
&- \frac{1}{\varrho} \operatorname{div}_x (\nabla_x \varrho \otimes \partial_x^\alpha \nabla_x \varrho) \\
&+ \mathbf{h}_\alpha^5.
\end{aligned}$$

Taking the scalar product of (39) with $\partial_x^\alpha \mathbf{v}$ we find

$$(40) \quad \partial_t |\partial_x^\alpha \mathbf{v}|^2 + \operatorname{div}_x (|\partial_x^\alpha \mathbf{v}|^2 \mathbf{v}) = -2\partial_x^\alpha \left(\frac{1}{\varrho} \nabla_x \Pi \right) \cdot (\partial_x^\alpha \mathbf{v})$$

$$(41) \quad + 2\frac{1}{\varrho} \nabla_x (\nabla_x \varrho \cdot \nabla_x \partial^\alpha \varrho) \cdot (\partial_x^\alpha \mathbf{v})$$

$$(42) \quad - \frac{2}{\varrho} \operatorname{div}_x (\nabla_x \varrho \otimes \partial_x^\alpha \nabla_x \varrho) \cdot (\partial_x^\alpha \mathbf{v})$$

$$(43) \quad + 2\mathbf{h}_\alpha^5 \cdot (\partial_x^\alpha \mathbf{v}).$$

We have the desired estimate for the remainder term

$$(44) \quad \int_\Omega |\mathbf{h}_\alpha^5 \cdot (\partial_x^\alpha \mathbf{v})| \leq c \|\mathbf{v}\|_{H^m(\Omega; \mathbb{R}^3)}^2 + c \|\varrho\|_{H^{m+1}(\Omega)}^2.$$

Similar estimates for other terms can only be obtained in combination with the density equation.

2.5. Velocity and density equations - synthesis. We now add the two balance equations (32) and (40) - (43) obtained above, sum up over $|\alpha| \leq m$, and $j = 1, 2, 3$, and integrate over the fluid domain Ω . Our observations will show that in this process the terms with the $m + 2$ derivative of ϱ and

$m + 1$ derivative of \mathbf{v} cancel out. First let us compare the term on the right-hand side of (32) with the term in display (42). Using the repeated index summation convention the former can be written as

$$(45) \quad - \int_{\Omega} \frac{2}{\varrho} \partial_x^\alpha \partial_j \mathbf{v} \cdot \nabla_x \varrho \partial_x^\alpha \partial_j \varrho \, dx = - \int_{\Omega} \frac{2}{\varrho} \partial_x^\alpha \partial_j v_i \partial_i \varrho \partial_x^\alpha \partial_j \varrho \, dx.$$

For the latter we have, after integration by parts and using the boundary conditions (4), (5) to cancel the boundary terms,

$$(46) \quad \begin{aligned} & \int_{\Omega} \frac{2}{\varrho} (\partial_x^\alpha \nabla_x \varrho \cdot \nabla_x \partial_x^\alpha \mathbf{v}) \cdot \nabla_x \varrho \, dx + \int_{\Omega} h_\alpha^6 \, dx \\ &= \int_{\Omega} \frac{2}{\varrho} \partial_x^\alpha \partial_j \varrho \partial_j \partial_x^\alpha v_i \partial_i \varrho \, dx + \int_{\Omega} h_\alpha^6 \, dx. \end{aligned}$$

We can see now that (45) and (46) add up to the remainder term, which again contains only at most $m + 1$ derivatives of ϱ and m derivatives of \mathbf{v} .

Next, the term on the right-hand side of (40) containing the pressure Π will be estimated by means of (30). As to (41) we can integrate by parts using the divergence-free condition on \mathbf{v} and the boundary condition (5). We obtain

$$\int_{\Omega} \frac{1}{\varrho} \nabla_x (\nabla_x \varrho \cdot \nabla_x \partial_x^\alpha \varrho) \cdot (\partial_x^\alpha \mathbf{v}) \, dx = \int_{\Omega} \frac{1}{\varrho^2} (\nabla_x \varrho \cdot \partial_x^\alpha \mathbf{v}) (\nabla_x \varrho \cdot \nabla_x \partial_x^\alpha \varrho) \, dx,$$

which is bounded by $\|\mathbf{v}\|_{H^m} \|\varrho\|_{H^{m+1}}^3$.

In view of these arguments, we may therefore conclude that

$$(47) \quad \begin{aligned} & \frac{d}{dt} \left(\|\varrho\|_{H^{m+1}(\Omega)}^2 + \|\mathbf{v}\|_{H^m(\Omega; \mathbb{R}^3)}^2 \right) \lesssim \\ & \lesssim 1 + \left(\|\varrho\|_{H^{m+1}(\Omega)}^2 + \|\mathbf{v}\|_{H^m(\Omega; \mathbb{R}^3)}^2 \right)^{N(m)}, \end{aligned}$$

for any $m \geq 3$. Estimate (47), together with a suitable choice of approximate problem, yields existence of (classical) solutions to problem (1) - (6) defined on a (possibly) short time interval $(0, T)$, where the specific value of T can be determined in terms of the structural constants appearing in (47).

3. APPROXIMATE PROBLEMS AND EXISTENCE OF LOCAL-IN-TIME SOLUTIONS

Approximate solutions will be constructed by means of the Galerkin method. More specifically, we consider the family of finite-dimensional spaces

$$Y_N = \text{span}\{\mathbf{w}_i \mid i = 1, \dots, N\}$$

where $\{\mathbf{w}_i\}_{i=1}^\infty \subset X^m$ is a complete system of eigenvectors to the eigenvalue problem

$$\int_{\Omega} \sum_{|\alpha| \leq m} \partial^\alpha \mathbf{w} \cdot \partial^\alpha \mathbf{v} \, dx = \lambda \int_{\Omega} \mathbf{w} \cdot \mathbf{v} \, dx \text{ for any } \mathbf{v} \in X^m,$$

where

$$X^m = \{\mathbf{v} \in H^m(\Omega; \mathbb{R}^3) \mid \operatorname{div}_x \mathbf{v} = 0, \mathbf{v} \cdot \mathbf{n}|_{\partial\Omega} = 0\}.$$

The same system of approximate spaces was used by Temam [14] (see also [15]). As shown by Ghidaglia [8], the eigenvectors \mathbf{w}_i belong to the space $H^{2m}(\Omega; \mathbb{R}^3) \subset H^{m+1}(\Omega; \mathbb{R}^3)$, $m \geq 1$.

