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LECTURES ON THE ONSAGER CONJECTURE

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ABSTRACT. These lectures give a detailed account of recent results pertaining to the celebrated Onsager conjecture. The conjecture states that the minimal space regularity needed for a weak solution of the Euler equation to conserve energy is $1/3$. Our presentation is based on the Littlewood-Paley method. We start with quasi-local estimates on the energy flux, introduce Onsager criticality, find a positive solution to the conjecture in Besov spaces of smoothness $1/3$. We illuminate important connections with the scaling laws of turbulence. Results for diadic models and a complete resolution of the Onsager conjecture for those is discussed, as well as recent attempts to construct dissipative solutions for the actual equation.

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"...in three dimensions a mechanism for complete dissipation of all kinetic energy, even without the aid of viscosity, is available."

L. Onsager, 1949

1. Lecture 1: motivation, Onsager criticality.

1.1. **Onsager's original conjecture.** The motion of an ideal homogeneous (with constant density 1) incompressible fluid is described by the system of Euler equations given by

$$\frac{\partial u}{\partial t} + (u\cdot\nabla)u = -\nabla p, \tag{1}$$

$$\nabla\cdot u = 0, \tag{2}$$

where u is a divergence-free velocity field, and p is the internal pressure. We assume that the fluid domain Ω here is either periodic or the entire space. It is an easy consequence of the antisymmetry of the nonlinear term in (1) and the incompressibility of the fluid that the law of energy conservation holds for smooth solutions:

$$\int_\Omega |u(t)|^2 dx = \int_\Omega |u_0|^2 dx, \text{ for all } t\geq 0. \tag{3}$$

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Although the well-posedness theory for the system (1) - (2) is well-developed for local in time smooth three-dimensional and global two-dimensional fluids, the Euler equations are also implicated in modeling a wide range of singular fluids or fluids with limited smoothness such as vortex sheets, filaments, vortices, and most importantly for us, turbulent flows in the limit of vanishing viscosity. We therefore are lead to considering the system (1) – (2) in the sense of distributions imposing only minimal assumption on the velocity field, namely $u \in L^\infty_t L^2_x$, or more physically relevant, u weakly continuous in time with values in $L^2(\Omega)$. Under these assumptions the law (3) becomes non-trivial, and one can ask under what minimal regularity assumptions on the velocity field the relation (3) still holds. A naive argument gets us quickly to the answer. Let us multiply (1) by u as we would have done if u was smooth and consider the total energy flux

$$\Pi = \int_\Omega (u \cdot \nabla) u \cdot u dx. \tag{4}$$

In the absence of necessary smoothness we cannot integrate by parts to obtain $\Pi = 0$. However, treating for a moment ∇ as a multiplication operator, which in many ways the Littlewood-Paley theory teaches us to do, we deduce

$$\Pi \sim \int_\Omega (|\nabla|^{1/3} u)^3.$$

It appears that if u has Hölder continuity $\frac{1}{3}$ we can at least make sense of the flux Π, and any better regularity would be sufficient to justify integration by parts in (4) to show that $\Pi = 0$. It is exactly what Onsager conjectured in his seminal paper on statistical fluid dynamics [41]: a) every weak solution to the Euler equations with smoothness $h > 1/3$ does not dissipate energy; b) and conversely, there exists a weak solution to (1) - (2) of smoothness of exactly $1/3$ which does not conserve energy. Energy dissipation due to irregularity of a flow is called *anomalous dissipation*.

Onsager's own justification of $1/3$ was not completely unrelated to the one above and it was based on laws of turbulence. A turbulent motion of fluid is believed to be described by solutions of the Navier-Stokes equation given by

$$\partial_t u + (u \cdot \nabla) u = -\nabla p + \nu \Delta u + f, \tag{5}$$

where ν is a small kinematic viscosity and f is an external force which supplies energy into the system (see Frisch [27]). Let us suppose that we have an ensemble average $\langle \cdot \rangle$ which measures statistical quantities of the system (5). In his unpublished notes (see Eyink and Sreenivasan [25] for a detailed historical account) Onsager reproduces a formula similar to what is known as Kármán-Howarth-Monin relation:

$$\partial_t \langle u(r) \cdot u(r+\ell) \rangle_{\mathrm{NL}} = \frac{1}{2} \operatorname{div}_\ell \langle |\delta u(\ell)|^2 \delta u(\ell) \rangle, \tag{6}$$

where $\delta u(\ell) = u(r+\ell) - u(r)$. Setting $\ell = 0$ in (6) one formally obtains $\partial_t \langle u^2 \rangle_{\mathrm{NL}} = 0$ provided u has average smoothness better than $1/3$. Since in the Euler equation (1) the nonlinear term is the only term present in the energy budget we again are led to Onsager's claim.

Due to this special relevance to turbulence it makes sense to also state the Onsager conjecture more broadly for a forced equation. Specifically, is there a force $f \in \mathcal{S}$ (or f with finite Fourier support) and a weak solution u of critical smoothness $1/3$ such that the energy balance relation

$$\|u(t)\|_2^2 = \|u(0)\|_2^2 + 2 \int_0^t (f, u(s)) ds, \tag{7}$$

is violated at some t. In particular, a stationary solution u_0 violates (7) if $(f, u_0) \neq 0$. In this form Onsager conjecture has been extensively studied for example in shell models, which we will discuss in Lecture 2.

1.2. **Onsager criticality and Besov spaces.** We now define suitable spaces to state Onsager's conjecture precisely. Let us recall that eventually we are after vanishing of the total energy flux (4). So, let us fix a scalar mollifier $h \in C_0^\infty(\mathbb{R}^n)$, and denote

$$h_\delta(y) = \delta^{-n} h(y\delta^{-1}), \tag{8}$$

$$u_\delta(x,t) = \int_{\mathbb{R}^n} h_\delta(y) u(x-y,t) dy. \tag{9}$$

Let us multiply equation (1) with $(u_\delta)_\delta$ and integrate in time. We obtain the relation

$$\frac{1}{2}(|u_\delta(t)|_2^2 - |u_\delta(0)|_2^2) = \int_0^t \int_{\mathbb{R}^n} (u \otimes u)_\delta : \nabla u_\delta dx ds, \tag{10}$$

where we denote $A : B = \mathrm{Tr}[AB]$. The right hand side of (10) is the energy flux through scales of order δ. We would like to find a bound on it in terms of $\|u\|_B^3$ for some space-time function space B. The optimal such B will have dimension that of the cube root of the energy. Fixing some units for velocity – U, length – X and time – T we thus arrive at relationship

$$(\dim \|\cdot\|_B)^3 = TU^3 X^{n-1}. \tag{11}$$

Any space B satisfying (11) will be called Onsager-critical, although of course not every Onsager-critical space is strong enough to produce a bound on (10). There are several example of Onsager-critical spaces that will be of particular interest to us. These include $L^3_t L^{9/2}_x$ and $L^3_t H^{5/6}_x$ in 3D, $L^3_t L^6_x$ in 2D, and most importantly a scale of Besov spaces which we introduce now.

We will fix the notation for scales $\lambda_q = 2^q$ in some inverse length units. Let us fix a nonnegative radial function $\chi \in C_0^\infty(\mathbb{R}^n)$ such that $\chi(\xi) = 1$ for $|\xi| \leq 1/2$, and $\chi(\xi) = 0$ for $|\xi| \geq 1$. We define $\varphi(\xi) = \chi(\lambda_1^{-1}\xi) - \chi(\xi)$, and $\varphi_q(\xi) = \varphi(\lambda_q^{-1}\xi)$ for $q \geq 0$, and $\varphi_{-1} = \chi$. For a tempered distribution vector field u on $\mathbb{R}^n$ we consider the Littlewood-Paley projections

$$u_q(x) = \int_{\mathbb{R}^n} \hat{u}(\xi)\varphi_q(\xi) e^{i\xi\cdot x} d\xi, \quad q \geq -1. \tag{12}$$

So, we have $u = \sum_{q=-1}^\infty u_q$ in the sense of distributions. We also use the following notation $u_{\leq q} = \sum_{p=-1}^q u_p$, and $\tilde{u}_q = u_{q-1} + u_q + u_{q+1}$. We say that a tempered distribution u belongs to the Besov space $B^s_{p,r}$ for $s \in \mathbb{R}$, $1 \leq p, r \leq \infty$ if

$$\|u\|_{B^s_{p,r}} = \left(\sum_{q \geq -1} (\lambda_q^s \|u_q\|_p)^r \right)^{1/r} < \infty.$$

We also define

$$B^s_{p,c_0} = \{u \in B^s_{p,\infty} : \lim_{q\to\infty} \lambda_q^s \|u_q\|_p = 0\}.$$

In the range of Besov spaces the Onsager-critical ones are $L^3 B_{p,r}^{\frac{n(3-p)+p}{3p}}$. In particular, $L^3 B^{\frac{1}{3}}_{3,l}$, $1 \leq l \leq \infty$ are critical in any spacial dimension. One can further define weaker versions of space-time critical spaces by reversing the order of time and space. We will make this precise in the next section.

We finish here by noting that without the use of local symmetries in the trilinear term, one can only obtain the following bound:

$$\left| \int_0^t \int_{\mathbb{R}^n} (u \otimes v) : \nabla w dx ds \right| \leq \|u\|_B \|v\|_B \|w\|_B,$$

where $B = L^3 B^{\frac{1}{2}}_{\frac{18}{7},2}$ and this bound is sharp (see [8]). Even though the space B is Onsager-critical, it does not reflect the critical smoothness $1/3$.

