

# ILL-POSEDNESS OF BASIC EQUATIONS OF FLUID DYNAMICS IN BESOV SPACES

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**ABSTRACT.** We give a construction of a divergence-free vector field  $u_0 \in H^s \cap B_{\infty, \infty}^{-1}$ , for all  $s < 1/2$ , with arbitrarily small norm  $\|u_0\|_{B_{\infty, \infty}^{-1}}$  such that any Leray-Hopf solution to the Navier-Stokes equation starting from  $u_0$  is discontinuous at  $t = 0$  in the metric of  $B_{\infty, \infty}^{-1}$ . For the Euler equation a similar result is proved in all Besov spaces  $B_{r, \infty}^s$  where  $s > 0$  if  $r > 2$ , and  $s > n(2/r - 1)$  if  $1 \leq r \leq 2$ . This includes the space  $B_{3, \infty}^{1/3}$ , which is known to be critical for the energy conservation in ideal fluids.

## 1. INTRODUCTION

In recent years numerous results appear in the literature on well-posedness theory of the Euler and Navier-Stokes equations in Besov spaces (see for example, [1, 4, 5, 10, 13] and references therein). The best local existence and uniqueness result known for the Euler equation states that for any initial condition  $u_0 \in B_{r, 1}^{\frac{n}{r}+1}$  with  $1 < r \leq \infty$ , where  $n$  is the dimension of the fluid domain, there exists a unique weak solution  $u$  in space  $C([0, T]; B_{r, 1}^{\frac{n}{r}+1})$ , for some  $T = T(u_0) > 0$ , such that  $u(t) \rightarrow u_0$  in  $B_{r, 1}^{\frac{n}{r}+1}$ . The case of  $r = 2$ ,  $n = 3$  is especially interesting for it constitutes the borderline space for applicability of the standard energy method in proving such a result (see [9]). Notice that  $B_{2, 1}^{5/2}$  is a proper subspace of the Sobolev space  $H^{5/2} = B_{2, 2}^{5/2}$ , where local existence is an outstanding open problem. In this paper we present a construction of  $u_0$  which demonstrates ill-posedness of the Euler equations in a range of Besov spaces with the opposite extreme summation index, namely in  $B_{r, \infty}^s$ . In particular, there exists a  $u_0 \in B_{r, \infty}^{\frac{n}{r}+1}$  such that any energy bounded weak solution to the Euler equation that starts from  $u_0$  does not converge back to  $u_0$  in the metric of  $B_{r, \infty}^{\frac{n}{r}+1}$  as time goes to zero. Another particular instance includes the space  $B_{3, \infty}^{1/3}$  which defines a critical

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regularity of solutions to obey the energy conservation law in ideal fluids (see [6]).

In the second part of this note we address the question of ill-posedness for the Navier-Stokes equations in the critical Besov space  $X = B_{\infty, \infty}^{-1}$ . We recall that the homogeneous space  $\dot{X} = \dot{B}_{\infty, \infty}^{-1}$  is invariant with respect to the natural scaling of the equation in  $\mathbb{R}^3$ . Moreover it is the largest such space (see [4]). The non-homogeneous space considered in this note is even larger although quasi-invariant only with respect to the small scale dilations. In a recent work of Bourgain and Pavlovic [3] a mild solution to NSE was constructed with initial condition  $\|u_0\|_{\dot{X}} < \delta$  such that at a time  $t < \delta$  the solution satisfies  $\|u(t)\|_{\dot{X}} > 1/\delta$ . This shows the evolution under NSE is not continuous from  $\dot{X}$  into  $C([0, T]; \dot{X})$ . In our Proposition 3.2, similar to the case of the Euler equation, we construct an initial condition  $U$  which belongs to all Besov spaces  $B_{r, \infty}^{3/r-1}$  in the range  $1 < r \leq \infty$ , – in particular  $U$  has finite energy – such that any Leray-Hopf weak solution starting from  $U$  does not return to  $U$  as  $t \rightarrow 0$  in the metric of the inhomogeneous space  $X$ . This demonstrates an even more dramatic breakdown of NSE evolution in  $X$  as there is no continuous trajectory in  $X$  at all. More importantly our construction gives a simple model for the forward energy cascade, which is typically observed in turbulent flows [8]. Incidentally, the result proved in [7] shows that any left-continuous Leray-Hopf solution in  $X$  is necessarily regular.

