Math 310: **Hour Exam 2** (Solutions)
Prof. S. Smith: Fri 9 Apr 1999

You must SHOW WORK to receive credit.

Problem 1:
(a) Do the vectors $(1, 1, 2)$, $(3, 2, 1)$, and $(5, 3, 0)$ form a basis of $\mathbf{R}^3$? (indicate why/why not)

No. One way: The matrix with these rows has determinant 0, so the vectors are not LI.

(b) Inside the vector space $\mathcal{P}_4$ (of poynomials of degree less than 4), what is the dimension of the subspace spanned by $1 - x, x - x^2, 1 - 2x + x^2$? (Indicate a basis for this subspace).

The dimension is 2: since $(1 - x) - (x - x^2) = 1 - 2x + x^2$, we see the three polynomials are LD. But the first two $1 - x, x - x^2$ are LI, so form a basis for the span of all 3.

Problem 2:
(a) Let V be the space of all differentiable functions on $\mathbf{R}$; and define a mapping

$$L(f(x)) = f'(x) - f(x)$$

on V. Show that L is a LINEAR transformation.

(for +:) $L(f + g) = (f + g)' - (f + g) = f' + g' - f - g$ *while*
$L(f) + L(g) = (f' - f) + (g' - g)$ *... equal.*
(for scalar mult.:) $L(cf) = (cf)' - cf = cf' - cf$ *while*
$cL(f) = c(f' - f)$ *...equal. So L is linear.*

(b) Give the matrix representing (in the standard basis) the linear transformation $L : \mathbf{R}^3 \to \mathbf{R}^3$ defined by $L(x_1, x_2, x_3) = (x_1 - 7x_2 + 3x_3, 5x_1 - 3x_3, 4x_1 - x_2 + 4x_3)$.

Apply L to standard basis, put into columns to get $\begin{pmatrix} 1 & -7 & 3 \\ 5 & 0 & -3 \\ 4 & -1 & 4 \end{pmatrix}$

Problem 3:
(a) In $\mathbf{R}^3$ with the usual dot product, determine the orthogonal complement to the row space of the matrix $\begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$.

Just solve $Ax = 0$: row-reduce with $A_2^{-1\times 1}$ to $\begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & 2 \end{pmatrix}$.

Then x_3 is free, and solutions have form $(-x_3, -x_3, x_3)^T$

(b) Verify that the vectors $(1, -1, 0)$, $(1, 1, -2)$ and $(1, 1, 1)$ are orthogonal (in the usual dot product) to each other. This guarantees they are LI, and so form a basis of $\mathbf{R}^3$. Determine the coordinates of the vector $b = (1, 2, 3)$ in this basis (easier, if you take advantage of the orthogonality).

The dot product for each pair is indeed 0. So dividing them by their lengths gives and orthonormal basis.

So we do not have to solve systems $Ax = b$ as in Section 3.5; as a shortcut we can just find the vector projections as in Section 5.1. For example, the first is

$$\frac{(1, 2, 3) \cdot (1, -1, 0)}{(1, -1, 0) \cdot (1, -1, 0)}(1, -1, 0) = -\frac{1}{2}(1, -1, 0)$$

So the first coordinate is $-\frac{1}{2}$. Similarly check that the second coordinate is $-\frac{3}{6} = -\frac{1}{2}$ and the third is $\frac{6}{3} = 2$.

Problem 4:
Find the best possible straight line to fit the 3 data points $(0,1)$, $(1,2)$, and $(2,4)$. That is, use the least-squares method to find unknown (m,b) in the inconsistent system $Ax = y$ given by:

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}.$$

Multiply A^T by the augmented matrix $[A|b]$ to get

$$\begin{pmatrix} 5 & 3 & 10 \\ 3 & 3 & 7 \end{pmatrix}$$

Row operation $A_2^{-\frac{3}{5}\times 1}$ leads to $\left(\begin{array}{cc|c} 5 & 3 & 10 \\ 0 & \frac{6}{5} & 1 \end{array}\right)$. *This gives $b = \frac{5}{6}$ and then $m = \frac{3}{2}$.*

Problem 5:
(a) Let S be the subspace of $\mathbf{R}^3$ spanned by $v_1 = (2,-1,-1)$ and $v_2 = (-1,2,-1)$. Use the Gram-Schmidt process to find an orthonormal basis for S.

First get orthogonal: use $x_1 = (2,-1,-1)$ and then
$x_2 = v_2 - [(v_2 \cdot x_1)/(x_1 \cdot x_1)]x_1 = (-1,2,-1) - [-3/6](2,-1,-1) = \frac{1}{2}(0,3,-3)$.

To make orthoNORMAL, divide by lengths to get $u_1 = \frac{1}{\sqrt{6}}(2,-1,-1)$ and $u_2 = \frac{1}{\sqrt{2}}(0,1,-1)$.

(b) Now find an orthogonormal basis for the orthogonal complement $S^{\perp}$, for S in (a). (That is, extend your answer in (a) to an orthonormal basis of $\mathbf{R}^3$).

Working as in Problem 3(a) we find S^* is spanned by $(1,1,1)$, so an orhtonormal basis is given by $\frac{1}{\sqrt{3}}(1,1,1)$.