Given $N \geq 1$, we look for approximate solutions in the form

$$\mathbf{v}_N = \sum_{i=1}^N a_i \mathbf{w}_i$$

such that ϱ satisfies the equation of continuity

$$(48) \quad \partial_t \varrho_N + \mathbf{v}_N \cdot \nabla_x \varrho_N = 0 \text{ in } (0, T) \times \Omega,$$

while \mathbf{v}_N solves a system of ordinary differential equations

$$(49) \quad \begin{aligned} & \int_{\Omega} \left(\partial_t \mathbf{v}_N + \operatorname{div}_x (\mathbf{v}_N \otimes \mathbf{v}_N) \right) \cdot \mathbf{w}_i \, dx \\ &= - \int_{\Omega} \frac{1}{\varrho_N} \operatorname{div}_x (\nabla_x \varrho_N \otimes \nabla_x \varrho_N) \cdot \mathbf{w}_i \, dx \\ & - \int_{\Omega} \frac{1}{\varrho_N} \nabla_x \Pi_N \cdot \mathbf{w}_i \, dx \\ & + \int_{\Omega} \frac{1}{\varrho_N} \nabla_x |\nabla_x \varrho_N|^2 \cdot \mathbf{w}_i \, dx, \end{aligned}$$

for $i = 1, \dots, N$. Here, the pressure Π_N satisfies

$$(50) \quad \begin{aligned} -\operatorname{div}_x \left(\frac{1}{\varrho_N} \nabla_x \Pi_N \right) &= \nabla_x \mathbf{v}_N : (\nabla_x \mathbf{v}_N)^\top \\ & + \frac{|\Delta \varrho_N|^2 - |\nabla_x^{(2)} \varrho_N|^2}{\varrho_N} \\ & + \frac{1}{\varrho_N^2} \left(\frac{1}{2} \nabla_x \varrho_N \cdot \nabla_x |\nabla_x \varrho_N|^2 - |\nabla_x \varrho_N|^2 \Delta \varrho_N \right), \end{aligned}$$

Problem (48) - (50) is supplemented with the boundary conditions

$$(51) \quad \nabla_x \varrho_N|_{\partial\Omega} = 0,$$

$$(52) \quad \nabla_x \Pi_N \cdot \mathbf{n}|_{\partial\Omega} = \varrho_N \nabla_x \mathbf{n} : [\mathbf{v}_N \otimes \mathbf{v}_N]|_{\partial\Omega},$$

the initial conditions

$$(53) \quad \varrho_N(0, \cdot) = \varrho_0,$$

$$(54) \quad \int_{\Omega} \mathbf{v}_N(0, \cdot) \cdot \mathbf{w}_i \, dx = \int_{\Omega} \mathbf{v}_0 \cdot \mathbf{w}_i \, dx, \quad i = 1, \dots, N,$$

and with zero mean value condition on Π_N , i.e.,

$$(55) \quad \int_{\Omega} \Pi_N = 0, \quad \text{for a.a. } t \in (0, T).$$

The existence of local-in-time solutions to problem (48) - (55) defined on a time interval $(0, T_N)$ can be shown by means of the standard Schauder fixed point argument. Now, the crucial observation is that the *a priori* bounds established in Section 2 apply to the approximate solutions ϱ , \mathbf{v}_N as a direct consequence of the choice of \mathbf{w}_i .

With the uniform bounds stated in Section 2 (that are valid provided that $\Omega \in \mathcal{C}^{m+2}$), it is easy to pass to the limit for $N \rightarrow \infty$ to deduce existence of local-in-time solutions to problem (1) - (6) belonging to the class

$$\varrho \in C([0, T]; H^{m+1}(\Omega)), \quad \mathbf{v} \in C([0, T]; X^m), \quad m \geq 4.$$

In view of the standard Sobolev embedding relations, the solutions ϱ , \mathbf{v} are classical. Moreover, it is easy to show that they are unique as soon as $m \geq 4$. Thus we have shown the following result.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^3$ be an open bounded domain with boundary $\partial\Omega \in \mathcal{C}^{m+2}$. Suppose that*

$$\varrho_0 \in H^{m+1}(\Omega), \quad 0 < m_0 \leq \varrho_0(x) \leq M_0 < \infty, \quad \nabla_x \varrho_0|_{\partial\Omega} = 0,$$

$$\mathbf{v}_0 \in H^m(\Omega; \mathbb{R}^3), \quad \operatorname{div}_x \mathbf{v}_0 = 0, \quad \mathbf{v}_0 \cdot \mathbf{n}|_{\partial\Omega} = 0,$$

for some integer $m \geq 3$.

Then there exists time $T > 0$ such that problem (1) - (6) possesses a unique solution in the class

$$\varrho \in C([0, T]; H^{m+1}(\Omega)), \quad \mathbf{v} \in C([0, T]; H^m(\Omega; \mathbb{R}^3)).$$

4. LOCAL WELL-POSEDNESS IN FRACTIONAL SMOOTHNESS SPACES

In this section we develop the Littlewood-Paley approach to well-posedness of system (1) - (6). We pursue the analogue of Theorem 3.1 in the simpler open space or periodic settings but for Sobolev spaces of fractional smoothness s down to the critical value $s_0 = \frac{n}{2} + 1$, where n is the spacial dimension of the fluid. The same smoothness $s = \frac{n}{2} + 1$ is known to be critical for the homogeneous fluids as well. The ill-posedness results found in the Besov space $B_{2,\infty}^{\frac{n}{2}+1}$, [6], and in the supercritical Hölder classes [1] apply in our case too to suggest that there might be an initial condition in $H^{\frac{n}{2}+1}$ for

which there is no continuous trajectory in $H^{\frac{n}{2}+1}$. This however remains an open question.