1.3. **Proof of the Onsager conjecture.** The end result of a series of works by Constantin et al [8, 14], Eyink [21], and Duchon and Robert [20] was the proof of the Onsager conjecture in the Besov class $B^{1/3}_{3,c_0}$. In other words, a solution needs to be "a little" smoother then just $1/3$. First, let us take a few preparatory steps to make the above discussion more rigorous (details can be found in [48]).

Definition 1.1. A vector field $u \in C_w([0,T]; L^2(\mathbb{R}^n))$, ("w" stand for weak continuity), is a weak solution of the Euler equations with initial data $u_0 \in L^2(\mathbb{R}^n)$ if for every compactly supported test function $\psi \in C_0^\infty([0,T] \times \mathbb{R}^n)$ with $\nabla_x \cdot \psi = 0$ and for every $0 \leq t \leq T$, we have

$$\int_{\mathbb{R}^n \times \{t\}} u \cdot \psi - \int_{\mathbb{R}^n \times \{0\}} u_0 \cdot \psi - \int_0^t \int_{\mathbb{R}^n} u \cdot \partial_s \psi = \int_0^t \int_{\mathbb{R}^n} (u \otimes u) : \nabla \psi, \tag{13}$$

and $\nabla_x \cdot u(t) = 0$ in the sense of distributions.

One can equivalently incorporate the associated pressure

$$p = - \sum_{l,k=1}^n R_l R_k (u_l u_k)$$

into (13) without requiring the test function to be divergence free.

The purpose of the following lemma is to show that one can pass from the weak formulation of the equation to the mollified or to the integral equations.

Lemma 1.2. *Let u be a weak solution. Then for each fixed $\delta > 0$, $u_\delta : [0,T] \to W^{s,q}$ (as defined in (9)) is absolutely continuous for all $s \geq 0$ and $q \geq 2$, and moreover*

$$\partial_t u_\delta = -\nabla \cdot (u \otimes u)_\delta - \nabla p_\delta, \tag{14}$$

for a.e. $t \in [0,T]$. Furthermore, (13) is equivalent to the integral equation

$$u(t) = u_0 - \int_0^t [\nabla \cdot (u \otimes u) + \nabla p] ds \tag{15}$$

in the sense of distributions for all $t \in [0,T]$.

For a divergence-free vector field $u \in L^2$ we introduce the Littlewood-Paley energy flux through a wave number λ_q by

$$\Pi_q = - \int_{\mathbb{R}^3} (u \otimes u)_{\leq q} : \nabla u_{\leq q} dx. \tag{16}$$

If $u(t)$ is a weak solution to the Euler equation, then from (14) we have

$$\partial_t u_{\leq q} = -\nabla \cdot (u \otimes u)_{\leq q} - \nabla p_{\leq q}, \tag{17}$$

Multiplying by $u_{\leq q}$ we obtain due to absolute continuity

$$\frac{1}{2}(|u(t)_{\leq q}|_2^2 - |u(0)_{\leq q}|_2^2) = - \int_0^t \Pi_q(s) ds. \tag{18}$$

So, positivity of the flux means the energy is flowing from large to small scales.

Let us introduce the following localization kernel

$$K_q = \begin{cases} \lambda_q^{2/3}, & q \leq 0; \\ \lambda_q^{-4/3}, & q > 0, \end{cases} \tag{19}$$

We now prove a kinematic bound on the flux. The important feature of the bound is that it is quasi-local in the sense that nonlocal interactions present in Π_q are diminished by the exponentially decaying tails of the kernel K.

Lemma 1.3. *The energy flux of a divergence-free vector field $u \in B^{\frac{1}{3}}_{3,\infty}$ satisfies the following estimate*

$$|\Pi_q| \leq C \sum_{p \geq 1} K_{q-p} \lambda_p \|u_p\|_3^3, \tag{20}$$

where $C > 0$ is an absolute constant.

Immediately from (18) and (20) we obtain the following result.

Theorem 1.4 ([8]). *Every weak solution u to the Euler equation on a time interval $[0, T]$ which satisfies*

$$\lim_{q \to \infty} \int_0^T \lambda_q \|u_q(t)\|_3^3 dt = 0 \tag{21}$$

conserves energy on the entire interval $[0, T]$. In particular, energy is conserved for every solution in the class $L^3([0, T]; B^{1/3}_{3,c_0})$.

Indeed, from (20) we have

$$\limsup_{q \to \infty} \int_0^t |\Pi_q(s)| ds \leq \limsup_{q \to \infty} \int_0^t \lambda_q \|u_q(s)\|_3^3 ds. \tag{22}$$

In order to see more transparent connection of condition to (21) to Onsager predicted smoothness $1/3$ we equivalently rewrite it as follows:

$$\lim_{|y| \to 0} \frac{1}{|y|} \int_0^T \int_{\mathbb{R}^n} |u(x) - u(x-y)|^3 dx dt = 0. \tag{23}$$

So, loosely speaking, the solution needs to be a little better than $1/3$ Hölder continuous in the space-time averaged sense in order to conserve energy.

Proof of Lemma 1.3. In the argument below all the inequalities should be understood up to a constant multiple. Let us denote for convenience the sequence

$$d_p = \lambda_p^{1/3} \|u_p\|_3.$$

Following [14] we write

$$(u \otimes u)_{\leq q} = r_q(u, u) - u_{>q} \otimes u_{>q} + u_{\leq q} \otimes u_{\leq q}, \tag{24}$$

where

$$r_q(u, u)(x) = \int_{\mathbb{R}^n} (\chi_q)^\vee(y)(u(x-y) - u(x)) \otimes (u(x-y) - u(x)) dy.$$

After substituting (24) into (16) we find

$$\Pi_q = \int_{\mathbb{R}^n} r_q(u, u) : \nabla u_{\leq q} dx - \int_{\mathbb{R}^n} (u_{>q} \otimes u_{>q}) : \nabla u_{\leq q} dx. \tag{25}$$

We can estimate the first term in (25) using the Hölder inequality by

$$\left| \int_{\mathbb{R}^n} r_q(u,u) : \nabla u_{\leq q} dx \right| \leq \|r_q(u,u)\|_{3/2} \|\nabla u_{\leq q}\|_3,$$

whereas

$$\|r_q(u,u)\|_{3/2} \leq \int_{\mathbb{R}^n} \left| \tilde{h}_\Lambda(y) \right| \|u(\cdot - y) - u(\cdot)\|_3^2 dy.$$

Let us now use Bernstein's and Minkowski's inequalities

$$\begin{aligned}\|u(\cdot - y) - u(\cdot)\|_3^2 &\leq \sum_{p\leq q} |y|^2 \lambda_p^2 \|u_p\|_3^2 + \sum_{p>q} \|u_p\|_3^2 \\ &= \lambda_q^{4/3} |y|^2 \sum_{p\leq q} \lambda_{q-p}^{-4/3} d_p^2 + \lambda_q^{-2/3} \sum_{p>q} \lambda_{q-p}^{2/3} d_q^2 \\ &\leq (\lambda_q^{4/3}|y|^2 + \lambda_q^{-2/3})(K * d^2)(q).\end{aligned}$$

Collecting the obtained estimates we find

$$\begin{aligned}&\left| \int_{\mathbb{R}^n} r_q(u,u) : \nabla u_{\leq q} dx \right| \\ &\leq (K * d^2)(q) \left(\int_{\mathbb{R}^n} |(\chi_q)^\vee(y)| \, \lambda_q^{4/3} |y|^2 dy + \lambda_q^{-2/3} \right) \left[\sum_{p\leq q} \lambda_p^2 \|u_p\|_3^2 \right]^{1/2} \\ &\leq (K * d^2)(q) \lambda_q^{-2/3} \left[\sum_{p\leq q} \lambda_p^{4/3} d_p^2 \right]^{1/2} \leq (K * d^2)^{3/2}(q) \leq (K * d^3)(q).\end{aligned}$$

Analogously we estimate the second term in (25)

$$\begin{aligned}&\int_{\mathbb{R}^n} (u_{>q} \otimes u_{>q}) : \nabla u_{\leq q} dx \leq \|u_{>q}\|_3^2 \|\nabla u_{\leq q}\|_3 \\ &\leq \left(\sum_{p>q} \|u_p\|_3^2 \right) \left(\sum_{p\leq q} \lambda_p^2 \|u_p\|_3^2 \right)^{1/2} \leq (K * d^2)^{3/2}(q) \leq (K * d^3)(q).\end{aligned}$$

This finishes the proof. □

Let us note that Lemma 1.3 shows that the energy transfer from one scale to another is controlled mostly by local interactions. It is a basic principle underlying turbulent motion for large Reynolds numbers. We will discuss more connections to turbulence in the next lecture.

As a consequence of Theorem 1.4 and the embedding $H^{5/6} \subset B^{1/3}_{3,2} \subset B^{1/3}_{3,c_0}$ we recover the result of Frisch and Sulem [28].

Corollary 1. *If u is a weak solution to the Euler equation which belongs to the class $L^3([0,T]; H^{5/6})$, then u conserves energy.*

Finally, we remark that the same regularity assumption of Theorem 1.4 automatically gives the energy balance (7) for the forced equation with a smooth force.