We consider periodic boundary conditions for two reasons. Firstly, we do not make use of infinitesimally small frequencies in our analysis, and secondly, our construction is more transparent when the frequency space is a lattice. However with the technique developed in [6] the results can be carried over to the case of  $\mathbb{R}^n$  as well.

Let us now introduce the notation and spaces used in this paper. We will fix the notation for scales  $\lambda_q = 2^q$  in some inverse length units. Let us fix a nonnegative radial function  $\chi \in C_0^\infty(\mathbb{R}^n)$  such that  $\chi(\xi) = 1$  for  $|\xi| \leq 1/2$ , and  $\chi(\xi) = 0$  for  $|\xi| \geq 1$ . We define  $\varphi(\xi) = \chi(\lambda_1^{-1}\xi) - \chi(\xi)$ , and  $\varphi_q(\xi) = \varphi(\lambda_q^{-1}\xi)$  for  $q \geq 0$ , and  $\varphi_{-1} = \chi$ . For a tempered distribution vector field  $u$  on the torus  $\mathbb{T}^n$  we consider the Littlewood-Paley projections

$$(1) \quad u_q(x) = \sum_{k \in \mathbb{Z}^n} \hat{u}(k) \varphi_q(k) e^{ik \cdot x}, \quad q \geq -1.$$

So, we have  $u = \sum_{q=-1}^{\infty} u_q$  in the sense of distributions. We also use the following notation  $u_{\leq q} = \sum_{p=-1}^q u_p$ , and  $\tilde{u}_q = u_{q-1} + u_q + u_{q+1}$ .

Let us recall the definition of Besov spaces. A tempered distribution  $u$  belongs to  $B_{r,l}^s(\mathbb{T}^n)$  for  $s \in \mathbb{R}$ ,  $1 \leq l, r \leq \infty$  iff

$$\|u\|_{B_{r,l}^s} = \left[ \sum_{q \geq -1} (\lambda_q^s \|u_q\|_r)^l \right]^{1/l} < \infty.$$

We use  $\|u\|_p$  to denote the norm in the Lebesgue space  $L^p(\mathbb{T}^n)$ .

## 2. INVISCID CASE

The Euler equations for the evolution of ideal fluid are given by

$$(2) \quad u_t + (u \cdot \nabla)u = -\nabla p,$$

$$(3) \quad \operatorname{div} u = 0.$$

As noted in the introduction we assume throughout that the fluid domain is the torus  $\mathbb{T}^n$ ,  $n \geq 2$ . By a weak solution to (2) we understand an  $L^2$ -valued weakly continuous field  $u$  satisfying (2)–(3) in the distributional sense. Let us recall that all such solutions have absolutely continuous in time Fourier coefficients (see for example [11]).

We denote by  $\vec{e}_1, \dots, \vec{e}_n$  the vectors of the standard unit basis. Let us fix an  $s > 0$  and define

$$u_0(x_1, \dots, x_n) = \vec{e}_1 \cos(x_2) + \vec{e}_2 \sum_{q=0}^{\infty} \frac{1}{\lambda_q^s} \cos(\lambda_q x_1).$$

Notice that  $u_0$  is divergence free, and  $u_0 \in B_{r,\infty}^s(\mathbb{T}^n)$  for any  $r \in [1, \infty]$ . Furthermore,  $u_0$  is in fact two dimensional.