We will rewrite the system (2) - (3) as follows

$$(56) \quad \varrho \partial_t \mathbf{v} + \varrho \mathbf{v} \cdot \nabla_x \mathbf{v} = -\nabla_x \Pi - \operatorname{div}_x (\boldsymbol{\xi} \otimes \boldsymbol{\xi})$$

$$(57) \quad \boldsymbol{\xi}_t + \mathbf{v} \cdot \nabla_x \boldsymbol{\xi} = -(\nabla_x \mathbf{v})^\top \boldsymbol{\xi},$$

where $\boldsymbol{\xi} = \nabla_x \varrho$ and $\Pi = p + \frac{1}{3} |\nabla_x \varrho|^2$. We will work on the fluid domain Ω which is either the open space \mathbb{R}^n or the torus \mathbb{T}^n . Since the case of \mathbb{R}^n requires extra care due to the presence of infinitesimally small frequencies, we will from now on assume that $\Omega = \mathbb{R}^n$, and $n \geq 2$. Our basic assumptions on the initial conditions remain the same:

$$(58) \quad \operatorname{div} \mathbf{v}_0 = 0, \quad 0 < m_0 \leq \varrho_0(x) \leq M_0 < \infty, \quad \text{for all } x \in \mathbb{R}^n.$$

The uniformity condition on the density is clearly inconsistent with the requirement $\varrho_0 \in H^{s+1}(\mathbb{R}^n)$. The proper analogue of the regularity condition on ϱ will thus become $\nabla_x \varrho \in H^s(\mathbb{R}^n)$.

Theorem 4.1. *Suppose \mathbf{v}_0 and ϱ_0 satisfy (58) and*

$$\mathbf{v}_0, \nabla_x \varrho_0 \in H^s, \quad \text{for some } s > \frac{n}{2} + 1.$$

Then there exists time $T > 0$ such that the initial value problem (56) - (58) has a unique solution on the interval $[0, T)$ satisfying

$$\mathbf{v}, \nabla_x \varrho \in C([0, T); H^s).$$

The associated pressure can be restored as before from the elliptic equation

$$(59) \quad -\operatorname{div}_x (\varrho^{-1} \nabla_x \Pi) = \operatorname{div}_x \operatorname{div}_x (\mathbf{v} \otimes \mathbf{v}) + \operatorname{div}_x (\varrho^{-1} \operatorname{div}_x (\boldsymbol{\xi} \otimes \boldsymbol{\xi})).$$

Unlike in the case of a bounded domain, our estimates for \mathbb{R}^n below will tolerate the loss of one derivative of Π over ϱ . Therefore, by the same argument as in Section 2.2.1 along with [11, Theorem 5.4] for elliptic estimates for fractional smoothness we obtain

$$(60) \quad \|\varrho^{-1} \nabla_x \Pi\|_{H^{s-1}} \leq C \langle \|\mathbf{v}\|_{H^s}^2 + \|\boldsymbol{\xi}\|_{H^s}^2 \rangle^N.$$

The proof of Theorem 4.1 is based upon obtaining a priori estimates on the energy inside each individual dyadic shell in the Fourier space. The closing argument however uses the Galerkin method presented in the previous sections, so we will not repeat it here. Thus, our aim is to find estimate of the form

$$(61) \quad \frac{d}{dt} (\|\mathbf{v}\|_{H^s}^2 + \|\boldsymbol{\xi}\|_{H^s}^2) \leq C \langle \|\mathbf{v}\|_{H^s}^2 + \|\boldsymbol{\xi}\|_{H^s}^2 \rangle^N,$$

for some $N \in \mathbb{N}$, assuming the solution is smooth.

4.1. Product estimates. In this section we introduce basic elements of the Littlewood-Paley theory and obtain product estimates that are at the core of the proof of Theorem 4.1. We will fix the notation for scales $\lambda_q = 2^q$ in some inverse length units. Let us fix a nonnegative radial function $\chi \in C_0^\infty(\mathbb{R}^n)$ such that $\chi(\xi) = 1$ for $|\xi| \leq 1/2$, and $\chi(\xi) = 0$ for $|\xi| \geq 1$. We define $\varphi(\xi) = \chi(\lambda_1^{-1}\xi) - \chi(\xi)$, and $\varphi_q(\xi) = \varphi(\lambda_q^{-1}\xi)$ for $q \geq 0$, and $\varphi_{-1} = \chi$. For a tempered distribution u on \mathbb{R}^n we consider the Littlewood-Paley projections

$$(62) \quad u_q(x) = \int_{\mathbb{R}^n} \mathcal{F}[u](\xi) \varphi_q(\xi) e^{i\xi \cdot x} d\xi, \quad q \geq -1,$$

where $\mathcal{F}[u]$ denotes the Fourier transformation of u . So, we have $u = \sum_{q=-1}^{\infty} u_q$ in the sense of distributions. We also use the following notation $u_{\leq q} = \sum_{p=-1}^q u_p$ and $u_{> q} = \sum_{p>q} u_p$. Let us recall the following classical inequalities:

$$(63) \quad \|f\|_{L^p} \leq \|f\|_{H^s}, \quad s \geq n \left(\frac{1}{2} - \frac{1}{p} \right), \quad p \geq 2, \text{ (Sobolev),}$$

$$(64) \quad \|\nabla|^s f_q\|_{L^r} \sim \lambda_q^s \|f_q\|_{L^r}, \text{ (Bernstein),}$$

$$(65) \quad \|\nabla^k f\|_{L^{\frac{2m}{k}}} \leq \|f\|_{L^\infty}^{1-\frac{k}{m}} \|\nabla^m f\|_{L^2}^{\frac{k}{m}}, \quad k \leq m, \text{ (Gagliardo-Nirenberg).}$$

In what follows we need the following detailed product estimate, which follows from the standard Littlewood-Paley estimates. We present the argument in the Appendix for the convenience of the reader.