1.4. **Sharpness of the Besov conditions.** No known example of a weak dissipative solution belongs to the class $B^{1/3}_{3,\infty}$. Neither do we know any example of a solution to the forced Euler equation with a smooth force, which violates the energy balance (7). Under weaker regularity requirements, however, such examples have been discovered some time ago. In [45] Scheffer constructed a weak solution $u \in L^2L^2$ compactly supported in space-time (his definition of course is weaker than ours). Later in [46] Shnirelman gave a simpler example with the same regularity which emerges from a backward energy cascade. Very recently more regular solutions have been found by De Lellis and Székelyhidi in the class $u \in L^\infty L^\infty$ also with compact support in space-time, [15].

We will show that firstly $B^{1/3}_{3,\infty}$ is sharp for the argument of Theorem 1.4. Secondly, the Euler equation itself is ill-posed in this space. Namely, there is an initial condition u_0 in $B^{1/3}_{3,\infty}$ for which there exists no continuous Euler trajectory starting from u_0. It suggests that the conventional approach to existence based on a fixed point argument may not be sufficient to resolve the other half of the Onsager conjecture.

For simplicity we will work on the torus $\mathbb{T}^n$ so that the dual group is $\mathbb{Z}^n$. The technique developed in [8] allows one to carry over the constructions to the open space $\mathbb{R}^n$. Let $\vec{e}_1$, $\vec{e}_2$ be the standard basis vectors in $\mathbb{R}^2$. Let us fix a sequence of indexes $q_1 < q_2 < \ldots < q_j < \ldots$ far a part from each other. Let us define vector field U as follows

$$\begin{aligned} U &= \sum_{j=1} (U_{q_j} + U_{q_j+1}), \\ U_{q_j} &= \lambda_{q_j}^{-1/3} \cos(y\lambda_{q_j})\vec{e}_1, \\ U_{q_j+1} &= \lambda_{q_j}^{-1/3} \left(\cos(x\lambda_{q_j+1})\vec{e}_1 + \sin(x\lambda_{q_j+1} + y\lambda_{q_j})(-\vec{e}_1 + 2\vec{e}_2)\right). \end{aligned}$$

Thus, $U \in B^{1/3}_{3,\infty}$. One can see that the flux through the λ_{q_j}-th shell contains only local interactions, and we have

$$\Pi_{q_j} = U_{q_j+1} \cdot \nabla U_{q_j} \cdot U_{q_j+1} \sim 1.$$

Another earlier example of a 3D vector field $U \in B^{1/3}_{3,\infty}$ with non-vanishing energy flux was presented by Eyink in [21].

Let us consider the following vector field:

$$u_0(x,y) = \vec{e}_1 \cos(y) + \vec{e}_2 \sum_{q=0}^{\infty} \frac{1}{\lambda_q^s} \cos(\lambda_q x).$$

We have $u_0 \in B^s_{r,\infty}$, for any $r \geq 1$.

Proposition 1 ([12])**.** *If u is a weak solution to the Euler equation with initial condition $u(0) = u_0$. Then there is $\delta = \delta(n,r,s) > 0$ independent of u such that we have*

$$\limsup_{t \to 0^+} \|u(t) - u_0\|_{B^s_{r,\infty}} \geq \delta, \tag{26}$$

where $s > 0$ if $r > 2$, and $s > n(2/r - 1)$ if $1 \leq r \leq 2$.

This result does not preclude existence of a weak solution from our initial condition u_0 with $r = 3$, $s = 1/3$. After all, among members of $B^{1/3}_{3,\infty}$ there are some, such as vortex sheets, which do allow for local solutions (see Section 3). It simply suggests that solutions will not possess enough time regularity to tie a non-vanishing flux at the origin to the flux at near positive times.

1.5. **Helicity and enstrophy.** A number of other conservation laws can be treated with the Littlewood-Paley method. We will highlight two of them.

For a divergence-free vector field $u \in H^{1/2}$ with vorticity $\omega = \nabla \times u \in H^{-1/2}$ we define the helicity and truncated helicity flux in $\mathbb{R}^3$ as follows

$$\mathcal{H} = \int_{\mathbb{R}^3} u \cdot \omega \, dx \tag{27}$$

$$\mathcal{H}_q = \int_{\mathbb{R}^3} (u \otimes u)_{\leq q} : \nabla \omega_{\leq q} + (u \wedge \omega)_{\leq q} : \nabla u_{\leq q} \, dx, \tag{28}$$

where $u \wedge \omega = u \otimes \omega - \omega \otimes u$. If u is a regular solution to the Euler equation, then the helicity is conserved and $\mathcal{H}_q$ is the rate of helicity transfer through the wavenumber λ_q,

$$\frac{d}{dt}(u_{\leq q} \cdot \omega_{\leq q}) = \mathcal{H}_q.$$

Similar to the energy, helicity flux is local in frequency space. More precisely, with the same localization kernel as defined in (19) we obtain the following result.

Proposition 2 ([8]). *The helicity flux of a divergence-free vector field $u \in H^{1/2}$ satisfies the following estimate*

$$|\mathcal{H}_q| \leq C \sum_{p \geq -1} K_{q-p} \lambda_p^2 \|u_p\|_3^3. \tag{29}$$

Consequently, every weak solution to the Euler equation that belongs to the class $L^3([0,T]; B^{2/3}_{3,c_0}) \cap L^\infty([0,T]; H^{1/2})$ conserves helicity.

An example of a field $U \in B^{2/3}_{3,\infty}$ can be constructed to show that Proposition 2 is sharp in the same sense as Theorem 1.4. We refer to [2, 24, 7] and references therein for more on helicity conservation and its role in turbulence and topology.

For regular solutions to the two dimensional Euler equation the vorticity is transported by fluid particles, so a number of additional conservation laws come into place, in particular the enstrophy $\|\omega\|_2$. For less regular weak solutions such as DiPerna-Majda solutions ([19]) with only $\omega \in L^\infty L^p$ the flow may not be properly defined and therefore questions of enstrophy conservation become nontrivial. The answer was given by DiPerna and Lions [18] via the method of so called renormalized solutions which applies to every weak solution with finite enstrophy. So, as long as $\omega \in L^\infty L^2$, the enstrophy is conserved. However, unlike in the case of energy and helicity the enstrophy transfer from low to high modes is essentially non-local (see Kraichnan [36], Eyink [23]). We can see it through our Littlewood-Paley approach too. Let us define the enstrophy flux as follows

$$\Omega_q = -\int_{\mathbb{R}^2} (u \otimes u)_{\leq q} : \nabla \nabla^\perp \omega_{\leq q} \, dx. \tag{30}$$

Notice that for regular solutions one has

$$\frac{d}{dt}\|\omega_{\leq q}\|_2^2 = -2\Omega_q. \tag{31}$$

Let us define the kernel

$$W(q) = \begin{cases} \lambda_q^2, & q \leq 0; \\ \lambda_q^{-4}, & q > 0, \end{cases} \tag{32}$$

Then following estimate holds (see [8]):

$$|\Omega_q| \leq \|\omega_{\leq q}\|_3^2 (W * c^2)^{1/2}(q) + (W * c^2)^{3/2}(q), \tag{33}$$

where $c = \{\|\omega_p\|_3\}_{p\geq -1}$. An example of a field presented in [8] shows sharpness of estimate (33) which is consistent with the infrared non-locality of enstrophy transfer observed in two dimensional turbulence. Finally, let us notice that DiPerna-Majda solutions conserve energy as long as $p \geq 3/2$.

2. Lecture 2: anomalous dissipation in turbulence; intermittency; Onsager conjecture for shell models.

In this lecture we will review some basic facts from the turbulence theory. The power laws of small scale turbulence predict that energy dissipative solutions to the Euler equation may come in the limit of vanishing viscosity of ”generic” viscous flows. This prediction has been successfully verified for so-called dyadic models which describe energy behavior in a simplified form. We will once again see that the Besov space $B^{1/3}_{3,\infty}$ plays the special role of an ambient space that houses individual realizations of turbulent flows.