**Proposition 2.1.** *Suppose  $u$  is a weak solution to the Euler equation (2) with initial condition  $u(0) = u_0$ . Then there is  $\delta = \delta(n, r, s) > 0$  independent of  $u$  such that*

$$(4) \quad \limsup_{t \rightarrow 0^+} \|u(t) - u_0\|_{B_{r,\infty}^s} \geq \delta,$$

where  $s > 0$  if  $r > 2$ , and  $s > n(2/r - 1)$  in the case if  $1 \leq r \leq 2$ .

The rest of the section is devoted to the proof of Proposition 2.1.

Let us denote  $X = B_{r,\infty}^s$  for notational convenience. We can assume without loss of generality that for some  $t_0 > 0$ ,  $u \in L^\infty([0, t_0]; X)$ . Indeed, otherwise (4) follows immediately. Further proof is based on the fact that  $u_0$  produces a strong forward energy transfer which forces  $u$  to actually escape from  $B_{r,\infty}^s$  unless (4) is satisfied. Let us consider frequencies

$$\xi_q = (\lambda_q, 1, 0, \dots, 0), \text{ for } q = 0, 1, \dots$$

Let  $p(\xi)$  be the symbol of the Leray-Hopf projection, i.e.

$$p(\xi) = I - \frac{\xi \otimes \xi}{|\xi|^2}.$$

By a direct computation we obtain

$$(5) \quad f_q = p(\xi_q)(u_0 \cdot \nabla u_0)^\wedge(\xi_q) = i\lambda_q^{1-s}\vec{e}_2 + O(1/\lambda_q^s).$$

We will prove the following estimate for the nonlinear term

$$(6) \quad \|(u \cdot \nabla v)_q\|_1 \lesssim \lambda_q^{1-s} \|u\|_X \|v\|_X,$$

for all  $u, v \in X$  and  $q \geq -1$ . First, let us assume that  $r \leq 2$ , and let  $r^*$  be the conjugate of  $r$ , i.e.  $\frac{1}{r} + \frac{1}{r^*} = 1$ . Using the identity  $\operatorname{div}(u \otimes v) = u \cdot \nabla v$  and the Bernstein inequality we obtain

$$(7) \quad \|\operatorname{div}(u \otimes v)_q\|_1 \lesssim \lambda_q \|(u \otimes v)_q\|_1 \leq \lambda_q \sum_{\substack{p', p'' \geq q \\ |p' - p''| \leq 2}} \|u_{p'}\|_r \|v_{p''}\|_{r^*}$$

$$(8) \quad + \lambda_q \|u_q\|_r \sum_{p \leq q} \|v_p\|_{r^*} + \lambda_q \|v_q\|_r \sum_{p \leq q} \|u_p\|_{r^*}.$$

Using that

$$\|w_p\|_{r^*} \lesssim \lambda_p^{n(2/r-1)} \|w_p\|_r,$$

we have for the first sum

$$\begin{aligned} \lambda_q \sum_{\substack{p', p'' \geq q \\ |p' - p''| \leq 2}} \|u_{p'}\|_r \|v_{p''}\|_{r^*} &\lesssim \lambda_q \sum_{\substack{p', p'' \geq q \\ |p' - p''| \leq 2}} \|u_{p'}\|_r \lambda_{p'}^s \|v_{p''}\|_r \lambda_{p''}^s \lambda_{p'}^{n(2/r-1)-2s} \\ &\lesssim \lambda_q^{1+n(2/r-1)-2s} \|u\|_X \|v\|_X. \end{aligned}$$

For the second sum we obtain

$$\lambda_q \|u_q\|_r \sum_{p \leq q} \|v_p\|_{r^*} \lesssim \lambda_q^{1-s} \lambda_q^s \|u_q\|_r \sum_{p \leq q} \|v_p\|_r \lambda_p^s \lambda_p^{n(2/r-1)-s} \lesssim \lambda_q^{1-s} \|u\|_X \|v\|_X.$$

Similar estimate holds for the third term. We thus obtain (6).

In the case  $r > 2$ , we use the basic embedding  $L^r \subset L^{r^*}$  instead of Bernstein's inequalities in (7)–(8). The rest of the argument is similar.