Lemma 4.2. *The following product estimate holds for all $s > 0$, and $q \geq 0$:*

$$(66) \quad \|fg\|_{H^s} \leq \|(fg)_{\leq q}\|_{H^s} + \|f\|_{L^\infty} \|\nabla|^s g_{>q-2}\|_{L^2} \\ + \|g\|_{L^\infty} \|\nabla|^s f_{>q-2}\|_{L^2}.$$

Throughout we assume that $0 < m_0 \leq \varrho(x) \leq M_0 < \infty$ holds for all $x \in \mathbb{R}^n$. We denote $\nabla_x \varrho = \boldsymbol{\xi}$. We will use the traditional Japanese brackets $\langle x \rangle = (1 + x^2)^{1/2}$. As the density often enters estimates as a multiplier we prove the following auxiliary lemma.

Lemma 4.3. *Let $F \in C^\infty([m_0, M_0])$ and $f \in H^s$, $s \geq 0$. Then the following inequality holds:*

$$(67) \quad \|F(\varrho)f\|_{H^s} \leq C \langle \|\boldsymbol{\xi}\|_{H^s} \rangle \|f\|_{H^s},$$

where C depends only on m_0, M_0 and F .

Proof. Let us assume $s > 0$, otherwise the statement is trivial. Using (66) with $q = 0$ we obtain

$$\|F(\varrho)f\|_{H^s} \leq \|F(\varrho)f\|_{L^2} + \|f\|_{L^\infty} \|\nabla|^s F(\varrho)_{>0}\|_{L^2} + \|F(\varrho)\|_{L^\infty} \|\nabla|^s f_{>0}\|_{L^2}.$$

This implies

$$\|F(\varrho)f\|_{H^s} \leq C(1 + \|\nabla|^s F(\varrho)_{>0}\|_{L^2})\|f\|_{H^s}.$$

So, it remains to estimate $\|\nabla|^s F(\varrho)_{>0}\|_{L^2}$. Let $d = [s]$. We have

$$\|\nabla|^s F(\varrho)_{>0}\|_{L^2} \leq \|\nabla|^{d+1} F(\varrho)_{>0}\|_{L^2} \leq \|\nabla|^{d+1} F(\varrho)\|_{L^2} \sim \|\nabla_x^{d+1} F(\varrho)\|_{L^2}.$$

For any multi-index β of order $d + 1$ we have

$$|\partial_x^\beta F(\varrho)| \leq C(F) \sum_{\sum_{i=1}^{d+1} |\beta_i| = d+1} \prod_{i=1}^{d+1} |\partial_x^{\beta_i} \varrho|^{l_i}.$$

Consequently, using Hölder and Gagliardo-Nirenberg inequalities we obtain

$$\begin{aligned} \|\nabla_x^{d+1} F(\varrho)\|_{L^2} &\leq C \sum_{\sum_{i=1}^{d+1} |\beta_i| = d+1} \prod_{i=1}^{d+1} \|\partial_x^{\beta_i} \varrho\|_{L^{\frac{2(d+1)}{|\beta_i|}}}^{l_i} \\ &\leq C \sum_{\sum_{i=1}^{d+1} |\beta_i| = d+1} \prod_{i=1}^{d+1} \|\varrho\|_{L^\infty}^{l_i \left(1 - \frac{|\beta_i|}{d+1}\right)} \|\nabla_x^{d+1} \varrho\|_{L^2}^{\frac{l_i |\beta_i|}{d+1}} \\ &\leq C(F, m_0, M_0) \|\nabla_x^{d+1} \varrho\|_{L^2} \sim \|\nabla_x^d \boldsymbol{\xi}\|_{L^2} \leq C \|\boldsymbol{\xi}\|_{H^s}. \end{aligned}$$

The proof is over. \square

Let a, b, c be distributions. Let us denote $\delta_y a(x) = a(x - y) - a(x)$. We have for $q \geq 0$

$$(ab)_q = a_q b + ab_q + r_q(a, b),$$

where

$$r_q(a, b)(x) = \int \mathcal{F}^{-1}[\varphi_q](y) \delta_y a(x) \delta_y b(x) dy.$$

Lemma 4.4. *For all $q \geq 0$ one has the following estimate*

$$\begin{aligned} \left| \int_{\mathbb{R}^n} r_q(a, b) c_q \right| &\leq \frac{1}{\lambda_q} \|c_q\|_{L^2} \|\nabla a_{<q-2}\|_{L^\infty} \|b_{\geq q-2}\|_{L^2} \\ &\quad + \frac{1}{\lambda_q} \|c_q\|_{L^2} \|\nabla b_{<q-2}\|_{L^\infty} \|a_{\geq q-2}\|_{L^2} \\ &\quad + \lambda_q^{\frac{n}{2}} \|c_q\|_{L^2} \|a_{\geq q-2}\|_{L^2} \|b_{\geq q-2}\|_{L^2}. \end{aligned}$$

Proof. Notice that

$$\int_{\mathbb{R}^n} \delta_y a_{<q-2}(x) \delta_y b_{<q-2}(x) c_q(x) dx = 0.$$

So, we have

$$\begin{aligned} \int_{\mathbb{R}^n} r_q(a, b) c_q &= \int_{\mathbb{R}^n} r_q(a_{<q-2}, b_{\geq q-2}) c_q + \int_{\mathbb{R}^n} r_q(a_{\geq q-2}, b_{<q-2}) c_q \\ &\quad + \int_{\mathbb{R}^n} r_q(a_{\geq q-2}, b_{\geq q-2}) c_q. \end{aligned}$$