2.1. Kolmogorov 1941 theory.

Let us recall that in homogeneous isotropic turbulence the mean kinetic energy per unit mass is defined by $\mathcal{E} = \frac{1}{2}\langle |u|^2\rangle$ while the energy spectrum is defined by $E(\kappa) = \frac{1}{2}\frac{d}{d\kappa}\langle |u_{<\kappa}|^2\rangle$, where $u_{<\kappa}$ denotes the filtered velocity field containing all the frequencies below wavenumber κ (see [27]). So, $\mathcal{E} = \int_0^\infty E(\kappa)d\kappa$. The mean energy dissipation rate per unit mass is defined by

$$\epsilon^\nu = \langle \nu |\nabla u^\nu|^2\rangle. \tag{34}$$

It is a basic assumption of the Kolmogorov 1941 theory ([33, 34, 35]) supported by numerous experimental and numerical evidence that in the limit of vanishing viscosity ϵ^ν converges to a finite positive value,

$$\epsilon^\nu \to \epsilon > 0 \tag{35}$$

(see Eyink [24] for the latest account). Let us assume now that $f = f_{<\kappa_f}$ and solutions to (5) tend to a statistically stationary state, i.e. statistical properties are independent of time, and solutions have uniformly bounded mean energy. Let us scalar multiply (5) with $u_{<\kappa}$ and take the average. We obtain

$$\Pi(k) = -\nu\langle |\nabla u_{<\kappa}|^2\rangle + \langle f\cdot u_{<\kappa}\rangle, \tag{36}$$

where $\Pi(k)$ is the flux due to nonlinearity. If $\kappa > \kappa_f$ we have $\langle f\cdot u_{<\kappa}\rangle = \langle f\cdot u\rangle = \epsilon^\nu$. On the other hand, by Bernstein's inequality, $\nu\langle |\nabla u_{<\kappa}|^2\rangle \leq \nu\kappa^2\langle |u|^2\rangle$. Since the energy $\langle |u|^2\rangle$ is uniformly bounded by assumption, letting $\nu \to 0$ we obtain from (36)

$$\Pi(\kappa) = \epsilon. \tag{37}$$

So, under no direct influence of the injection force the energy flux is invariant and the energy simply propagates down from large to small scales with constant rate ϵ. Recent direct numerical simulations (see for example Keneda et al [30]) verified (37) to a high degree of accuracy. We interpret (37) as a reconciliation of Onsager and Kolmogorov's anomalies. Indeed, in the limit of vanishing viscosity an average solution to the (forced) Euler equation inherits the anomalous dissipation rate of size ϵ.

An additional self-similarity hypothesis on the structure functions in small scales (see [27]) implies that the energy spectrum distributes according to the law $E(\kappa) \propto \epsilon^{2/3}\kappa^{-5/3}$ in the inertial range $\kappa \in [\kappa_f, \kappa_d]$. Here κ_d is the Kolmogorov dissipation wavenumber given by

$$\kappa_d = \left(\frac{\epsilon}{\nu^3}\right)^{\frac{1}{4}}, \tag{38}$$

and κ_f is the integral wavenumber $\kappa_f = \max\{|\xi| : \xi \in \operatorname{supp}\hat{f}\}$. In spite of abundant empirical data supporting such distribution of the energy spectrum further experiments revealed that not every turbulence was consistent with the same regime (c.f. Anselmet et al [1]). In fact, Kolmogorov's 1941 theory demands that the local velocity fluctuations are uniformly distributed over the space. In order to achieve (35), however, certain regions of the flow must provide intense velocity gradients, or vorticity. As a consequence of the Helmholtz theorem a subsequent dynamical stretching of vortex filaments leaves some regions of the fluid domain with moderate turbulent activity and some with more intense. This so-called spacial intermittency, although fully consistent with the dissipation anomaly (35), must be accounted for in the description of other scaling laws and, in particular, of the energy spectrum. Generally, one obtains the expressions

$$E(\kappa) \propto \frac{\epsilon^{2/3}}{\kappa^{\frac{8-D}{3}}}, \tag{39}$$

$$\kappa_d = \left(\frac{\epsilon}{\nu^3}\right)^{\frac{1}{1+D}}, \tag{40}$$

where $D \in [0,3]$ is the dimension of the effective dissipation region. Thus, the classical K41 model corresponds to $D = 3$, while $D = 0$ corresponds to a fully intermittent model where energy cascades through inertial scales and dissipates on points.

Let us now find interpretation of the above for individual realizations of the NSE flow. Suppose u^ν is a Leray-Hopf solution to the Navier-Stokes equation (5). Let us denote by $\langle \cdot \rangle$ the long time average, a choice consistent with the Ergodic Hypothesis, or a finite time average. We define the Littlewood-Paley energy spectrum of u^ν by

$$E_{LP}(\kappa) = \frac{\langle \|u_q\|_2^2 \rangle}{\kappa}, \tag{41}$$

for all $\kappa \in [\lambda_q, \lambda_{q+1})$. One similarly defines the mean energy dissipation rate by

$$\epsilon_\nu = \nu \langle \|\nabla u^\nu\|_2^2 \rangle. \tag{42}$$

If a family of individual realizations $\{u^\nu\}_{\nu<\nu_0}$ verifies Kolmogorov's hypothesis (35), then the locality of energy flux expressed in (20) suggests the following analogue of (37)

$$\lambda_q \langle \|u_q^0\|_3^3 \rangle \propto \epsilon \tag{43}$$

for all q large enough. In other words the limiting solution to the Euler equation u^0 finds itself "on average" in the Onsager-critical Besov class $B^{1/3}_{3,\infty}$. Eyink showed in [22] that the Besov space $B^{1/3}_{3,\infty}$ is consistent with the multifractal intermittency models of Frisch and Parisi [26] (see also [5]). It speaks of sufficient capacity of this class to house turbulent solutions with singularity sets of dimension less then 3 (we will discuss this in much more detail in the next lecture). Within the Littlewood-Paley framework we can alternatively model intermittency correction consistent with (43) by fixing a saturation level of Bernstein's inequalities. Specifically, in three spacial dimensions we have

$$\|u_q\|_3 \lesssim \lambda_q^{\frac{1}{2}} \|u_q\|_2,$$

and assuming that the region of active turbulence is bounded (say, on the torus), one also has

$$\|u_q\|_2 \lesssim \|u_q\|_3.$$

Suppose that for $\lambda_q \in [\kappa_f, \kappa_d]$ we have the relation

$$\|u_q\|_3 \sim \lambda_q^{\alpha} \|u_q\|_2, \tag{44}$$

for some $0 \leq \alpha \leq \frac{1}{2}$. In view of (43) and (44) we recover the energy spectrum law

$$E_{LP}(\kappa) \sim \frac{\epsilon^{2/3}}{\kappa^{\frac{5}{3}+2\alpha}} \tag{45}$$

with α is related to D via $D = 3 - 6\alpha$. So, the fully saturated Bernstein's inequalities, i.e. $\alpha = 1/2$, correspond to uniform distribution of modes of u in each dyadic shell and hence strong localization in the physical space ($D = 0$). On the other extreme, $\alpha = 0$ corresponds to uniform distribution of u_q in physical space ($D = 3$) and localization in frequency space, which brings us back to the the classical K41 regime. By fixing saturation levels between higher L^p-norms one can achieve intermittency corrections for higher order structure functions, but we will not get into details.

We also mention here an important space-time local interpretation of Kolmogorov's dissipation anomaly (35) given by Duchon and Robert in [20]. They introduce a measure of anomalous energy dissipation as a distribution $D(u) \in \mathcal{D}'((0,T) \times \mathbb{R}^3)$ defined by the limit

$$D(u) = \lim_{\varepsilon \to 0} \frac{1}{4} \int \nabla \phi_\varepsilon(\xi) \cdot \delta u |\delta u|^2 d\xi,$$

where $\delta u = u(x + \xi) - u(x)$, and ϕ_ε is a mollifier. If u^ν is a solution to the NSE, or Euler for $\nu = 0$, then $D(u^\nu)$ does not depend of a particular choice of the mollifier and the following energy balance holds in the distributional sense (we drop forces for simplicity)

$$\partial_t(\frac{1}{2}(u^\nu)^2) + \operatorname{div}(u^\nu(\frac{1}{2}(u^\nu)^2 + p^\nu)) - \frac{\nu}{2}\Delta(u^\nu)^2 + \nu(\nabla u^\nu)^2 + D(u^\nu) = 0,$$

Similar balance holds for solutions of the Euler equation:

$$\partial_t(\frac{1}{2}(u^0)^2) + \operatorname{div}(u^0(\frac{1}{2}(u^0)^2 + p^0)) + D(u^0) = 0. \tag{46}$$

If $\{u^\nu\}_{\nu \leq \nu_0}$ is a sequence of weak solutions to the NSE which tends to u^0 strongly in $L^3_t L^3_x$, then

$$\lim_{\nu \to 0} \nu(\nabla u^\nu)^2 + D(u^\nu) = D(u^0), \tag{47}$$

in the sense of distributions. So, assuming that u^ν's themselves are regular enough to disallow Onsager dissipation, i.e. $D(u^\nu) = 0$, then (47) becomes a space-time local version of (35) provided $D(u^0) > 0$. Integration in space and a finite time interval reveals that the average total anomalous flux is equal to the limit $\lim_{\nu \to 0} \nu \langle \|\nabla u^\nu\|_2^2 \rangle$, as suggested by (37). However, the relationship (47) does not seem to give a detailed shell-by-shell information as in (20) and (37). A sufficient condition for $D(u^0) = 0$ similar to (21) was also obtained in [20] as a direct generalization of [14]. It implies energy conservation via (46), provided $u(t)$ is strongly continuous in L^2 since (46) is a distributional relation (for more on this, see [8]).

The saturation hypothesis (44) and locality of energy transfer lay down a basis of so-called dyadic models of turbulence, which we will discuss next.