We have

$$(9) \quad \hat{u}(\xi_q, t) = \hat{u}(\xi_q, 0) + \int_0^t p(\xi_q)(u \cdot \nabla u)^\wedge(\xi_q, s) ds,$$

for all  $t > 0$ . By our construction,  $\hat{u}(\xi_q, 0) = 0$ . On the other hand we can estimate using (6)

$$\begin{aligned} |p(\xi_q)(u \cdot \nabla u)^\wedge(\xi_q, s) - f_q| &\leq |(u \cdot \nabla u)^\wedge(\xi_q, s) - (u_0 \cdot \nabla u_0)^\wedge(\xi_q)| \\ &= |(u \cdot \nabla u)_q^\wedge(\xi_q, s) - (u_0 \cdot \nabla u_0)_q^\wedge(\xi_q)| \\ &\leq \|(u \cdot \nabla u)_q(s) - (u_0 \cdot \nabla u_0)_q\|_1 \\ &\lesssim \lambda_q^{1-s} (\|u(s)\|_X + \|u_0\|_X) \|u(s) - u_0\|_X. \end{aligned}$$

Thus, from (9) we obtain

$$\lambda_q^s |\hat{u}(\xi_q, t)| \geq t\lambda_q - tO(1) - C\lambda_q \int_0^t (\|u(s)\|_X + \|u_0\|_X) \|u(s) - u_0\|_X ds.$$

We can see that if the limit in (4) does not exceed  $\delta = 1/(10C)$  then the integral becomes less than  $t/2$ . This implies that  $u(t) \notin X$ .

### 3. ILL-POSEDNESS OF NSE

Now we turn to the analogous question for the viscous model. The Navier-Stokes equations are given by

$$(10) \quad u_t + (u \cdot \nabla)u = \nu \Delta u - \nabla p,$$

$$(11) \quad \operatorname{div} u = 0.$$

Our fluid domain here is the three dimensional torus  $\mathbb{T}^3$ . We refer to [12] for the classical well-posedness theory for this equation. Let us recall that for every divergence free field  $U \in L^2(\mathbb{T}^3)$  there exists a weak solution  $u \in C_w([0, T]; L^2) \cap L^2([0, T]; H^1)$  to (10)-(11) satisfying the energy equality

$$(12) \quad \|u(t)\|_2^2 + 2\nu \int_0^t \|\nabla u(s)\|_2^2 ds \leq \|U\|_2^2,$$

for all  $t > 0$ , and such that  $u(t) \rightarrow U$  strongly in  $L^2$  as  $t \rightarrow 0$ . In what follows we do not actually use inequality (12) which allows us to formulate a more general statement below in Proposition 3.2.

Let us fix a small  $\epsilon > 0$ . Let us choose a sequence  $q_1 < q_2 < \dots$  with elements sufficiently far apart so that  $\lambda_{q_i}^2 / \lambda_{q_{i+1}} < \epsilon$ . Let us fix a small  $c > 0$  and consider the following blocks of integers:

$$\begin{aligned} A_j &= [(1-c)\lambda_{q_j}, (1+c)\lambda_{q_j}] \times [-c\lambda_{q_j}, c\lambda_{q_j}]^2 \cap \mathbb{Z}^3 \\ B_j &= [-c\lambda_{q_{j-1}}, c\lambda_{q_{j-1}}]^2 \times [(1-c)\lambda_{q_{j-1}}, (1+c)\lambda_{q_{j-1}}] \cap \mathbb{Z}^3 \\ C_j &= A_j + B_j \\ A_j^* &= -A_j, \quad B_j^* = -B_j, \quad C_j^* = -C_j. \end{aligned}$$