We estimate

$$\begin{aligned} \left| \int_{\mathbb{R}^n} r_q(a_{<q-2}, b_{\geq q-2}) c_q \right| &\leq \|c_q\|_{L^2} \int_{\mathbb{R}^n} |\mathcal{F}^{-1}[\varphi_q](y)| \|\delta_y a_{<q-2}\|_{L^\infty} \|\delta_y b_{\geq q-2}\|_{L^2} dy \\ &\leq \|c_q\|_{L^2} \|\nabla a_{<q-2}\|_{L^\infty} \|b_{\geq q-2}\|_{L^2} \int_{\mathbb{R}^n} |\mathcal{F}^{-1}[\varphi_q](y)| |y| dy \\ &\leq \frac{1}{\lambda_q} \|c_q\|_{L^2} \|\nabla a_{<q-2}\|_{L^\infty} \|b_{\geq q-2}\|_{L^2}. \end{aligned}$$

Similarly,

$$\left| \int_{\mathbb{R}^n} r_q(a_{\geq q-2}, b_{<q-2}) c_q \right| \leq \frac{1}{\lambda_q} \|c_q\|_{L^2} \|a_{\geq q-2}\|_{L^2} \|\nabla b_{<q-2}\|_{L^\infty},$$

and

$$\begin{aligned} \left| \int_{\mathbb{R}^n} r_q(a_{\geq q-2}, b_{\geq q-2}) c_q \right| &\leq \|c_q\|_{L^\infty} \|a_{\geq q-2}\|_{L^2} \|b_{\geq q-2}\|_{L^2} \\ &\leq \lambda_q^{\frac{n}{2}} \|c_q\|_{L^2} \|a_{\geq q-2}\|_{L^2} \|b_{\geq q-2}\|_{L^2}. \end{aligned}$$

□

Lemma 4.5. *For any $s > \frac{n}{2} + 1$ one has the following estimates*

$$(68) \quad \sum_{q \geq 0} \lambda_q^{2s} \int_{\mathbb{R}^n} |r_q(a, b) \cdot \nabla c_q| dx \leq \|a\|_{H^s} \|b\|_{H^s} \|c\|_{H^s},$$

$$(69) \quad \sum_{q \geq 0} \lambda_q^{2s} \int_{\mathbb{R}^n} |r_q(a, b) \cdot c_q| dx \leq \|\nabla a\|_{H^{s-1}} \|b\|_{H^{s-1}} \|c\|_{H^s}.$$

Proof. Let us establish (68) first. According to Lemma 4.4 we have to estimate three terms. Since the first two are symmetric we only focus on the

first and third ones. We have

$$\begin{aligned}
& \sum_{q \geq 0} \lambda_q^{2s} \lambda_q^{-1} \|\nabla c_q\|_{L^2} \|\nabla a_{<q-2}\|_{L^\infty} \|b_{\geq q-2}\|_{L^2} \\
& \leq \|a\|_{H^{\frac{n}{2}+1}} \left(\sum_q \lambda_q^{2s} \|c_q\|_{L^2}^2 \right)^{1/2} \left(\sum_q \lambda_q^{2s} \|b_{\geq q-2}\|_{L^2}^2 \right)^{1/2} \\
& \leq \|a\|_{H^s} \|c\|_{H^s} \left(\sum_q \lambda_q^{2s} \sum_{p \geq q-2} \|b_p\|_{L^2}^2 \right)^{1/2} \\
& \leq \|a\|_{H^s} \|c\|_{H^s} \left(\sum_q \sum_{p \geq q-2} \lambda_{q-p}^{2s} \lambda_p^{2s} \|b_p\|_{L^2}^2 \right)^{1/2} \leq \|a\|_{H^s} \|b\|_{H^s} \|c\|_{H^s}.
\end{aligned}$$

As to the third term we obtain

$$\begin{aligned}
& \sum_{q \geq 0} \lambda_q^{2s} \lambda_q^{n/2} \|\nabla c_q\|_{L^2} \|a_{\geq q-2}\|_{L^2} \|b_{\geq q-2}\|_{L^2} \\
& \leq \sup_q \lambda_q^{n/2+1} \|c_q\|_{L^2} \left(\sum_q \lambda_q^{2s} \|a_{\geq q-2}\|_{L^2}^2 \right)^{1/2} \left(\sum_q \lambda_q^{2s} \|b_{\geq q-2}\|_{L^2}^2 \right)^{1/2} \\
& \leq \|c\|_{H^s} \|a\|_{H^s} \|b\|_{H^s}.
\end{aligned}$$

In order to show (69) we rearrange powers of λ_q differently. We have

$$\begin{aligned}
\lambda_q^{2s} \int_{\mathbb{R}^n} |r_q(a, b) \cdot c_q| & \leq \lambda_q^s \|c_q\|_{L^2} \|\nabla a_{<q-2}\|_{L^\infty} \lambda_q^{s-1} \|b_{\geq q-2}\|_{L^2} \\
& \quad + \lambda_q^s \|c_q\|_{L^2} \lambda_q^s \|a_{\geq q-2}\|_{L^2} \lambda_q^{-1} \|\nabla b_{<q-2}\|_{L^\infty} \\
& \quad + \lambda_q^s \|c_q\|_{L^2} \lambda_q^s \|a_{\geq q-2}\|_{L^2} \lambda_q^{n/2} \|b_{\geq q-2}\|_{L^2} \\
& \leq \lambda_q^s \|c_q\|_{L^2} \lambda_q^{s-1} \|b_{\geq q-2}\|_{L^2} \|\nabla a\|_{H^{s-1}} \\
& \quad + \lambda_q^s \|c_q\|_{L^2} \lambda_q^{s-1} \|\nabla a_{\geq q-2}\|_{L^2} \|b\|_{H^{s-1}} \\
& \quad + \lambda_q^s \|c_q\|_{L^2} \lambda_q^{s-1} \|\nabla a_{\geq q-2}\|_{L^2} \lambda_q^{s-1} \|b_{\geq q-2}\|_{L^2}.
\end{aligned}$$