2.2. **Discrete shell models.** In late 70s–80's several "toy" models of turbulence appeared in mathematical and physical literature to test the power laws of Kolmogorov's theory (see for example [17, 40, 29]). In those models the total energy of the flow in a dyadic shell is replaced by a scalar quantity $a_q^2(t)$ while the nonlinear term of the full NSE or Euler equation is simplified to involve only local interactions. So, to derive a shell model we start with the fully local version of (20):

$$\Pi_q \propto \lambda_q \|u_q\|_3^3.$$

Next we include intermittency correction by adopting one of the Bernstein identities (44) for some α. We will use $c = 3\alpha + 1$ as a parameter instead of α. Thus, $1 \leq c \leq 5/2$. Denoting $a_q = \|u_q\|_2$ we obtain the expression

$$\Pi_q \propto \lambda_q^c a_q^3.$$

Subsequent analysis shows that in order to preserve positivity of the scalar quantities a_q's as suggested by the definition one can model the flux with

$$\Pi_q = \lambda_q^c a_q^2 a_{q+1}.$$

The shell analogue of the energy budget relation is

$$\frac{1}{2}\frac{d}{dt}\sum_{p=0}^{q} a_p^2 = -\Pi_q + \sum_{p=0}^{q}(-\nu\lambda_p^2 a_p^2 + f_p a_p),$$

and replacing q by $q-1$ we also have

$$\frac{1}{2}\frac{d}{dt}\sum_{p=0}^{q-1} a_p^2 = -\Pi_{q-1} + \sum_{p=0}^{q-1}(-\nu\lambda_p^2 a_p^2 + f_p a_p).$$

Subtracting and canceling a_q we obtain the Desnyansky-Novikov model

$$\frac{d}{dt}a_q = \lambda_{q-1}^c a_{q-1}^2 - \lambda_q^c a_q a_{q+1} - \nu\lambda_q^2 a_q + f_q, \quad q = 0, 1, \ldots \tag{48}$$

where $a_{-1} = 0$. We also consider the inviscid variant of (48) as a model for the Euler equation:

$$\frac{d}{dt}a_q = \lambda_{q-1}^c a_{q-1}^2 - \lambda_q^c a_q a_{q+1} + f_q, \quad q = 0, 1, \ldots \tag{49}$$

Long after the origination analytical studies of the models resumed in recent works of Katz and Pavlovic [31], Cheskidov et al [13, 10], Kiselev and Zlatoš [32] and others. The Onsager conjecture was first addressed in the works of Cheskidov et al in [10, 9]. We will describe the results below as they demonstrate a complete agreement with Onsager's prediction.

The energy of the system is defined by $\mathcal{E}(t) = \frac{1}{2}\sum_{q=0}^{\infty} a_q^2(t)$, while the Sobolev norm by $\|a\|_{H^s}^2 = \sum_{q=0}^{\infty} \lambda_q^{2s} a_q^2$. We assume that the force has only one non-zero mode, $f = (f_0, 0, \ldots)$.

Let us first discuss the inviscid model, i.e. $\nu = 0$. The Onsager-critical class for validity of the energy balance relation

$$\mathcal{E}(t) = \mathcal{E}(0) + \int_0^t (f, a(s)) ds \tag{50}$$

is $L^3([0,T); H^{c/3})$. However, no solution to (49) stays in $H^{c/3}$ for all times, and every solutions start to dissipate energy! The situation is the following.

1) For every $a_0 \in \ell^2$ there exists a global weakly continuous in ℓ^2 global solution to (49) with each $a_q(t)$ continuously differentiable in time. If $a_0 \geq 0$, then $a(t) > 0$ for all t, and the energy inequality

$$\mathcal{E}(t) \leq \mathcal{E}(t_0) + \int_{t_0}^t (f, a(s)) ds, \tag{51}$$

holds for all $0 \leq t_0 < t$.

2) The $H^{c/3}$-norm of any solution blows up in finite time, moreover, for some $T > 0$, $a \notin L^3([0,T); H^{c/3})$.

3) The unique equilibrium given by $\alpha^0 = \{\lambda_q^{-c/3}\sqrt{f_0}\}$ is an exponential attractor, i.e. every trajectory converges to α^0 exponentially fast in the metric of ℓ^2. Clearly, α^0 itself violates (50) since $(\alpha^0, f) = \alpha^0_0 f_0 > 0$.

4) For a trajectory $a(t)$ we define the anomalous dissipation rate by

$$\epsilon_0(t) = (a(t), f) - \frac{1}{2}\frac{d}{dt}|a(t)|_2^2$$

in the sense of distributions. In view of (51), ϵ_0 is a positive distribution, and hence a Borel measure. Define

$$\epsilon_0 = \lim_{T\to\infty} \frac{1}{T}\int_0^T d\epsilon_0(t).$$

Then, $\epsilon_0 = (\alpha^0, f) > 0$ because of fact 3). We see that every solution dissipates energy eventually. Moreover, the long time averaged flux through wavenumber κ satisfies $\Pi_\kappa = (\alpha^0_{\leq\kappa}, f) = (\alpha^0, f) = \epsilon_0$, thus confirming the law (37).

5) Since every solution converges to the equilibrium α^0, the energy spectrum defined by $E(\kappa) = \frac{\langle a_q^2\rangle}{\kappa}$, $\kappa \in [\lambda_q, \lambda_{q+1})$, satisfies $E(\kappa) \propto \lambda_q^{-1-2c/3}$. This is consistent with the description of the spectrum in the corresponding intermittency regime given by (39).

The viscous case was treated in [9] for a limited range of parameter $c \in (3/2, 5/2]$.

i) In this case there exists a unique smooth stationary positive solution α^ν, which is an exponential global attractor in ℓ^2. Moreover, $\alpha^\nu \to \alpha^0$ as $\nu \to 0$ the metric of any H^s, for $s < c/3$ (notice that α^0 does not belong to $H^{c/3}$). Thus, for small $\nu > 0$, $\alpha_q^\nu \propto \lambda_q^{-c/3}$ in the range of q consistent with the Kolmogorov dissipation wavenumber (40), beyond which α_q^ν decreases exponentially fast.

ii) For a fixed $\nu > 0$ and solution $a^\nu(t)$ there is no anomalous dissipation in the long time average sense, i.e.

$$\epsilon_\nu := \lim_{T\to\infty} \int_0^T \frac{1}{T}\nu\|a^\nu(t)\|^2_{H^1} dt = \lim_{T\to\infty}\int_0^T \frac{1}{T}(a^\nu(t), f)dt = (\alpha^\nu, f). \tag{52}$$

So, in view of i), letting $\nu \to 0$ we obtain

$$\epsilon_\nu \to (\alpha^0, f) = \epsilon_0 > 0,$$

which is a vivid manifestation of the Kolmogorov's dissipation anomaly law (35).

Due to the technical restriction $c \in (3/2, 5/2]$ it remains open whether the above results hold in the classical K41 regime $c = 1$.

3. **Lecture 3: local energy balance, organized singularities, vortex sheets.** As we argued previously the suitable functional class in Besov range to house turbulent solutions to the Euler equation is the Onsager-critical space $B^{1/3}_{3,\infty}$. In intermittent regimes individual realizations of Euler flow may in principle exhibit organized formations of the singular set not covering the entire fluid domain. In this lecture we will examine some of those cases in more detail. We obtain sufficient Onsager-critical conditions for energy conservation in terms of mixed L^p-spaces. Since a turbulent solution is expected to dissipate energy these conditions essentially tell us what a turbulent motion cannot be. Guided by these findings we will explore a possibility of obtaining dissipative stationary solutions with a point singularity in 2D. We will further establish that vortex sheets are not turbulent despite being in the critical class $B^{1/3}_{3,\infty}$ and no better. However at the time of first roll-up there is a possibility for energy drop or loss. The results of this lecture are mostly contained in [48] and are inspired by an earlier work of Caflisch et al [5].

First we will look at the problem locally in space-time. We define the appropriate local regularity class by taking (21) as a determining condition.

Definition 3.1. Let u be a weak solution to the Euler equations. We say that u is regular, and write $u \in \mathcal{R}$ if the condition (21) is satisfied. For a relatively open set $D \subset [0,T] \times \mathbb{R}^n$ we write $u \in \mathcal{R}(D)$ if $u\phi \in \mathcal{R}$ for all scalar $\phi \in C_0^\infty(D)$. The maximal such D is called the regular set of u while its complement is called the singular set.

For a set $A \subset \mathbb{R}^n \times [0,T]$ we denote by $A(t)$ the slice $A \cap (\mathbb{R}^n \times \{t\})$, and $A(t',t'') = A \cap (\mathbb{R}^n \times (t',t''))$.

Lemma 3.2. *Let D be the regular set of a weak solution u. Then for every $\phi \in C_0^\infty(D)$ one has*

$$\int_{D(t'')} |u|^2\phi - \int_{D(t')} |u|^2\phi = \int_{D(t',t'')} |u|^2\partial_t\phi + (|u|^2 + 2p)u \cdot \nabla\phi, \tag{53}$$

for all $t', t'' \in [0,T]$.

Definition 3.3. We say that a set $S \subset \mathbb{R}^n \times [0,T]$ admits a *k-dimensional $C^{\gamma,1}$-cover* if for every point (x_0,t_0) in the space-time there is an open neighborhood U of x_0 in $\mathbb{R}^n$ and a relatively open subinterval $I \subset [0,T]$ containing t_0 for which there exists a family of C^1-diffeomorphisms

$$\varphi_t : U \to B_1, \quad t \in I, \tag{54}$$

satisfying the following conditions

(a) $S(t) \cap U \subset \varphi_t^{-1}(\mathbb{R}^k \times \{0\}^{n-k} \cap B_1)$, for all $t \in I$;
(b) There is $C > 0$ such that

$$\sup_{x\in U} |\varphi_{t'}(x) - \varphi_{t''}(x)| \leq C|t'-t''|^\gamma,$$

for all $t', t'' \in I$;
(c) $\sup_{x\in U, t\in I} |\nabla_x \varphi_t(x)| \leq C$.