Thus,  $A_j, C_j$  and their conjugates lie in the  $\lambda_{q_j}$ -th shell, while  $B_j, B_j^*$  lie in the contiguous  $\lambda_{q_j-1}$ -th shell. Let us denote

$$\vec{e}_1(\xi) = p(\xi)\vec{e}_1, \quad \vec{e}_2(\xi) = p(\xi)\vec{e}_2,$$

for  $\xi \in \mathbb{Z}^3 \setminus \{0\}$ . We define

$$(13) \quad U = \sum_{j \geq 1} (U_{q_j} + U_{q_j-1}),$$

where

$$\widehat{U}_{q_j}(\xi) = \frac{1}{\lambda_{q_j}^2} \left( \vec{e}_2(\xi) \chi_{A_j \cup A_j^*} + i(\vec{e}_2(\xi) - \vec{e}_1(\xi)) \chi_{C_j} - i(\vec{e}_2(\xi) - \vec{e}_1(\xi)) \chi_{C_j^*} \right),$$

and

$$\widehat{U}_{q_j-1}(\xi) = \frac{1}{\lambda_{q_j}^2} \vec{e}_1(\xi) \chi_{B_j \cup B_j^*}.$$

Since  $U$  has no modes in the  $(q_j + 1)$ -st shell, we have  $\tilde{U}_{q_j} = U_{q_j-1} + U_{q_j}$ .

**Lemma 3.1.** *One has  $U \in B_{r,\infty}^{\frac{3}{r}-1}$ , for all  $1 < r \leq \infty$ . Henceforth,  $U \in H^s$  for any  $s < \frac{1}{2}$ .*

*Proof.* We give the estimate only for one block. Using boundedness of the Leray-Hopf projection, we have for  $1 < r < \infty$

$$\|\lambda_{q_j}^{-2} (\vec{e}_2(\cdot) \chi_{A_j})^\vee\|_r \lesssim \lambda_{q_j}^{-2} \|(\chi_{A_j})^\vee\|_r \leq \lambda_{q_j}^{-2} \|D_{c\lambda_{q_j}}\|_r^3,$$

where  $D_N$  denote the Dini kernel. By a well-known estimate, we have  $\|D_N\|_r \leq N^{1-\frac{1}{r}}$ , which implies the lemma.

If  $r = \infty$ , we simply use the triangle inequality to obtain

$$\|U_{q_j}\|_\infty \lesssim \lambda_{q_j}.$$

The conclusion  $U \in H^s$  is a consequence of the embedding  $B_{2,\infty}^{\frac{1}{2}} \subset H^s$  for any  $s < \frac{1}{2}$ .  $\square$

Let us now examine the trilinear term. We will use the following notation for convenience

$$(14) \quad u \otimes v : \nabla w = \int_{\mathbb{T}^3} v_i \partial_i w_j u_j dx.$$

Using the antisymmetry we obtain

$$\begin{aligned}
U \otimes U : \nabla U_{q_j} &= \sum_{k \geq j+1} \tilde{U}_{q_k} \otimes \tilde{U}_{q_k} : \nabla U_{q_j} + \tilde{U}_{q_j} \otimes \tilde{U}_{q_j} : \nabla U_{q_j} \\
&\quad + U_{\leq q_{j-1}} \otimes \tilde{U}_{q_j} : \nabla U_{q_j} + \tilde{U}_{q_j} \otimes U_{\leq q_{j-1}} : \nabla U_{q_j} \\
&= \sum_{k \geq j+1} \tilde{U}_{q_k} \otimes \tilde{U}_{q_k} : \nabla U_{q_j} + U_{q_{j-1}} \otimes U_{q_j} : \nabla U_{q_j} \\
&\quad - U_{q_j} \otimes U_{q_j} : \nabla U_{\leq q_{j-1}} = A + B - C.
\end{aligned}$$

Using Bernstein's inequalities we estimate

$$(15) \quad |A| \lesssim \lambda_{q_j} \|U_{q_j}\|_\infty \sum_{k \geq j+1} \|\tilde{U}_{q_k}\|_2^2 \lesssim \frac{\lambda_{q_j}^2}{\lambda_{q_{j+1}}} \leq \epsilon,$$

$$(16) \quad |C| \lesssim \|U_{q_j}\|_2^2 \sum_{k \leq j-1} \lambda_{q_k} \|\tilde{U}_{q_k}\|_\infty \lesssim \frac{\lambda_{q_{j-1}}^2}{\lambda_{q_j}} \leq \epsilon.$$