Proceeding with summation as before we obtain (69). \square

4.2. Proof of (61). Let us notice that it suffices to establish the bound only for the high frequencies,

$$(70) \quad \frac{d}{dt} (\|\mathbf{v}_{>0}\|_{H^s}^2 + \|\boldsymbol{\xi}_{>0}\|_{H^s}^2) \leq C \langle \|\mathbf{v}\|_{H^s}^2 + \|\boldsymbol{\xi}\|_{H^s}^2 \rangle^N,$$

as the low frequency a priori bound follows readily from the energy conservation (17). So, in order to prove (70) we test the momentum equation

against $(\mathbf{v}_q)_q$, and (57) against $(\boldsymbol{\xi}_q)_q$, for $q \geq 0$. For the former we obtain (dropping integral signs for short)

$$(71) \quad \begin{aligned} & \varrho_q \mathbf{v}_t \cdot \mathbf{v}_q + \frac{1}{2} \varrho (|\mathbf{v}_q|^2)_t + r_q(\varrho, \mathbf{v}_t) \cdot \mathbf{v}_q \\ & + \varrho_q (\mathbf{v} \cdot \nabla_x \mathbf{v}) \cdot \mathbf{v}_q + \varrho (\mathbf{v} \cdot \nabla_x \mathbf{v})_q \cdot \mathbf{v}_q + r_q(\varrho, \mathbf{v} \cdot \nabla_x \mathbf{v}) \cdot \mathbf{v}_q \\ & = \boldsymbol{\xi}_q \otimes \boldsymbol{\xi} : \nabla_x \mathbf{v}_q + \boldsymbol{\xi} \otimes \boldsymbol{\xi}_q : \nabla_x \mathbf{v}_q + r_q(\boldsymbol{\xi}, \boldsymbol{\xi}) : \nabla_x \mathbf{v}_q. \end{aligned}$$

Notice the following identity

$$\varrho (\mathbf{v} \cdot \nabla_x \mathbf{v})_q \cdot \mathbf{v}_q = \varrho \mathbf{v}_q \cdot \nabla_x \mathbf{v} \cdot \mathbf{v}_q + \frac{1}{2} \varrho \mathbf{v} \cdot \nabla_x |\mathbf{v}_q|^2 + \varrho r_q(\mathbf{v}, \nabla_x \mathbf{v}) \cdot \mathbf{v}_q,$$

while

$$\varrho \mathbf{v} \cdot \nabla_x |\mathbf{v}_q|^2 = -|\mathbf{v}_q|^2 \mathbf{v} \cdot \nabla_x \varrho = |\mathbf{v}_q|^2 \varrho_t.$$

On the other hand, using the original equation (56) we write

$$\begin{aligned} \varrho_q \mathbf{v}_t \cdot \mathbf{v}_q + \varrho_q (\mathbf{v} \cdot \nabla_x \mathbf{v}) \cdot \mathbf{v}_q &= -\varrho_q \varrho^{-1} \nabla_x \Pi \cdot \mathbf{v}_q \\ &\quad - \varrho_q \varrho^{-1} \operatorname{div}_x (\boldsymbol{\xi} \otimes \boldsymbol{\xi}) \cdot \mathbf{v}_q, \end{aligned}$$

and similarly,

$$\begin{aligned} r_q(\varrho, \mathbf{v}_t) \cdot \mathbf{v}_q + r_q(\varrho, \mathbf{v} \cdot \nabla_x \mathbf{v}) \cdot \mathbf{v}_q &= -r_q(\varrho, \varrho^{-1} \nabla_x \Pi) \cdot \mathbf{v}_q \\ &\quad - r_q(\varrho, \varrho^{-1} \operatorname{div}_x (\boldsymbol{\xi} \otimes \boldsymbol{\xi})) \cdot \mathbf{v}_q. \end{aligned}$$

Plugging the obtained identities back into (71) we obtain

$$(72) \quad \frac{d}{dt} \left(\frac{1}{2} \varrho |\mathbf{v}_q|^2 \right) = A_q + B_q,$$

where

$$\begin{aligned} A_q &= \varrho_q \varrho^{-1} \nabla_x \Pi \cdot \mathbf{v}_q + \varrho_q \varrho^{-1} \operatorname{div}_x (\boldsymbol{\xi} \otimes \boldsymbol{\xi}) \cdot \mathbf{v}_q - \varrho \mathbf{v}_q \cdot \nabla_x \mathbf{v} \cdot \mathbf{v}_q \\ &\quad + \boldsymbol{\xi} \cdot \nabla_x \mathbf{v}_q \cdot \boldsymbol{\xi}_q + \boldsymbol{\xi}_q \cdot \nabla_x \mathbf{v}_q \cdot \boldsymbol{\xi}, \end{aligned}$$

and the remainder terms are

$$\begin{aligned} B_q &= r_q(\varrho, \varrho^{-1} \nabla_x \Pi) \cdot \mathbf{v}_q + r_q(\varrho, \varrho^{-1} \operatorname{div}_x (\boldsymbol{\xi} \otimes \boldsymbol{\xi})) \cdot \mathbf{v}_q \\ &\quad - \varrho r_q(\mathbf{v}, \nabla_x \mathbf{v}) \cdot \mathbf{v}_q + r_q(\boldsymbol{\xi}, \boldsymbol{\xi}) : \nabla_x \mathbf{v}_q. \end{aligned}$$

From the continuity equation (57) we similarly obtain

$$(73) \quad \begin{aligned} \frac{d}{dt} \left(\frac{1}{2} |\boldsymbol{\xi}_q|^2 \right) &= -\boldsymbol{\xi}_q \cdot \nabla_x \mathbf{v}_q \cdot \boldsymbol{\xi} - \mathbf{v}_q \cdot \nabla_x \boldsymbol{\xi} \cdot \boldsymbol{\xi}_q - \boldsymbol{\xi}_q \cdot \nabla_x \mathbf{v} \cdot \boldsymbol{\xi}_q \\ &\quad + r_q(\mathbf{v}, \boldsymbol{\xi}) \operatorname{div}_x \boldsymbol{\xi}_q. \end{aligned}$$