In the case $k < n-1$ we have the following result.

Theorem 3.4 ([48]). *Let $u \in L^3(\mathbb{R}^n \times [0,T])$ be a weak solution to the Euler equation on the time interval $[0,T]$. Then u conserves energy provided the singular set S of u admits a k-dimensional $C^{\gamma,1}$-cover and $u \in L^3_t L^3_S L^{\frac{3(n-k)}{n-k-1}}_{S^\perp}$ locally, where the values of $\gamma, n, k > 0$ satisfy*

$$\gamma \geq \frac{3}{n-k+2}, \quad n > k+1. \tag{55}$$

The mixed L^p-space used in the theorem is defined as follows. Assuming that each slice $S(t)$ is a k-dimensional smooth submanifold of $\mathbb{R}^n$ we consider a local normal fiber bundle $S^\perp(t)$. Thus, each fiber $S^\perp(x,t)$ is a γ-smooth in time local tile orthogonal to the surface $S(t)$. We define the local space $L^3_t L^p_S L^q_{S^\perp}$ by assuming

$$\int_I \left(\int_{S(t)\cap U} \left(\int_{S^\perp(x,t)\cap U} |u(x,y,t)|^q d\sigma_t^{n-k}(y) \right)^{p/q} d\sigma_t^k(x) \right)^{3/p} dt < \infty,$$

over each coordinate neighborhood $U \times I$, where $d\sigma_t^r$ denotes the surface measure of dimension r. A straightforward computation shows that the space $L^3_t L^3_S L^{\frac{3(n-k)}{n-k-1}}_{S^\perp}$ is in fact Onsager-critical.

The argument used in the proof of Theorem 3.4 is local. We use the local energy balance relation (53) as a platform for approaching the singular set by an orderly cut-off procedure. Just to give an illustration, let us consider the point case $k = 0$ and assume the point singularity $s(t)$ is regular in time. Let us consider a scalar cut-off function $\eta \in C^\infty_{\mathbb{R}^+}$ with

$\eta(r) = 1$ for $r \in [0,1)$, and $\eta(r) = 0$ for $r \geq 2$. For $\epsilon > 0$ we consider $\phi_\epsilon(x,t) = 1 - \eta\left(\frac{|x-s(t)|}{\epsilon}\right)$. Since the support of ϕ_ϵ lies entirely in the regular region of u, we can substitute ϕ_ϵ into the local energy equality (53) (compactness of the support can be easily removed). As we let $\epsilon \to 0$ the left hand side clearly converges to $\|u(t'')\|_2^2 - \|u(t')\|_2^2$. Since $\partial_t \phi_\epsilon$ behaves like ϵ^{-1} we obtain

$$\left| \int_{D(t',t'')} |u|^2 \partial_t \phi_\epsilon \right| \leq \frac{1}{\epsilon} \int_{t'}^{t''} \int_{B_{2\epsilon}(s(t))} |u|^2.$$

This is smaller than the estimate we will obtain for the trilinear term. We have

$$\begin{aligned} \left| \int_{D(t',t'')} |u|^2 u \cdot \nabla \chi_\epsilon \right| &\lesssim \frac{1}{\epsilon} \int_{t'}^{t''} \int_{B_{2\epsilon}(s(t))} |u|^3 dx dt \\ &\lesssim \int_{t'}^{t''} \left(\int_{B_{2\epsilon}(s(t))} |u|^{\frac{3n}{n-1}} dx \right)^{\frac{n-1}{n}} dt \to 0. \end{aligned}$$

The pressure can be treated similarly.

Since the argument is local one can extend it to the case of locally finite unions of singular sets. Specifically, suppose that in every coordinate neighborhood $V = U \times I$

$$S = \bigcup_{j=1}^{N_V} S_j, \tag{56}$$

where S_j's are k_j-dimensionally $C^{\gamma_j,1}$-covered in V. The conclusions of Theorem 3.4 remain valid under the corresponding assumptions on u locally near each S_j.

Because of its local nature it is more convenient to view Theorem 3.4 as a statement about the local energy balance relation (53) rather than the total energy conservation. Thus if u and S are locally as described by Theorem 3.4 then as far as the energy balance is concerned the regular set D of u can be augmented to include points of S.

Example 1. Suppose u has a point singularity $s(t)$ at each time $t \in [0,T]$. Thus, $k = 0$ and the energy conservation is guaranteed if $s \in C^{\frac{3}{n+2}}$ and $u \in L^3 L^{\frac{3n}{n-1}}$. In 2D we obtain the condition $u \in L^3 L^6$, while in 3D $u \in L^3 L^{9/2}$. In the case of $n = 3$ and $k = 1$ we obtain $u \in L^3 L^3_S L^6_{S^\perp}$.

Example 2. As we discussed earlier, a physically meaningful formulation of Onsager's conjecture in stationary case consists of finding a smooth force $f \in \mathcal{S}$ and a field $u \in B^{1/3}_{3,\infty} \cap L^2$ such that

$$(u \cdot \nabla) u + \nabla p = f \tag{57}$$

holds in the distributional sense and

$$(f, u) \neq 0. \tag{58}$$

In view of Theorem 3.4 we look for a solution having one point singularity at the origin with $u \in L^6_{\text{weak}}$. A natural choice would be (in polar coordinates)

$$u = \begin{cases} \frac{1}{r^{1/3}} (\frac{2}{3} \Psi \vec{\tau} - \Psi' \vec{\nu}), & r \leq 1; \\ smooth, & 1 < r < 2; \\ 0, & r \geq 2. \end{cases}, \tag{59}$$

where $\Psi(\theta) \in C^2$. This is an example of a divergence-free field with $u \in L^6_{\text{weak}} \cap B^{1/3}_{3,\infty} \cap L^2$. In order to satisfy (57) and (58), function Ψ has to obey the following two conditions

$$3(\Psi')^2 + 4\Psi^2 + 6\Psi\Psi'' = P, \tag{60}$$

where P is a constant related to the value of the pressure near singularity, and

$$\int_0^{2\pi} (\Psi'(\theta))^3 d\theta \neq 0. \tag{61}$$

Unfortunately an ad hoc argument shows that (60) and (61) are inconsistent. Indeed, suppose $P \neq 0$. Let us test equation (60) against $\Psi^n \Psi'$ with $n \geq 1$. One obtains $\int (\Psi')^3 \Psi^n = 0$ for all $n \geq 1$. Thus for any analytic function G vanishing at 0 one also has $\int (\Psi')^3 G(\Psi) = 0$. Let us take $G_\epsilon(x) = 1 - e^{-x^2/\epsilon}$ and let $\epsilon \to 0$. We obtain $\int_{\Psi \neq 0} (\Psi')^3 = 0$. In order to stay consistent with (61) we necessarily conclude that $|\{\Psi = 0\}| > 0$. Hence, the set where $\Psi = 0$ has a cluster point θ_0. At that point $\Psi'(\theta_0) = 0$ which contradicts (60).

Suppose now that $P = 0$, and assume without loss of generality that $\Psi(\theta) > 0$ at some θ. Notice that at any such point θ, $\Psi''(\theta) < 0$. Let us consider a local maximum θ_0 so that $\Psi'(\theta_0) = 0$ and $\Psi(\theta_0) > 0$. Since Ψ cannot be identically constant, there is a $\theta_1 > \theta_0$ such that $\Psi'(\theta_1) < 0$, yet $\Psi(\theta_1) > 0$. Since $\Psi'' < 0$, the slope $\Psi'(\theta)$ will decrease to the right of θ_1. We eventually find a point $\theta_2 > \theta_1$ such that $\Psi(\theta_2) = 0$, yet $\Psi'(\theta_2) < 0$. This contradicts equation (60).

As a consequence, every solution exist in the form of (59) satisfies the energy balance relation, and yet this case is not covered by Theorem 3.4.

3.1. **Case $k = n - 1$: slits.** We now describe the hypersurface version of Theorem 3.4 which appears to be the critical case. Our settings will be somewhat stronger than those described in Definition 3.3. Namely, let S be a C^1 in time family of closed orientable C^2-submanifolds of $\mathbb{R}^n$. Denote by $\vec{n}(x,t)$ the positively oriented unit normal to $S(t)$ at $x \in S(t)$. Let us also define the normal segments for every $(x,t) \in S$:

$$\Gamma_+(x,t) = (x,t) + \vec{n}(x,t)[0, \epsilon_1],$$
$$\Gamma_-(x,t) = (x,t) + \vec{n}(x,t)[-\epsilon_1, 0],$$

(the length being uniform in every coordinate chart). Let $W \subset \mathbb{R}^n \times [0,T]$ be a relatively open set such that $W \cap S$ is a coordinate chart on S. For a function or field f on W we define the normal maximal function defined by

$$f^*_\pm(x,t) = \sup_{x' \in \Gamma_\pm(x,t)} |f(x',t)|,$$

and the normal limits by

$$f_\pm(x,t) = \lim_{\substack{x' \to x \\ x' \in \Gamma_\pm(x,t)}} f(x',t),$$

provided the latter exist. By $L^q(S)_{\text{loc}}$ we understand the local L^q-space with respect to the measure $d\sigma_t dt$, where $d\sigma_t$ is the surface measure on $S(t)$.