On the other hand, a straightforward computation shows that  $B \sim \lambda_{q_j}$ . Combining this with estimates (21), (23) we obtain

$$(17) \quad U \otimes U : \nabla U_{q_j} \sim \lambda_{q_j}.$$

**Proposition 3.2.** *Let  $u \in C_w([0, T]; L^2) \cap L^2([0, T]; H^1)$  be a weak solution to the NSE with initial condition  $u(0) = U$ . Then there is  $\delta = \delta(u) > 0$  such that*

$$(18) \quad \limsup_{t \rightarrow 0^+} \|u(t) - U\|_{B_{\infty, \infty}^{-1}} \geq \delta.$$

*If in addition  $u$  is a Leray-Hopf solution satisfying the energy inequality (12), then  $\delta$  can be chosen independent of  $u$ .*

*Proof.* Using  $u_{q_j}$  as a test function we can write

$$\partial_t (\tilde{u}_{q_j} \cdot u_{q_j}) = -\nu \nabla \tilde{u}_{q_j} \cdot \nabla u_{q_j} + u \otimes u : \nabla u_{q_j}.$$

Denote  $E(t) = \int_0^t \|\nabla u\|_2^2 ds$ . Using (17) and integrating in time we obtain

$$(19) \quad \|\tilde{u}_{q_j}(t)\|_2^2 \geq \|U_{q_j}\|_2^2 - \nu E(t) + c_1 \lambda_{q_j} t$$

$$(20) \quad -c_2 \int_0^t |u \otimes u : \nabla u_{q_j} - U \otimes U : \nabla U_{q_j}| ds,$$

for some positive constants  $c_1$  and  $c_2$ . We now show that if the conclusion of the proposition fails then for some small  $t > 0$  the integral term in (20) is less than  $c_1 \lambda_{q_j} t / 2$  uniformly for all large  $j$ . This forces the inequalities  $\|\tilde{u}_{q_j}(t)\|_2^2 \gtrsim \lambda_{q_j} t$  to hold for all large  $j$ . Hence,  $u$  has infinite energy, which is a contradiction.

So, let us suppose that for every  $\delta > 0$  there exists  $t_0 = t_0(\delta) > 0$  such that  $\|u(t) - U\|_{B_{\infty,1}^{-1}} < \delta$  for all  $0 < t \leq t_0$ . Denoting  $w = u - U$  we write

$$\begin{aligned} u \otimes u : \nabla u_{q_j} - U \otimes U : \nabla U_{q_j} &= w \otimes U : \nabla U_{q_j} + u \otimes w : \nabla U_{q_j} \\ &\quad + u \otimes u : \nabla w_{q_j} = A + B + C. \end{aligned}$$

We will now decompose each triplet into three terms according to the type of interaction (c.f. Bony [2]) and estimate each of them separately.

$$\begin{aligned} A &= \sum_{\substack{p', p'' \geq q_j \\ |p' - p''| \leq 2}} w_{p'} \otimes U_{p''} : \nabla U_{q_j} + w_{\leq q_j} \otimes \tilde{U}_{q_j} : \nabla U_{q_j} \\ &\quad + \tilde{w}_{q_j} \otimes U_{\leq q_j} : \nabla U_{q_j} - \text{repeated} = A_1 + A_2 + A_3. \end{aligned}$$

Using Lemma 3.1 along with Hölder and Bernstein inequalities we obtain

$$\begin{aligned} |A_1| &\leq \|\nabla U_{q_j}\|_4 \sum \|w_{p'}\|_{\infty} \|U_{p''}\|_{4/3} \lesssim \lambda_{q_j}^{5/4} \sum \|w_{p'}\|_{\infty} \lambda_{p''}^{-5/4} \lesssim \delta \lambda_{q_j}, \\ |A_2| &= |U_{q_j} \otimes \tilde{U}_{q_j} : \nabla w_{\leq q_j}| \leq \|\tilde{U}_{q_j}\|_2^2 \|\nabla w_{\leq q_j}\|_{\infty} \\ &\lesssim \lambda_{q_j}^{-1} \sum_{p \leq q_j} \lambda_p^2 \lambda_p^{-1} \|w_p\|_{\infty} < \delta \lambda_{q_j}. \end{aligned}$$