Adding (72) and (73) the last term of A_q and the first on the right hand side of (73) cancel out. Moreover, using that $\nabla_x \boldsymbol{\xi}_q = \nabla_x \nabla_x \varrho_q$ is symmetric, we obtain further cancelations:

$$\begin{aligned}
\boldsymbol{\xi} \cdot \nabla_x \mathbf{v}_q \cdot \boldsymbol{\xi}_q - \mathbf{v}_q \cdot \nabla_x \boldsymbol{\xi} \cdot \boldsymbol{\xi}_q &= -\boldsymbol{\xi} \cdot \nabla_x \boldsymbol{\xi}_q \cdot \mathbf{v}_q - (\operatorname{div}_x \boldsymbol{\xi}) \mathbf{v}_q \cdot \boldsymbol{\xi}_q \\
&\quad - \mathbf{v}_q \cdot \nabla_x \boldsymbol{\xi} \cdot \boldsymbol{\xi}_q \\
&= -\mathbf{v}_q \cdot \nabla_x \boldsymbol{\xi}_q \cdot \boldsymbol{\xi} - \mathbf{v}_q \cdot \nabla_x \boldsymbol{\xi} \cdot \boldsymbol{\xi}_q \\
&\quad - (\operatorname{div}_x \boldsymbol{\xi}) \mathbf{v}_q \cdot \boldsymbol{\xi}_q \\
&= -\mathbf{v}_q \cdot \nabla_x (\boldsymbol{\xi}_q \cdot \boldsymbol{\xi}) - (\operatorname{div}_x \boldsymbol{\xi}) \mathbf{v}_q \cdot \boldsymbol{\xi}_q \\
&= -(\operatorname{div}_x \boldsymbol{\xi}) \mathbf{v}_q \cdot \boldsymbol{\xi}_q.
\end{aligned}$$

Thus, we obtain

$$(74) \quad \frac{d}{dt} \frac{1}{2} (\varrho |\mathbf{v}_q|^2 + |\boldsymbol{\xi}_q|^2) = A'_q + B'_q,$$

where

$$\begin{aligned}
A'_q &= \varrho_q (\varrho^{-1} \nabla_x \Pi) \cdot \mathbf{v}_q + \varrho_q \varrho^{-1} \operatorname{div}_x (\boldsymbol{\xi} \otimes \boldsymbol{\xi}) \cdot \mathbf{v}_q - \varrho \mathbf{v}_q \cdot \nabla_x \mathbf{v} \cdot \mathbf{v}_q \\
&\quad - (\operatorname{div}_x \boldsymbol{\xi}) \mathbf{v}_q \cdot \boldsymbol{\xi}_q - \boldsymbol{\xi}_q \cdot \nabla_x \mathbf{v} \cdot \boldsymbol{\xi}_q, \\
B'_q &= B_q + r_q(\mathbf{v}, \boldsymbol{\xi}) \operatorname{div}_x \boldsymbol{\xi}_q.
\end{aligned}$$

We will now multiply both sides of (74) by λ_q^{2s} and sum up over $q \geq 0$. On the left hand side we obtain $\frac{1}{2} (\varrho \|\mathbf{v}_{>0}\|_{H^s}^2 + \|\boldsymbol{\xi}_{>0}\|_{H^s}^2)_t$, and we need to obtain an estimate on the right hand side in terms of powers of $\|\mathbf{v}\|_{H^s}$ and $\|\boldsymbol{\xi}\|_{H^s}$. Using (60) and the embedding $L^\infty \hookrightarrow H^{s-1}$, we obtain

$$\begin{aligned}
\sum_{q \geq 0} \lambda_q^{2s} A'_q &\lesssim \|\boldsymbol{\xi}\|_{H^{s-1}} \|\varrho^{-1} \nabla_x \Pi\|_{L^\infty} \|\mathbf{v}\|_{H^s} \\
&\quad + \|\boldsymbol{\xi}\|_{H^{s-1}} \|\varrho^{-1} \operatorname{div}_x (\boldsymbol{\xi} \otimes \boldsymbol{\xi})\|_{L^\infty} \|\mathbf{v}\|_{H^s} + \|\mathbf{v}\|_{H^s}^2 \|\nabla_x \mathbf{v}\|_{L^\infty} \\
&\quad + \|\operatorname{div}_x \boldsymbol{\xi}\|_{L^\infty} \|\mathbf{v}\|_{H^s} \|\boldsymbol{\xi}\|_{H^s} + \|\boldsymbol{\xi}\|_{H^s}^2 \|\nabla_x \mathbf{v}\|_{L^\infty} \\
&\lesssim \|\boldsymbol{\xi}\|_{H^{s-1}} \|\varrho^{-1} \nabla_x \Pi\|_{H^{s-1}} \|\mathbf{v}\|_{H^s} + \|\boldsymbol{\xi}\|_{H^s}^3 \|\mathbf{v}\|_{H^s} + \|\mathbf{v}\|_{H^s}^3 \\
&\quad + \|\mathbf{v}\|_{H^s} \|\boldsymbol{\xi}\|_{H^s}^2 + \|\boldsymbol{\xi}\|_{H^s}^2 \|\mathbf{v}\|_{H^s} \\
&\lesssim \langle \|\boldsymbol{\xi}\|_{H^s}^2 + \|\mathbf{v}\|_{H^s}^2 \rangle^N.
\end{aligned}$$

The remainder terms collected in B'_q can be all estimated by a direct application of Lemma 4.5. Indeed, from (69) and (60) we obtain

$$\begin{aligned}
\sum_{q \geq 0} \lambda_q^{2s} \int_{\mathbb{R}^n} |r_q(\varrho, \varrho^{-1} \nabla_x \Pi) \cdot \mathbf{v}_q| \, dx &\lesssim \|\boldsymbol{\xi}\|_{H^{s-1}} \|\varrho^{-1} \nabla_x \Pi\|_{H^{s-1}} \|\mathbf{v}\|_{H^s} \\
&\lesssim \langle \|\boldsymbol{\xi}\|_{H^s}^2 + \|\mathbf{v}\|_{H^s}^2 \rangle^N.
\end{aligned}$$