Definition 3.5. Let u be a weak solution to the Euler equations. The surface S is called a *slit* of u if

1) The limits $u_\pm, p_\pm$ exist for a.e. $t \in [0,T]$ and a.e. $x \in S(t)$,
2) $u^*_\pm \in L^3(S)_{\text{loc}}$ and $p^*_\pm \in L^{3/2}(S)_{\text{loc}}$

If S is a slit, then one can show directly from the weak formulation of the Euler equation and divergence-free condition that the normal component of u has no jump across the surface $S(t)$:

$$u_+ \cdot \vec{n} = u_- \cdot \vec{n} := u_n,$$

and the pressure is continuous through the surface $p_+ = p_-$ for a.e. $t \in [0,T]$. Moreover almost every particle $x \in S(t)$ through which the velocity has a jump, $u_+(x,t) \neq u_-(x,t)$, will remain on the surface S at almost every time (see [48] for precise statement). The latter condition is similar to the kinematic condition on the surface of a water wave.

The substitute of Theorem 3.4 can now be stated as follows.

Theorem 3.6. *Suppose that $u \in L^3(\mathbb{R}^n \times [0,T])$ is a weak solution to the Euler equations and the singular set of u is a slit. Then u conserves the total energy. Moreover the local energy balance equality (53) holds on the entire space $D = \mathbb{R}^n$.*

We note that a generic field with a slit-type singularity belongs exactly to the Onsager-critical class $B^{1/3}_{3,\infty}$, and no better class. As an example, consider $u(x,y) = (1,0)$, for $y > 0$, and $u(x,y) = (0,0)$ for $y \leq 0$, and assume that u is tapered far from the origin. It is a typical example of a vortex sheet which we will discuss next.

3.2. **Energy conservation for vortex sheets.** A typical example of a slit is a smooth vortex sheet. Vortex sheets in the classical sense are singular solutions to the Euler equations with vorticity concentrated on a hypersurface (see [42]). For notational convenience we will consider the two dimensional case. In 2D a vortex sheet is described by the graph of a regular function $\zeta(\alpha,t) = (\alpha, h(\alpha,t))$ and vorticity density $\gamma = \gamma(\alpha,t)$ on the graph. Typically, one assumes 2π-periodicity on h and γ. Thus, in complex variable notation the velocity field off the sheet is given by the Biot-Savart law

$$\bar{u}(z,t) = \frac{1}{4\pi i} \int_{-\pi}^{\pi} \cot\left(\frac{z - \zeta(\alpha,t)}{2}\right) \gamma(\alpha,t) d\alpha.$$

Provided γ has enough smoothness on a time interval $[0,T]$, the standard potential theoretical considerations imply that $u \in L^\infty_t L^\infty_x$, the non-tangential, and hence normal, limits exist and are given by

$$u_\pm(\alpha,t) = -\frac{1}{4\pi i} PV \int_{-\pi}^{\pi} \overline{\cot}\left(\frac{\zeta(\alpha,t) - \zeta(\alpha',t)}{2}\right) \gamma(\alpha',t) d\alpha' \mp \gamma(\alpha,t)\vec{s},$$

where $\vec{s}$ is the unit tangent vector oriented in the positive direction of the x-axis. The pressure can be recovered from Bernoulli's function, and is given by the double-layer potential formula

$$p = -\frac{1}{2}|u|^2 + \frac{1}{2}\mathcal{D}(|u_+|^2 - |u_-|^2).$$

From the classical jump relations for the double-layer potential $\mathcal{D}$ we see that the limits $p_\pm$ exist, $p_+ = p_- = \frac{1}{4}(|u_+|^2 + |u_-|^2)$ and $p^*_\pm \in L^q(S)_{\mathrm{loc}}$ for all $1 \leq q < \infty$.

Corollary 2. *According to our Definition 3.5, the classical vortex sheet is a slit.*

The kinematic condition on the surface discussed above is nothing but the well-known evolution law of the sheet:

$$\partial_t h = -U_1 \partial_\alpha h + U_2,$$

where $U = \frac{1}{2}(u_+ + u_-)$. In order for the total kinetic energy of the vortex sheet to be finite we assume vanishing of the total circulation:

$$\int_{-\pi}^{\pi} \gamma(\alpha,t) d\alpha = 0.$$

Under this condition, $u \in L^\infty L^2$, and by interpolation with $u \in L^\infty L^\infty$ we obtain $u \in L^3 L^3$.

Thus, the conditions of Theorem 3.6 are satisfied and we arrive at the following corollary.

Corollary 3 ([48]). *Suppose that $\gamma, h \in C^\infty([0,T] \times [-\pi,\pi])$, and the total circulation of γ is zero. Then the energy of the vortex sheet is conserved.*

Vortex sheets of this nature are known to exist in 2D and 3D locally in time in spaces of functions that admit analytic extension to a complex strip (see [4, 50]). In general, the global existence is precluded by occurrence of the roll-up singularity (see [37]). The conditions on Cauchy data stated in [50] that guarantee local existence allow for sheets with zero circulation. Thus, Corollary 3 applies to a variety of existing vortex sheets. However, just as for Theorem 3.4 the proof of Theorem 3.6 is completely local. So, it applies to obtain local energy balance relation even for sheets with infinite energy.

Unfortunately, as we note in the beginning of this lecture Theorem 3.6 closes the possibility of finding Osager-critical dissipative solutions among regular vortex sheets. The curvature blow-up of initially regular sheet constructed by Caflisch and Orellana [6] still retains the structure of a slit as defined above even at the critical time. Therefore this type of singularity sustains energy conservation again due to Theorem 3.6. However, the spiral roll-up that develops due to Kelvin-Helmoltz instability is severe enough to dismantle the geometric structure of the sheet thus opening a possibility of energy loss or gain. In [53] Wu proved regularity results implying that at the time of roll-up a vortex sheet fails to remain in the class of so-called cord-arch curves with moderate rate of roll-up. This is also suggestive of a steep velocity gradient near the core of the spiral. Since away from the core the vortex sheet remains regular, Theorem 3.6 implies that the energy flux is localized around the center. So, the question of energy concervation is reduced to a point singularity which falls into the lower dimensional settings of Theorem 3.4. According to Theorem 3.4 if u remains in $L^3([0,t^*];L^6)$ near the center then the energy is conserved, however it is not known whether this condition holds or not.

We note that if energy does not remain constant at the time of roll-up, this automatically implies non-uniqueness of solutions in so-called Delort's class. Those are solutions with vorticity being a measure-valued function of time (see [16]), of which vortex sheets are particular examples. Existence of such solutions with sign definite initial data was demonstrated by Delort in [16] via a weak limit of solutions with regularized initial data. As a result Delort's solutions can only dissipate energy at worst. So, if the energy of a classical vortex sheet increases at the roll-up time, this solution must differ from Delort's one. If the energy drops at the time of roll-up, the reversed in time solution proves the same result.

4. **Lecture 4: energy equality for Leray-Hopf solutions of the NSE.**

In this last lecture we will address the question of anomalous dissipation in viscous fluids (not to be confused with the dissipation anomaly $\epsilon > 0$), which carries similar difficulty as the inviscid case. Again, the Navier-Stokes equation governing evolution of a viscous fluid is given by

$$u_t + (u \cdot \nabla)u = \nu \Delta u - \nabla p + f. \tag{62}$$

Here u is a three dimensional divergence free field, and $f \in \mathcal{S}$ divergence free also. For the most part of this lecture we will assume fluid domain Ω to be $\mathbb{R}^3$ or $\mathbb{T}^3$. We refer to Temam [52] for the classical well-posedness theory for this equation. Let us recall that for every field $u_0 \in L^2(\Omega)$ there exists a weak solution $u \in C_w([0,T);L^2) \cap L^2([0,T);H^1)$

to (62) such that the energy inequality

$$|u(t)|_2^2 + 2\nu \int_{t_0}^t |\nabla u(s)|_2^2 ds \le |u(t_0)|_2^2 + 2\int_{t_0}^t f \cdot u(s) ds, \tag{63}$$

holds for all $0 < t$, and a.e. $0 \le t_0 < t$ including $t_0 = 0$. So, $\lim_{t\to 0+} u(t) \to u_0$ strongly in L^2. If (63) holds in the local sense too, the solution is called suitable. Existence of such solutions was demonstrated by Scheffer [44] in $\mathbb{R}^3$ and other domains by [3, 51]. In fact in the case of $\mathbb{R}^3$ the Leray scheme already produces a suitable weak solution.

It is believed that for $\nu > 0$ any Leray-Hopf solution has no anomalous dissipation, i.e. the energy equality rather than inequality (63) holds, i.e.

$$|u(t)|_2^2 + 2\nu \int_0^t |\nabla u(s)|_2^2 ds = |u_0|_2^2 + 2\int_0^t f \cdot u(s) ds, \quad t > 0. \tag{64}$$

The Leray-Hopf regularity class $C_w([0,T); L^2) \cap L^2([0,T); H^1)$ is not sufficient, at least in a direct sense, to establish (64). Under the additional assumption $u \in L^4 L^4$, the energy equality was proved by Lions [39]. Recently, Kukavica [38] relaxed Lions' criterion to $p \in L^2 L^2$, although this condition is dimensionally the same. By a direct application of Theorem 1.4 we obtain a dimensionally different sufficient condition for (64).