As to  $A_3$ , we choose an  $r > 1$  close enough to 1 and  $r^*, \frac{1}{r} + \frac{1}{r^*} = 1$ , and estimate using Lemma 3.1

$$\begin{aligned} |A_3| &\leq \lambda_{q_j} \|U_{\leq q_j}\|_{r^*} \|U_{q_j}\|_r \|\tilde{w}_{q_j}\|_{\infty} \lesssim \lambda_{q_j} \lambda_{q_j}^{1 - \frac{3}{r^*}} \lambda_{q_j}^{1 - \frac{3}{r}} \|\tilde{w}_{q_j}\|_{\infty} \\ &\leq \|\tilde{w}_{q_j}\|_{\infty} \leq \delta \lambda_{q_j}. \end{aligned}$$

So, we have proved the following estimate:

$$(21) \quad |A| \lesssim \delta \lambda_{q_j}.$$

As to  $B$  we decompose analogously,

$$\begin{aligned} B &= \sum_{\substack{p', p'' \geq q_j \\ |p' - p''| \leq 2}} u_{p'} \otimes w_{p''} : \nabla U_{q_j} + u_{\leq q_j} \otimes \tilde{w}_{q_j} : \nabla U_{q_j} \\ &\quad + \tilde{u}_{q_j} \otimes w_{\leq q_j} : \nabla U_{q_j} - \text{repeated} = B_1 + B_2 + B_3. \end{aligned}$$

Again, using Lemma 3.1, Bernstein and Hölder inequalities we obtain

$$\begin{aligned}
|B_1| &\lesssim \lambda_{q_j} \|U_{q_j}\|_2 \sum \|u_{p'}\|_2 \|w_{p''}\|_\infty \leq \delta \lambda_{q_j}^{1/2} \|\nabla u\|_2. \\
|B_2| &= |U_{q_j} \otimes \tilde{w}_{q_j} : \nabla u_{\leq q_j}| \leq \|U_{q_j}\|_2 \|\tilde{w}_{q_j}\|_\infty \|\nabla u_{\leq q_j}\|_2 \\
&\leq \lambda_{q_j}^{-1/2} \|\tilde{w}_{q_j}\|_\infty \|\nabla u\|_2 \leq \delta \lambda_{q_j}^{1/2} \|\nabla u\|_2. \\
|B_3| &\leq \|\tilde{u}_{q_j}\|_2 \|w_{\leq q_j}\|_\infty \|\nabla U_{q_j}\|_2 \lesssim \lambda_{q_j}^{1/2} \|\tilde{u}_{q_j}\|_2 \sum_{p \leq q_j} \lambda_p^{-1} \|w_p\|_\infty \lambda_p \\
&\lesssim \delta \lambda_{q_j}^{1/2} \|\nabla u\|_2.
\end{aligned}$$

We thus obtain

$$(22) \quad |B| \lesssim \delta \lambda_{q_j}^{1/2} \|\nabla u\|_2.$$

Continuing in a similar fashion we write

$$\begin{aligned}
C &= \sum_{\substack{p', p'' \geq q_j \\ |p' - p''| \leq 2}} u_{p'} \otimes u_{p''} : \nabla w_{q_j} + u_{\leq q_j} \otimes \tilde{u}_{q_j} : \nabla w_{q_j} \\
&\quad + \tilde{u}_{q_j} \otimes u_{\leq q_j} : \nabla w_{q_j} - \text{repeated} = C_1 + C_2 + C_3.
\end{aligned}$$