Again, using (69) and Lemma 4.3,

$$\begin{aligned} & \sum_{q \geq 0} \lambda_q^{2s} \int_{\mathbb{R}^n} |r_q(\varrho, \varrho^{-1} \operatorname{div}_x(\boldsymbol{\xi} \otimes \boldsymbol{\xi})) \cdot \mathbf{v}_q| dx \\ & \lesssim \|\boldsymbol{\xi}\|_{H^{s-1}} \|\varrho^{-1} \operatorname{div}_x(\boldsymbol{\xi} \otimes \boldsymbol{\xi})\|_{H^{s-1}} \|\mathbf{v}\|_{H^s} \lesssim \langle \|\boldsymbol{\xi}\|_{H^s}^2 + \|\mathbf{v}\|_{H^s}^2 \rangle^N, \end{aligned}$$

and likewise,

$$\sum_{q \geq 0} \lambda_q^{2s} \int_{\mathbb{R}^n} |\varrho r_q(\mathbf{v}, \nabla_x \mathbf{v}) \cdot \mathbf{v}_q| dx \lesssim M_0 \|\mathbf{v}\|_{H^s}^3 \lesssim \langle \|\boldsymbol{\xi}\|_{H^s}^2 + \|\mathbf{v}\|_{H^s}^2 \rangle^N.$$

Applying (68) to the remaining terms we obtain

$$\begin{aligned} & \sum_{q \geq 0} \lambda_q^{2s} \int_{\mathbb{R}^n} |r_q(\boldsymbol{\xi}, \boldsymbol{\xi}) : \nabla_x \mathbf{v}_q| dx \lesssim \|\boldsymbol{\xi}\|_{H^s}^2 \|\mathbf{v}\|_{H^s} \lesssim \langle \|\boldsymbol{\xi}\|_{H^s}^2 + \|\mathbf{v}\|_{H^s}^2 \rangle^N \\ & \sum_{q \geq 0} \lambda_q^{2s} \int_{\mathbb{R}^n} |r_q(\mathbf{v}, \boldsymbol{\xi}) \operatorname{div}_x \boldsymbol{\xi}_q| dx \lesssim \|\mathbf{v}\|_{H^s} \|\boldsymbol{\xi}\|_{H^s}^2 \lesssim \langle \|\boldsymbol{\xi}\|_{H^s}^2 + \|\mathbf{v}\|_{H^s}^2 \rangle^N. \end{aligned}$$

Thus,

$$\sum_{q \geq 0} \lambda_q^{2s} B'_q \leq \langle \|\boldsymbol{\xi}\|_{H^s}^2 + \|\mathbf{v}\|_{H^s}^2 \rangle^N.$$

This finishes the proof.

5. APPENDIX I: PROOF OF LEMMA 4.2

We have

$$\|fg\|_{H^s} \leq \|(fg)_{\leq q}\|_{H^s} + \|(fg)_{> q}\|_{H^s}.$$

Using the Littlewood-Paley decomposition we have

$$\|(fg)_{> q}\|_{H^s}^2 \sim \sum_{p > q} \lambda_p^{2s} \|(fg)_p\|_{L^2}^2.$$

Let us notice

$$(fg)_p = \left(f_{< p-2} g_{p-2 \leq \cdot \leq p+2} + f_{p-2 \leq \cdot \leq p+2} g_{< p-2} + \sum_{k > p, |a|, |b| < 2} f_{k+a} g_{k+b} \right)_p.$$

So,

$$\begin{aligned} \|(fg)_p\|_{L^2}^2 & \lesssim \|f_{< p-2} g_{p-2 \leq \cdot \leq p+2}\|_{L^2}^2 + \|f_{p-2 \leq \cdot \leq p+2} g_{< p-2}\|_{L^2}^2 + \\ & + \left\| \sum_{k > p, |a|, |b| < 2} f_{k+a} g_{k+b} \right\|_{L^2}^2. \end{aligned}$$

We thus obtain

$$\begin{aligned} \|(fg)_{>q}\|_{H^s}^2 &\leq \sum_{p>q} \lambda_p^{2s} \|f_{<p-2} g_{p-2 \leq \cdot \leq p+2}\|_{L^2}^2 + \sum_{p>q} \lambda_p^{2s} \|f_{p-2 \leq \cdot \leq p+2} g_{<p-2}\|_{L^2}^2 \\ &\quad + \sum_{p>q} \lambda_p^{2s} \left\| \sum_{k>p, |a|, |b| < 2} f_{k+a} g_{k+b} \right\|_{L^2}^2. \end{aligned}$$

To estimate the first term we notice that $\|f_{<p-2}\|_{L^\infty} \leq \|f\|_{L^\infty}$. So, the first term is bounded by $C\|f\|_{L^\infty}^2 \|\nabla|^s g_{>q-2}\|_{L^2}^2$. Similar estimate holds for the second term. As to the third term we have

$$\begin{aligned} \sum_{p>q} \lambda_p^{2s} \left\| \sum_{k>p, |a|, |b| < 2} f_{k+a} g_{k+b} \right\|_{L^2}^2 &\leq \sum_{p>q} \left\| \sum_{k>p, |a|, |b| < 2} \lambda_{p-k}^s (\lambda_k^s f_{k+a}) g_{k+b} \right\|_{L^2}^2 \\ &\leq \sum_{p>q} \sum_{k>p, |a|, |b| < 2} \lambda_{p-k}^s \|\lambda_k^s f_{k+a} g_{k+b}\|_{L^2}^2 \\ &\leq \|g\|_{L^\infty}^2 \sum_{p>q} \sum_{k>p, |a|, |b| < 2} \lambda_{p-k}^s \|\lambda_k^s f_{k+a}\|_{L^2}^2 \\ &\leq \|g\|_{L^\infty}^2 \|\nabla|^s f_{>q-2}\|_{L^2}^2. \end{aligned}$$

This finishes the proof.

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