Theorem 4.1. *Let $u \in C_w([0,T); L^2(\mathbb{R}^3)) \cap L^2([0,T); H^1(\mathbb{R}^3))$ be a weak solution to the 3D incompressible Navier-Stokes equations with*

$$\lim_{q\to\infty} \int_0^T \lambda_q \|u_q(t)\|_3^3 dt = 0 \tag{65}$$

Then u satisfies the energy equality (64). In particular, (64) holds if $u \in L^3([0,T); H^{\frac{5}{6}})$.

Let us notice that by interpolation with $L^2([0,T); H^1)$ Lions' condition implies $u \in L^3([0,T); H^{\frac{5}{6}})$. In a bounded domain the analogue of Theorem 4.1 can be stated in terms of the domain of the Stokes operator A. Thus, (64) holds provided $u \in L^3([0,T); D(A^{\frac{5}{12}}))$, which is again a condition of Onsager's physical criticality (see [11]).

The analogue of Theorem 3.4 for point singularities requires a weaker assumption on the Hölder continuity in time of the singular set.

Theorem 4.2 ([47]). *Let $s \in C^{1/2}([0,T]; \mathbb{R}^3)$ and u be a weak solution to the NSE satisfying the following conditions*

(i) $u \in C_w L^2 \cap L^2 H^1 \cap L^3 L^{9/2}$;

(ii) $u \in \mathcal{R}((0,T) \times \mathbb{R}^3 \backslash \mathrm{Graph(s)})$,

Then u satisfies the generalized energy equality:

$$\int_{\mathbb{R}^3\times\{t''\}} |u|^2\phi - \int_{\mathbb{R}^3\times\{t'\}} |u|^2\phi - \int_{\mathbb{R}^3\times(t',t'')} |u|^2 \partial_t\phi$$
$$= \int_{\mathbb{R}^3\times(t',t'')} (|u|^2 + 2p) u\cdot\nabla\phi - 2\nu \int_{\mathbb{R}^3\times(t',t'')} |\nabla u|^2\phi + \nu \int_{\mathbb{R}^3\times(t',t'')} |u|^2 \Delta\phi. \tag{66}$$

for all $\phi \in C_0^\infty([0,T] \times \mathbb{R}^3)$, and $0 \le t' < t'' \le T$. In particular u is a suitable weak solution.

Conditions stated in the previous too theorems, although weaker than the classical ones, still do not bring us much closer to the Leray-Hopf regularity. Figure 1 gives a graphic illustration of the known results relative to the Leray-Hopf line in the $(\frac{1}{p}, \frac{1}{q})$ coordinates. However, one can still hope that the Onsager critical scaling can be broken with the use of extra information about the singular set. Recall that the end result of the partial regularity

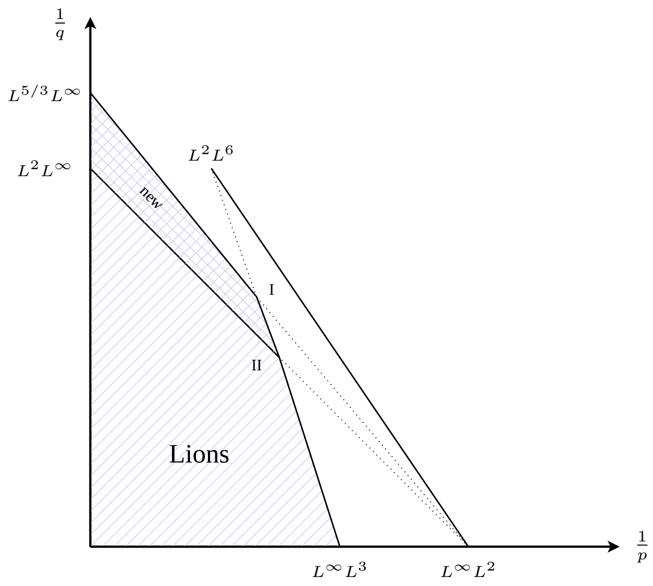

FIGURE 1. Here, $I = L^3L^{9/2}$, $II = L^4L^4$

theory advanced by Scheffer [43], Caffarelli, Kohn and Nirenberg [3], Sohr, von Wahl [49] and others states that the singular set of a suitable weak solution is 1-dimensional in the sense of the parabolic Hausdorff dimension. Moreover, the 1-dimensional measure of the set is zero. Let us examine how one can overcome the Onsager scaling, for example, when the singular set is just one point at time $t = 0$ and the solution is regular for $t \in [-1, 0)$. We can assume without loss of generality that the singular point is $0 \in \mathbb{R}^3$.

Lemma 4.3. *Suppose that under the conditions described above the weak solution u satisfies*

$$\lim_{\epsilon \to 0} \sup_{t \in (-1,0)} \int_{|x|<\epsilon} |u(x,t)|^2 dx = 0. \tag{67}$$

Then the energy equality (64) as well as its local variant (66) hold for all $-1 \leq t' < t'' \leq 0$.

We see that (67) is already of the same dimension as $L^\infty L^2$. For example, it holds if $u \in L^\infty L^p$, for some $p > 2$. Let us sketch the argument of Lemma 4.3.

Proof. Let us fix $\epsilon > 0$ and $\delta > 0$ and fix a non-decreasing $\phi_0 \in C^\infty(\mathbb{R}^+)$ defined for $r \geq 0$, $\phi_0(r) = 1$ for $r > 1$ and $\phi_0(r) = 0$ for $r < 1/2$. Let us consider

$$\phi_{\epsilon,\delta}(x,t) = \phi_0\left(\left(\frac{|x|^2}{\epsilon^2} + \frac{t^2}{\delta^2}\right)^{1/2}\right).$$

Let us plug it into (66) assuming $t'' = 0$ and $-1 \leq t' < 0$. We write the obtained identity using labels for respective terms in the same order as they appear in (66):

$$A_{\epsilon,\delta} - B_{\epsilon,\delta} - C_{\epsilon,\delta} = D_{\epsilon,\delta} - 2\nu E_{\epsilon,\delta} + \nu F_{\epsilon,\delta}. \tag{68}$$

Clearly we have

$$A_{\epsilon,\delta} \to \int_{\mathbb{R}^3\times\{0\}} |u|^2, \tag{69}$$

$$B_{\epsilon,\delta} \to \int_{\mathbb{R}^3\times\{t'\}} |u|^2, \tag{70}$$

$$E_{\epsilon,\delta} \to \int_{\mathbb{R}^3\times(0,t')} |\nabla u|^2, \tag{71}$$

as $\epsilon, \delta \to 0$ regardless of the relative speeds of ϵ and δ.

Next notice that all partial derivatives of $\phi_{\epsilon,\delta}$ are supported in the ellipsoidal shell

$$1/2 \le \left(\frac{|x|^2}{\epsilon^2} + \frac{t^2}{\delta^2} \right)^{1/2} \le 1,$$

where in particular $|t| \le \delta$ and $|x| \le \epsilon$. Thus, we have up to absolute constant multiples

$$\chi_{(t',0)}(t)|\partial_t \phi_{\epsilon,\delta}(x,t)| \le \frac{1}{\delta}\chi_{(-\delta,0)}(t)\chi_{|x|\le\epsilon}(x), \tag{72}$$

$$\chi_{(t',0)}(t)|\partial_x \phi_{\epsilon,\delta}(x,t)| \le \frac{1}{\epsilon}\chi_{(-\delta,0)}(t)\chi_{|x|\le\epsilon}(x), \tag{73}$$

$$\chi_{(t',0)}(t)|\partial_x^2 \phi_{\epsilon,\delta}(x,t)| \le \frac{1}{\epsilon^2}\chi_{(-\delta,0)}(t)\chi_{|x|\le\epsilon}(x). \tag{74}$$

These imply the following estimates on the remaining terms C, D, F:

$$|C_{\epsilon,\delta}| \le \frac{1}{\delta}\int_{-\delta}^0 \int_{|x|<\epsilon} |u|^2 dxdt, \tag{75}$$

$$|D_{\epsilon,\delta}| \le \frac{1}{\epsilon}\int_{-\delta}^0 \int_{|x|<\epsilon} (|u|^3 + |p||u|) dxdt, \tag{76}$$

$$|F_{\epsilon,\delta}| \le \frac{1}{\epsilon^2}\int_{-\delta}^0 \int_{|x|<\epsilon} |u|^2 dxdt. \tag{77}$$

We now let $\delta \to 0$ first, and then $\epsilon \to 0$. Since every Leray-Hopf solution remains in $u \in L^3 L^3_{loc}$ and $p \in L^{3/2} L^{3/2}_{loc}$ we clearly have $D_{\epsilon,\delta} \to 0$ and $F_{\epsilon,\delta} \to 0$ already as $\delta \to 0$. As to $C_{\epsilon,\delta}$ we use the condition of the lemma to obtain

$$|C_{\epsilon,\delta}| \le \sup_{t\in(-1,0)} \int_{|x|<\epsilon} |u(x,t)|^2 dx \to 0$$

as $\epsilon \to 0$. □

Balancing the rates of decay between δ and ϵ one can obtain further conditions for energy equality up to time $t = 0$. These include $u \in L^5 L^3$ (take $\delta = \epsilon^{5/2}$) and the line of spaces connecting $L^{14} L^{42/17}$ and $L^\infty L^2$, excluding the latter end (take $\delta = \epsilon^4$).

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