$$|C_1| \leq \|\nabla w_{q_j}\|_\infty \sum_{p \geq q_j - 2} \|\tilde{u}_p\|_2^2 \lesssim \lambda_{q_j} \|w_{q_j}\|_\infty \lambda_{q_j}^{-2} \|\nabla u\|_2^2 \leq \delta \|\nabla u\|_2^2,$$

$$|C_2| \leq \|\nabla u\|_2 \|\tilde{u}_{q_j}\|_2 \|w_{q_j}\|_\infty \lesssim \lambda_{q_j}^{-1} \|\nabla u\|_2^2 \|w_{q_j}\|_\infty \leq \delta \|\nabla u\|_2^2.$$

Now using a uniform bound on the energy  $\|u(t)\|_2^2 \lesssim 1$  for almost all  $t \geq 0$ , we estimate

$$|C_3| \lesssim \lambda_{q_j} \|w_{q_j}\|_\infty \|\tilde{u}_{q_j}\|_2 \leq \delta \lambda_{q_j} \|\nabla \tilde{u}_{q_j}\|_2.$$

Thus,

$$(23) \quad |C| \lesssim \delta \|\nabla u\|_2^2 + \delta \lambda_{q_j} \|\nabla \tilde{u}_{q_j}\|_2.$$

Now combining estimates (21), (22), (23) along with the boundedness of  $E(t_0)$  we obtain

$$\begin{aligned}
(24) \quad \int_0^{t_0} |u \otimes u : \nabla u_{q_j} - U \otimes U : \nabla U_{q_j}| ds &\lesssim \delta \lambda_{q_j} t_0 + \delta \lambda_{q_j}^{1/2} t_0^{1/2} \\
&\quad + \delta + \delta \lambda_{q_j} \int_0^{t_0} \|\nabla \tilde{u}_{q_j}(s)\|_2 ds.
\end{aligned}$$

Using that

$$\int_0^{t_0} \|\nabla \tilde{u}_{q_j}(s)\|_2 ds \rightarrow 0$$

as  $j \rightarrow \infty$  we can chose  $\delta$  small enough and  $j_0$  large enough so that the right-hand side of (24) is less than  $\frac{c_1}{2c_2} \lambda_{q_j} t_0$ , for all  $j \geq j_0$ . Going back to (19) this implies

$$\|\tilde{u}_{q_j}(t_0)\|_2^2 \geq \|U_{q_j}\|_2^2 - \nu E(t_0) + c_1 \lambda_{q_j} t_0 / 2,$$

for all  $j > j_0$ , which shows that  $u(t_0)$  has infinite energy, a contradiction.

The last statement of the proposition follows from the fact that we have the bounds on  $\|u(t)\|_2 \leq \|U\|_2$  and  $E(t_0) \leq (2\nu)^{-1} \|U\|_2^2$  which remove dependence of the constants on  $u$ .  $\square$

*Remark 3.3.* Using our vector field  $U$  one can obtain the approximate equality (up to a constant independent of  $q$ )

$$\lambda_q^5 U \otimes U : \nabla U_q \sim \|U\|_{B_{2,1}^{5/2}} \lambda_q^{5/2} \|U_q\|_2.$$

This shows that the conventional energy argument along is not sufficient to prove local well-posedness for the 3D Euler equations in  $H^{5/2}$ . It is tempting to conjecture therefore that there is an example of initial condition  $u_0$  in any Besov space  $B_{2,l}^{5/2}$ , for  $l > 1$ , for which the analogue of Proposition 2.1 is true. However, this remains an open question.

*Remark 3.4.* To make a closer parallel with the result of Bourgain and Pavlović, [3], we can consider as our initial condition the field  $\epsilon U$  for  $\epsilon > 0$  as small as we like. The conclusion of Proposition 3.2 will remain the same, although  $\delta$  in (18) will naturally become dependent on  $\epsilon$ . This also shows that a possible small initial data result in  $B_{\infty,\infty}^{-1}$  will not include the condition of continuity in time, as for example is the case with  $H^{1/2}$  or  $L^3$ .

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